A Study on a Subset of Absolutely
Convergent Sequence Space

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Abstract: In this paper, we define the section sequence space $\ell_s$ which is called the section sequence space of $\ell$ and study the inclusion $\ell_s \subset \ell$. Further AK-property, Dual space of $\ell_s$ are studied.

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1. Introduction and preliminaries

Let $\ell$ be an BK-space. We denote $\ell_s$ as the sequence consisting of all those sequences $\ell_s = \{ x = (x_k) : (y_k) \in \ell \}$, where

$$y_k = x_1 + x_2 + x_3 + \ldots + x_k \quad \text{for each fixed } k = 1, 2, 3, \ldots.$$

For a sequence $(y_k) \in \ell_s$, we can calculate the sequence $(x_k)$ by
\[ x_1 = y_1, \]
\[ x_2 = y_2 - x_1 = y_2 - y_1, x_3 = y_3 - x_1 - x_2 = y_3 - y_1 - (y_2 - y_1) = y_3 - y_2, \ldots \]
\[ x_n = y_n - y_{n-1}. \]

For any \( x \in \ell_s \), we define
\[ \|x\| = \left( |x_1|^2 + |x_1 + x_2|^2 + \ldots + |x_1 + x_2 + \ldots + x_k + x_{k+1}|^2 + \ldots \right)^{1/2} < \infty. \]

For a given a sequence \( x = \{x_k\} \), we define the \( n^{th} \) section as the sequence
\[ x^{(n)} = \{x_1, x_2, \ldots, x_n, 0, 0, \ldots\}. \]

Let
\[ \delta^{(n)} = (0, 0, \ldots, 1, -1, 0, 0, 0, \ldots), \]
where 1 is in the \( n^{th} \) place and -1 in the \( (n+1)^{th} \) place.

An FK-space \( X \) is said to have AK-property if \( \{\delta^{(n)}\} \) is a Schauder basis for \( X \). The space \( X \) is said to have AD if \( \Phi \) is dense in \( X \). We note that \( \text{AK} \Rightarrow \text{AD} \) by [1].

For the sequence space \( X \), we define
\[ X^\beta = \left\{ a = \{a_k\} : \sum_{k=1}^\infty a_kx_k \text{ is convergent, for each } x \in X \right\}. \]

We called \( X^\alpha, X^\beta, X^\gamma \) as the \( \alpha \)-dual of \( X \), \( \beta \)-dual of \( X \), \( \gamma \)-dual of \( X \), respectively. Note that \( X^\alpha \subset X^\beta \subset X^\gamma \). If \( X \subset Y \) then \( Y^\mu \subset X^\mu \), for \( \mu = \alpha, \beta \) and \( \gamma \).

We have the following known results.

**Lemma 1**: (See Theorem 7.2.7.in [3])

Let \( X \) be an FK-space \( \supset \Phi \). Then

(i) \( X^\gamma \subset X^f \)

(ii) If \( X \) has AK, \( X^\beta = X^f \)

(iii) If \( X \) has AD, \( X^\beta = X^\gamma \)

**Lemma 2** (Page 69, 2.3.1 in [2]):
If a Normed space $X$ has a Schauder basis, then $X$ is separable.

2. Main Results:

In this section we study some of the property of $\ell_s$.

**Proposition-1:** $\ell_s$ has Schauder basis namely $(e_1, e_2, e_3, \ldots)$, where $e_k = \{0,0,0,\ldots,1,-1,0,0,\ldots\}$, 1 is in the $k^{th}$ place and -1 is at the $(k+1)^{th}$ for $k=1,2,\ldots$.

**Proof:** We know that $\{\delta^{(1)}, \delta^{(2)}, \ldots\}$ is a Schauder basis for $\ell$ transformations given in the introduction. It follows that $(e_1, e_2, e_3, \ldots)$ is a Schauder basis for $\ell_s$.

**Theorem-1:** $\ell_s$ has AK-property.

**Proof.** Let $x = (x_1) \in \ell_s$. Then $T(y_n) \in \ell$ with $y_n = x_1 + x_2 + \ldots + x_k$. Put $x^{(n)} = (x_1, x_2, x_3, \ldots, x_n, 0, 0, \ldots)$. Then

$$
\|x - x^{(n)}\| = \|0,0,0,\ldots,x_{n+1},x_{n+2},\ldots\| = |x_{n+1}| + |x_{n+1} + x_{n+2}| + \ldots.
$$

$$
= |y_{n+1} - y_n| + |y_{n+2} - y_{n+1}| + |y_{n+3} - y_{n+2}| + \ldots.
$$

$$
= \sum_{k=n+1}^{\infty} |y_k - y_n| \to 0, \text{ as } n \to \infty.
$$

Thus we have $0 \leq \|x - x^{(n)}\| \leq 0$, for sufficiently large $n$. Hence

$$
\|x - x^{(n)}\| \to 0, \text{ as } n \to \infty.\text{ Therefore the space } \ell_s \text{ has AK. This completes the proof.}
$$

