Generalized Laguerre Polynomials and Rational Chebyshev Collocation Method for Solving Unsteady Gas Equation

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Abstract

In this paper we propose, a collocation method to solve unsteady gas equation which is a nonlinear ordinary differential equation on semi-infinite interval. This approach is based on generalized Laguerre polynomials and rational Chebyshev functions. This method reduces the solution of this problem to the solution of a system of algebraic equations. We also present the comparison of this work with some other numerical results. It shows that the present solution is highly accurate.

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Keywords: Unsteady gas equation, Rational Chebyshev, Generalized Laguerre polynomials, Collocation method, Nonlinear ODE

1 Introduction

In the study of the unsteady flow of a gas through a semi-infinite porous medium [7],[9] initially filled with gas at a uniform pressure \( p_0 \geq 0 \), at time \( t = 0 \), the pressure at the outflow face is suddenly reduced from \( p_0 \) to \( p_1 \geq 0 \) (\( p_1 = 0 \) is the case of diffusion into a vacuum) and is, thereafter, maintained at this lower pressure. The unsteady isothermal flow of gas is described by a nonlinear partial differential equation. The nonlinear partial differential equation that describes the unsteady flow of gas through a semi-infinite porous medium has been derived by Muskat [8] in the form

\[
\nabla^2 (P^2) = (2\Phi \mu/k) \frac{\partial P}{\partial t}
\]  

(1)
where \( P \) is the pressure within porous medium, \( \Phi \) the porosity, \( \mu \) the viscosity, 
\( k \) the permeability, and \( t \) the time. New variables were introduced by Kidder [7] 
and Davis [3] to transform the nonlinear partial differential equation (1) to the 
nonlinear ordinary differential equation. The nonlinear ordinary differential 
equation due to Kidder [7] given by (unsteady gas equation)
\[
y'' + 2xy'/\sqrt{1-\alpha y} = 0, \quad x > 0, \quad 0 < \alpha < 1
\]
(2)
The typical boundary conditions imposed by the physical properties are
\[
y(0) = 1, \quad y(\infty) = 0.
\]
(3)
A substantial amount of numerical and analytical work has been invested so far 
[7],[10] on this model. The main reason of this interest is that the approxima-
tion can be used in many engineering purposes. As stated before, the problem 
(2) was handled by Kidder [7] where a perturbation technique is carried out 
to include terms of the second order. Recently Wazwaz[12] has applied the 
modified decomposition method for solving nonlinear this equation. The base 
of his approach is modification of the Adomian decomposition method. The 
diagonal Pade approximants are effectively used in the analysis to capture the 
essential behavior of \( y(x) \) and to determine the initial slope \( y'(0) \).

2 Properties of generalized Laguerre polyno-
mials and rational Chebyshev functions

This section is devoted to the introduction of the basic notions and working 
tools concerning orthogonal generalized Laguerre polynomials and rational 
Chebyshev functions.

\( L^\alpha_n(x) \) (generalized Laguerre polynomial) is the \( n \)th eigenfunction of the 
Sturm-Liouville problem [4],[5]:
\[
x \frac{d^2}{dx^2} L^\alpha_n(x) + (\alpha + 1 - x) \frac{d}{dx} L^\alpha_n(x) + nL^\alpha_n(x) = 0,
\]
\[ x \in I = [0, \infty), \quad n = 0, 1, 2, \ldots . \quad (4) \]

The generalized Laguerre polynomials are defined with the following recurrence 
formula:
\[
L^\alpha_0(x) = 1, \quad L^\alpha_1(x) = 1 + \alpha - x,
\]
\[
nL^\alpha_n(x) = (2n - 1 + \alpha - x)L^\alpha_{n-1}(x) - (n + \alpha - 1)L^\alpha_{n-2}(x), \quad n \geq 2, \alpha > -1(5)
\]
with the normalizing condition:
\[
L^\alpha_n(0) = \binom{n + \alpha}{n}.
\]
(6)
These are orthogonal polynomials for the weight function \( w_\alpha = x^\alpha e^{-x} \):
\[
\int_0^{+\infty} L_n^\alpha(x)L_m^\alpha(x)w_\alpha(x)dx = \left( \frac{\Gamma(n + 1 + \alpha)}{n!} \right) \delta_{nm},
\]
(7)

