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Generalized Laguerre Polynomials and Rational Chebyshev Collocation Method for Solving Unsteady Gas Equation

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Abstract

In this paper we propose, a collocation method to solve unsteady gas equation which is a nonlinear ordinary differential equation on semi-infnite interval. This approach is based on generalized Laguerre polynomials and rational Chebyshev functions. This method reduces the solution of this problem to the solution of a system of algebraic equations. We also present the comparison of this work with some other numerical results. It shows that the present solution is highly accurate.

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1 Introduction

In the study of the unsteady flow of a gas through a semi-infinite porous medium [7],[9] initially filled with gas at a uniform pressure $p_0 \geq 0$, at time $t = 0$, the pressure at the outflow face is suddenly reduced from p_0 to $p_1 \geq 0$ ($p_1 = 0$ is the case of diffusion into a vacuum) and is, thereafter, maintained at this lower pressure. The unsteady isothermal flow of gas is described by a nonlinear partial differential equation. The nonlinear partial differential equation that describes the unsteady flow of gas through a semi-infinite porous medium has been derived by Muskat [8] in the form

$$\nabla^2(P^2) = (2\Phi\mu/k)\,\frac{\partial P}{\partial t} \tag{1}$$

where P is the pressure within porous medium, Φ the porosity, μ the viscosity, k the permeability, and t the time. New variables were introduced by Kidder [7] and Davis [3] to transform the nonlinear partial differential equation (1) to the nonlinear ordinary differential equation. The nonlinear ordinary differential equation due to Kidder [7] given by (unsteady gas equation)

$$y'' + 2xy'/(1-\alpha y)^{1/2} = 0, \qquad x > 0, \qquad 0 < \alpha < 1 \tag{2}$$

The typical boundary conditions imposed by the physical properties are

$$y(0) = 1, \qquad y(\infty) = 0. \tag{3}$$

A substantial amount of numerical and analytical work has been invested so far [7],[10] on this model. The main reason of this interest is that the approximation can be used in many engineering purposes. As stated before, the problem (2) was handled by Kidder [7] where a perturbation technique is carried out to include terms of the second order. Recently Wazwaz[12] has applied the modified decomposition method for solving nonlinear this equation. The base of his approach is modification of the Adomian decomposition method. The diagonal Pade approximants are effectively used in the analysis to capture the essential behavior of $y(x)$ and to determine the initial slope $y'(0)$.

2 Properties of generalized Laguerre polynomials and rational Chebyshev functions

This section is devoted to the introduction of the basic notions and working tools concerning orthogonal generalized Laguerre polynomials and rational Chebyshev functions.

$L_n^\alpha(x)$ (generalized Laguerre polynomial) is the nth eigenfunction of the Sturm-Liouville problem [4],[5]:

$$x\frac{\mathrm{d}^2}{\mathrm{d}^2x}L_n^\alpha(x) + (\alpha+1-x)\frac{\mathrm{d}}{\mathrm{d}x}L_n^\alpha(x) + nL_n^\alpha(x) = 0,$$
$$x \in I = [0,\infty), \qquad n = 0,1,2,\ldots. \tag{4}$$

The generalized Laguerre polynomials are defined with the following recurrence formula:

$$\begin{aligned} L_0^\alpha(x) &= 1, \qquad L_1^\alpha(x) = 1+\alpha-x, \\ nL_n^\alpha(x) &= (2n-1+\alpha-x)L_{n-1}^\alpha(x) - (n+\alpha-1)L_{n-2}^\alpha(x), n \geq 2, \alpha > -1 \end{aligned} \tag{5}$$

with the normalizing condition:

$$L_n^\alpha(0) = \binom{n+\alpha}{n}. \tag{6}$$

These are orthogonal polynomials for the weight function $w_\alpha = x^\alpha e^{-x}$:

$$\int_0^{+\infty} L_n^\alpha(x) L_m^\alpha(x) w_\alpha(x) \mathrm{d}x = \left(\frac{\Gamma(n+1+\alpha)}{n!} \right) \delta_{nm}, \tag{7}$$

The rational Chebyshev functions, denoted by $R_n(x)$, are defined by [2, 6]

$$R_n(x) = T_n(y) = \cos(nt), \tag{8}$$

where L is a constant parameter, $T_n(y)$ is the well-known Chebyshev polynomial and

$$y = \frac{x-L}{x+L}; \quad y \in [-1,1], \qquad t = 2\cot^{-1}\left(\sqrt{\frac{x}{L}}\right); \quad t \in [0,\pi]. \tag{9}$$