The rational Chebyshev functions, denoted by \( R_n(x) \), are defined by [2, 6]
\[
R_n(x) = T_n(y) = \cos(nt),
\]
(8)
where \( L \) is a constant parameter, \( T_n(y) \) is the well-known Chebyshev polynomial and
\[
y = \frac{x - L}{x + L}; \quad y \in [-1, 1], \quad t = 2 \cot^{-1}\left( \sqrt{\frac{x}{L}} \right); \quad t \in [0, \pi].
\]
(9)
The constant parameter \( L \) sets the length scale of the mapping. Boyd [1] offered guidelines for optimizing the map parameter \( L \). \( R_n(x) \) is the \( n \)th eigenfunction of the singular Sturm-Liouville problem
\[
(x + L)\sqrt{\frac{x}{L}}R'_n(x) + n^2R_n(x) = 0,
\]
\[
x \in I = [0, \infty), \quad n = 0, 1, 2, \ldots,
\]
(10)
and satisfies in the following recurrence relation:
\[
R_0(x) = 1, \quad R_1(x) = \frac{x - L}{x + L},
\]
\[
R_{n+1}(x) = 2 \left( \frac{x - L}{x + L} \right) R_n(x) - R_{n-1}(x), \quad n \geq 1.
\]
(11)
These functions are orthogonal with respect to the weight function \( w_R(x) = \sqrt{L/\sqrt{x(x + L)}} \)
\[
\int_0^\infty R_n(x)R_m(x)w_R(x) = \begin{cases} 
\pi \delta_{nm} & m = 0, \\
\frac{\pi}{2} \delta_{nm} & m \geq 1.
\end{cases}
\]
(12)

Let \( w(x) \) denotes either generalized Laguerre polynomials’ or rational Chebyshev functions’ weight function over the interval \( I = [0, \infty) \). We define
\[
L_w^2(I) = \{ v : I \to \mathbb{R} \mid v \text{ is measurable and } \|v\|_w < \infty \},
\]
(13)
where
\[
\|v\|_w = \left( \int_0^{+\infty} |v(x)|^2w(x)dx \right)^{\frac{1}{2}},
\]
(14)
is the norm induced by the scalar product
\[
\langle u, v \rangle_w = \int_0^{+\infty} u(x)v(x)w(x)dx.
\]
(15)
Let
\[ \mathcal{R}_N = \text{span}\{ \phi_0, \phi_1, \ldots, \phi_N \}, \] (16)

where \( \phi_i \)s are either generalized Laguerre polynomials or rational Chebyshev functions.

For generalized Laguerre polynomials we show by \( x_j, j = 0, \ldots, N \) the generalized Laguerre-Gauss-Radau nodes which are zeroes of \( \frac{d}{dx} L_\alpha^N \) and the point \( x = 0 \); For rational Chebyshev functions \( x_j \)s are zeros of the function \( R_{N+1}(x) + R_N(x) \) and are named as rational Chebyshev-Gauss-Radau nodes.

By Gauss-Radau integration we have:
\[
\int_0^\infty u(x)w(x)dx = \sum_{j=0}^N u(x_j)w_j \quad \forall u \in \mathcal{R}_{2N}, \tag{17}
\]

where \( w_j \) is the corresponding weight of the node \( x_j \). For generalized Laguerre polynomials the corresponding weights are:
\[
w_0 = \frac{(\alpha + 1)\Gamma^2(\alpha + 1)(N - 1)!}{\Gamma(N + \alpha + 1)}
\]
\[
w_j = \frac{\Gamma(\alpha + N)}{N!} \left( L_\alpha^N(x_j)\frac{d}{dx}L_{\alpha-1}^N(x_j) \right)^{-1}, \quad j = 1, 2, \ldots, N. \tag{18}
\]

also for rational Chebyshev functions the weights are
\[
w_0 = \frac{\pi}{2N + 1}, \quad w_j = \frac{\pi}{N + 1}, \quad j = 1, 2, \ldots, N. \tag{19}
\]

We define
\[
I_Nu(x) = \sum_{j=0}^N a_j \phi_j(x), \tag{20}
\]

such that \( I_Nu(x_j) = u(x_j), \quad j = 0, \ldots, N \). \( I_Nu \) is the orthogonal projection of \( u \) upon \( \mathcal{R}_N \) with respect to the discrete inner product and discrete norm as:
\[
\langle u, v \rangle_{w,N} = \sum_{j=0}^N u(x_j)v(x_j)w_j, \tag{21}
\]
\[
\|u\|_{w,N} = \langle u, u \rangle_{w,N}^{1/2}. \tag{22}
\]

The Gauss-Radau integration formula implies that
\[
\langle I_Nu, v \rangle_{w,N} = \langle u, v \rangle_{w,N}, \quad \forall u,v \in \mathcal{R}_{2N}. \tag{23}
\]


3 Solution of unsteady gas equation

To apply a collocation approximation to the standard unsteady gas equation introduced in (2) with boundary conditions (3), we approximate $y$ as the truncated series $I_Ny$. The solution $I_Ny$ is represented by its values at the grid points $x_j$. The grid values of $I_Ny$ are related to the discrete expansion coefficients $a_k$s. Thus, our goal is to find the coefficients $a_k$, $0 \leq k \leq N$.

For generalized Laguerre collocation method, at first we expand $y(x)$, as follows:

$$I_Ny(x) = \sum_{j=0}^{N} a_j L_j^1(x/k). \quad k > 0. \quad (24)$$

where $k > 0$ is a constant. To find the unknown coefficients $a_j$s, we substitute the truncated series into the (2) and boundary conditions in (3). So we have

$$\sum_{j=0}^{N} a_j L_j^1(x/k) + 2x \frac{\sum_{j=0}^{N} a_j L_j^1(x/k)}{[1 - \alpha \sum_{j=0}^{N} a_j L_j^1(x/k)]^{1/2}} = 0, \quad (25)$$

$$\sum_{j=0}^{N} a_j L_j^1(0) = 1. \quad (26)$$

It is clear that $\lim_{x \to \infty} \sum_{j=0}^{N} a_j L_j^1(x) = 0$. By replacing $x$ in (25) with the $N$ collocation points which are roots of functions $\frac{d}{dx} L_N^1$, we have $N$ equations that generates a set of $N + 1$ nonlinear equations with boundary equation in (3).

By applying the above discussion for rational Chebyshev collocation method and using rational Chebyshev-Gauss-Radau points, we have

$$\left( \frac{d^2 I_Ny}{dx^2} + 2x \frac{dI_Ny}{dx} (1 - \alpha I_Ny)^{-\frac{1}{2}} \right) \bigg|_{x=x_j} = 0, \quad j = 1, \ldots, N, \quad (27)$$

such that

$$I_Ny(x_0) = 1, \quad \lim_{x \to \infty} I_Ny(x) = 0. \quad (28)$$

By omitting the last equation in (27) and with two boundary conditions in (28), we have $N + 1$ equations that generate a set of $N + 1$ nonlinear equations that can be solved by Newton method for unknown coefficients $a_k$s.

Table 1 shows the comparison of the $y'(0)$, between generalized Laguerre polynomials (GLP), rational Chebyshev functions (RCF) for $N = 6, 7$ and Padé approximation used by [12].

Table 2 shows the approximations of $y(x)$ for standard unsteady gas with $\alpha = 0.5$ obtained by the methods proposed in this paper for $N = 7$, the perturbation method used by [7] and Padé approximation by Wazwaz [12].
Figure 1 shows the resulting graph of unsteady gas for $N = 7$ obtained by present methods.

Table 1. Comparison of initial slope $y'(0)$ for $\alpha = 0.5$.

<table>
<thead>
<tr>
<th>$N$</th>
<th>GLP</th>
<th>RCF</th>
<th>Padé[2,2]</th>
<th>Padé[3,3]</th>
</tr>
</thead>
<tbody>
<tr>
<td>6</td>
<td>-1.26417503</td>
<td>-1.10805718</td>
<td>-1.37317809</td>
<td>-1.02552970</td>
</tr>
<tr>
<td>7</td>
<td>-1.28213483</td>
<td>-1.26250357</td>
<td>-1.37317809</td>
<td>-1.02552970</td>
</tr>
</tbody>
</table>

Table 2. Values of $y(x)$ for $\alpha = 0.5$ for $x = 0.1$ to 1.0

<table>
<thead>
<tr>
<th>$x$</th>
<th>GLP</th>
<th>RCF</th>
<th>Perturbation</th>
<th>Padé[2,2]</th>
<th>Padé[3,3]</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.1</td>
<td>0.90035366</td>
<td>0.88042558</td>
<td>0.88165883</td>
<td>0.86330606</td>
<td>0.89791670</td>
</tr>
<tr>
<td>0.2</td>
<td>0.80020256</td>
<td>0.76963630</td>
<td>0.76630768</td>
<td>0.73012623</td>
<td>0.79852282</td>
</tr>
<tr>
<td>0.3</td>
<td>0.70810133</td>
<td>0.66402932</td>
<td>0.65653800</td>
<td>0.60330541</td>
<td>0.70411297</td>
</tr>
<tr>
<td>0.4</td>
<td>0.61792075</td>
<td>0.55870984</td>
<td>0.55440240</td>
<td>0.48488987</td>
<td>0.61650379</td>
</tr>
<tr>
<td>0.5</td>
<td>0.53394788</td>
<td>0.46068185</td>
<td>0.46136503</td>
<td>0.37616039</td>
<td>0.53705338</td>
</tr>
<tr>
<td>0.6</td>
<td>0.45698352</td>
<td>0.37504968</td>
<td>0.37831093</td>
<td>0.27773116</td>
<td>0.46656257</td>
</tr>
<tr>
<td>0.7</td>
<td>0.40743340</td>
<td>0.30320332</td>
<td>0.30559765</td>
<td>0.18968434</td>
<td>0.40624260</td>
</tr>
<tr>
<td>0.8</td>
<td>0.32539068</td>
<td>0.24431600</td>
<td>0.24313255</td>
<td>0.11171052</td>
<td>0.35608017</td>
</tr>
<tr>
<td>0.9</td>
<td>0.27070888</td>
<td>0.19666414</td>
<td>0.19046237</td>
<td>0.04323673</td>
<td>0.31799666</td>
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<tr>
<td>1.0</td>
<td>0.22306464</td>
<td>0.15835106</td>
<td>0.15876898</td>
<td>0.01646751</td>
<td>0.29002550</td>
</tr>
</tbody>
</table>

4 Conclusions

The fundamental goal of this paper has been to construct an approximation to the solution of nonlinear Unsteady gas equation. A set of orthognal
polynomials and rational functions are proposed to provide an effective but simple way to improve the convergence of the solution by collocation method.

References


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