The constant parameter L sets the length scale of the mapping. Boyd [1] offered guidelines for optimizing the map parameter L. $R_n(x)$ is the nth eigenfunction of the singular Sturm-Liouville problem

$$(x+L)\frac{\sqrt{x}}{L}\Big((x+L)\sqrt{x}R_n'(x)\Big)' + n^2 R_n(x) = 0,$$
$$x \in I = [0,\infty), \quad n = 0,1,2,\dots, \tag{10}$$

and satisfies in the following recurrence relation:

$$R_0(x) = 1, \qquad R_1(x) = \frac{x-L}{x+L},$$
$$R_{n+1}(x) = 2\left(\frac{x-L}{x+L}\right) R_n(x) - R_{n-1}(x), \qquad n \geq 1. \tag{11}$$

These functions are orthogonal with respect to the weight function $w_R(x) = \sqrt{L}/[\sqrt{x}(x+L)]$

$$\int_0^{\infty} R_n(x) R_m(x) w_R(x) = \begin{cases} \pi\delta_{nm} & m = 0, \\ \frac{\pi}{2}\delta_{nm} & m \geq 1. \end{cases} \tag{12}$$

Let $w(x)$ denotes either generalized Laguerre polynomials' or rational Chebyshev functions' weight function over the interval $I = [0,\infty)$. We define

$$L_w^2(I) = \{v : I \to \mathbb{R} \mid v \text{ is measurable and } \|v\|_w < \infty\}, \tag{13}$$

where

$$\|v\|_w = \left(\int_0^{\infty} |v(x)|^2 w(x) dx\right)^{\frac{1}{2}}, \tag{14}$$

is the norm induced by the scalar product

$$\langle u, v\rangle_w = \int_0^{+\infty} u(x)v(x)w(x)dx. \tag{15}$$

Let

$$\mathfrak{R}_N = span\{\phi_0, \phi_1, \ldots, \phi_N\}, \tag{16}$$

where ϕ_is are either generalized Laguerre polynomials or rational Chebyshev functions.

For generalized Laguerre polynomials we show by $x_j,\ j = 0, \ldots, N$ the generalized Laguerre-Gauss-Radau nodes which are zeroes of $\frac{\mathrm{d}}{\mathrm{d}x} L_N^\alpha$ and the point $x = 0$; For rational Chebyshev functions x_js are zeros of the function $R_{N+1}(x) + R_N(x)$ and are named as rational Chebyshev-Gauss-Radau nodes.

By Gauss-Radau integration we have:

$$\int_0^\infty u(x)w(x)\mathrm{d}x = \sum_{j=0}^{N} u(x_j)w_j \quad \forall u \in \mathfrak{R}_{2N}, \tag{17}$$

where w_j is the corresponding weight of the node x_j. For generalized Laguerre polynomials the corresponding weights are:

$$\begin{aligned} w_0 &= \frac{(\alpha+1)\Gamma^2(\alpha+1)(N-1)!}{\Gamma(N+\alpha+1)} \\ w_j &= \frac{\Gamma(\alpha+N)}{N!}\left(L_N^\alpha(x_j)\frac{\mathrm{d}}{\mathrm{d}x}L_{N-1}^\alpha(x_j)\right)^{-1}, \quad j = 1, 2, \ldots, N., \end{aligned} \tag{18}$$

also for rational Chebyshev functions the weights are

$$w_0 = \frac{\pi}{2N+1}, \qquad w_j = \frac{\pi}{N+1}, \quad j = 1, 2, \ldots, N. \tag{19}$$

We define

$$I_N u(x) = \sum_{j=0}^{N} a_j \phi_j(x), \tag{20}$$

such that $I_N u(x_j) = u(x_j),\ j = 0, \ldots, N$. $I_N u$ is the orthogonal projection of u upon $\mathfrak{R}_N$ with respect to the discrete inner product and discrete norm as:

$$\langle u, v\rangle_{w,N} = \sum_{j=0}^{N} u(x_j)v(x_j)w_j, \tag{21}$$

$$\|u\|_{w,N} = \langle u, u\rangle_{w,N}^{1/2}. \tag{22}$$

The Gauss-Radau integration formula implies that

$$\langle I_N u, v\rangle_{w,N} = \langle u, v\rangle_{w,N}, \qquad \forall u.v \in \mathfrak{R}_{2N}. \tag{23}$$

3 Solution of unsteady gas equation

To apply a collocation approximation to the standard unsteady gas equation introduced in (2) with boundary conditions (3), we approximate y as the truncated series $I_N y$. The solution $I_N y$ is represented by its values at the grid points x_j. The grid values of $I_N y$ are related to the discrete expansion coefficients a_ks. Thus, our goal is to find the coefficients a_k, $0 \le k \le N$.

For generalized Laguerre collocation method, at first we expand $y(x)$, as follows:

$$I_N y(x) = \sum_{j=0}^{N} a_j L_j^1(x/k). \quad k > 0. \tag{24}$$

where $k > 0$ is a constant. To find the unknown coefficients a_js, we substitute the truncated series into the (2) and boundary conditions in (3). So we have

$$\sum_{j=0}^{N} a_j L_j''^1(x/k) + 2x \frac{\sum_{j=0}^{N} a_j L_j'^1(x/k)}{[1 - \alpha \sum_{j=0}^{N} a_j {L_j}^1(x/k)]^{1/2}} = 0, \tag{25}$$

$$\sum_{j=0}^{N} a_j {L_j}^1(0) = 1. \tag{26}$$

It is clear that $\lim_{x\to\infty} \sum_{j=0}^{N} a_j {L_j}^1(x) = 0$. By replacing x in (25) with the N collocation points which are roots of functions $\frac{\mathrm{d}}{\mathrm{d}x} L_N^1$, we have N equations that generates a set of $N+1$ nonlinear equations with boundary equation in (3).

By applying the above discussion for rational Chebyshev collocation method and using rational Chebyshev-Gauss-Radau points, we have

$$\left(\frac{\mathrm{d}^2 I_N y}{\mathrm{d}x^2} + 2x \frac{\mathrm{d} I_N y}{\mathrm{d}x} (1 - \alpha I_N y)^{-\frac{1}{2}} \right) \Bigg|_{x=x_j} = 0, \quad j = 1, \dots, N, \tag{27}$$

such that

$$I_N y(x_0) = 1, \qquad \lim_{x\to\infty} I_N y(x) = 0. \tag{28}$$

By omitting the last equation in (27) and with two boundary conditions in (28), we have $N+1$ equations that generate a set of $N+1$ nonlinear equations that can be solved by Newton method for unknown coefficients a_ks.

Table 1 shows the comparison of the $y'(0)$, between generalized Laguerre polynomials (GLP), rational Chebyshev functions (RCF) for $N = 6, 7$ and Padé approximation used by [12].

Table 2 shows the approximations of $y(x)$ for standard unsteady gas with $\alpha = 0.5$ obtained by the methods proposed in this paper for $N = 7$, the perturbation method used by [7] and Padé approximation by Wazwaz [12].

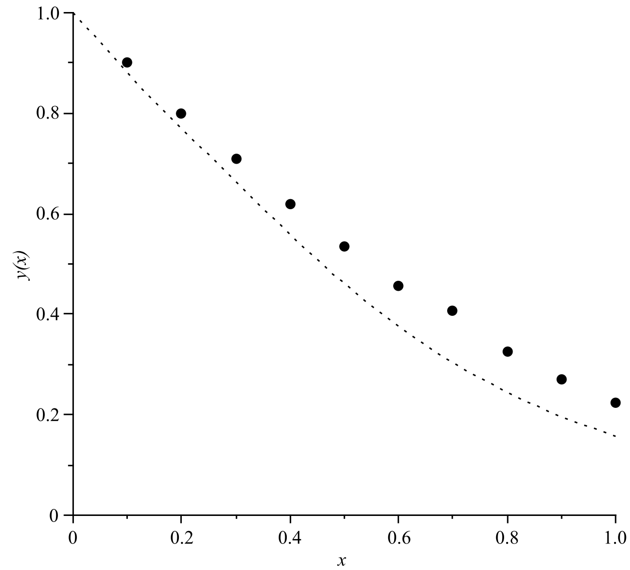

Figure 1: Unsteady gas equation graph obtained by GLP (circles) and RCF (dotted line).

Figure 1 shows the resulting graph of unsteady gas for $N = 7$ obtained by present methods.

Table 1. Comparison of initial slope $y'(0)$ for $\alpha = 0.5$.

N	GLP	RCF	Padé[2,2]	Padé[3,3]
6	-1.26417503	-1.10805718	-1.37317809	-1.02552970
7	-1.28213483	-1.26259357	-1.37317809	-1.02552970

Table 2. values of $y(x)$ for $\alpha = 0.5$ for $x = 0.1$ to 1.0

x	GLP	RCF	Perturbation	Padé[2,2]	Padé[3,3]
0.1	0.90035366	0.88042558	0.88165883	0.86330606	0.89791670
0.2	0.80020256	0.76963630	0.76630768	0.73012623	0.79852282
0.3	0.70810133	0.66402932	0.65653800	0.60330541	0.70411297
0.4	0.61792075	0.55870984	0.55440240	0.48488987	0.61650379
0.5	0.53394788	0.46068185	0.46136503	0.37616039	0.53705338
0.6	0.45698352	0.37504968	0.37831093	0.27773116	0.46656257
0.7	0.40743340	0.30320332	0.30559765	0.18968434	0.40624260
0.8	0.32539068	0.24431600	0.24313255	0.11171052	0.35608017
0.9	0.27070888	0.19666414	0.19046237	0.04323673	0.31799666
1.0	0.22306464	0.15835106	0.15876898	0.01646751	0.29002550

4 Conclusions

The fundamental goal of this paper has been to construct an approximation to the solution of nonlinear Unsteady gas equation . A set of orthoghonl polynomials and rational functions are proposed to provide an effective but simple way to improve the convergence of the solution by collocation method.

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