**Corollary-1:** The set $\{\delta^{(1)}, \delta^{(2)}, \ldots\}$ is a Schauder basis for $\ell_s$.

**Proof:** By p.59,4.2.13 in [3].

**Proposition-2:** $\ell_s \subset \ell$ and the inclusion is strict.
Proof. Let \( x_k \in \ell_x \). Then \( y_k \in \ell \). Hence \( \sum_{k=1}^{\infty} |y_k| < \infty \). But as \( x_k = y_k - y_{k-1} \). We have
\[
|x_k| = |y_k - y_{k-1}| \leq |y_k| + |y_{k-1}|
\]
Then
\[
\sum_{k=1}^{\infty} |x_k| \leq \sum_{k=1}^{\infty} |y_k| + \sum_{k=1}^{\infty} |y_{k-1}|
\]
Hence \( x_k \in \ell \). Consequently \( \ell_x \subset \ell \).

Next we show that the above inclusion is strict. For this take the sequence \( \delta^{(k)} = (1,0,0,\ldots) \). Then \( \delta^{(k)} \in \ell \) and thus we have
\[
y_1 = 1, \ y_2 = 1 + 0 = 1, y_3 = 1 + 0 + 0 = 1, \ldots, y_k = 1 + 0 + \ldots + 0 = 1.
\]
Now, \( |y_k| = 1 \) for all \( k \). Hence \( \{|y_k|\} \) does not tend to zero as \( k \to \infty \). Hence \( \delta^{(k)} \not\in \ell_x \).
Thus the inclusion \( \ell_x \subset \ell \). This completes the proof.

Theorem-2: The dual of space \( \ell_x \) is \( \ell_\infty \).

Proof: A Schauder basis for \( \ell_x \) is \( \{e_k\} \) where \( e_k = (s^k) \) has \( 1 \) in the \( k \)-th place and \(-1\) in the \((k+1)\)-th place and zero’s elsewhere. Let \( x \in \ell_x \). Then there exist scalars \( \alpha_1, \alpha_2, \ldots \) such that \( x = \sum_{k=1}^{\infty} \alpha_k e_k \) is unique. Now for any bounded linear operator \( f \) on \( \ell_x \) we have
\[
f(x) = f(\sum_{k=1}^{\infty} \alpha_k e_k) = \sum_{k=1}^{\infty} \alpha_k f(e_k) = \sum_{k=1}^{\infty} \alpha_k \gamma_k,
\]
where the numbers \( \gamma_k = f(e_k) \) are uniquely determined by \( f \). Also \( \gamma_k = f(e_k), |\gamma_k| = |f(e_k)| \). Since \( f \) is linear and bounded \( |\gamma_k| = |f(e_k)| \leq \|f\| \|e_k\| \). But
\[
\|e_k\| = \|s^{(k)}\| = \|(0,0,\ldots,1,-1,0,0,\ldots)\| = |0| + |0 + 0| + \ldots + |0 + 1| + |1 - 1| + \ldots
\]
(sum of the first \( k \) terms) and \( \|s^{(k)}\| = \|e_k\| = |1| = 1 \). Thus
\[ |\gamma_k| \leq \|f\| \|k_k\| \leq \|f\| \cdot 1 \]
\[ \Rightarrow |\gamma_k| \leq \|f\| \Rightarrow \sup_{(k)} |\gamma_k| \leq \|f\| = M. \]

Hence \((\gamma_k) \in \ell_\infty\). Therefore

(2.1) \[ \ell'_s \subset \ell_\infty. \]

But by Proposition-2, \( \ell_s \subset \ell \). Hence \( \ell' \subset \ell'_s \). As \( \ell' = \ell_\infty \),

(2.2) \[ \ell_\infty \subset \ell'_s. \]

Hence from (2.1) and (2.2) \( \ell'_s = \ell_\infty \). This completes the proof.

**Theorem-3**: The β-dual of \( \ell_s \) is \( \ell_\infty \).

**Proof**: By Proposition-2 we get \( \ell_s \subset \ell \). Hence \( \ell^\beta \subset (\ell_s)^\beta \). But \( \ell^\beta = \ell_\infty \). Hence

(2.3) \[ \ell_\infty \subset (\ell_s)^\beta. \]

Next, let \( y \in (\ell_s)^\beta \) and \( f(x) = \sum_{k=1}^{\infty} x_k y_k \) with \( x \in \ell_s \). Take \( x = s^{(k)} \in \ell_s \), where \( s^{(k)} = (0,0,\ldots,1,-1,0,\ldots) \), \( \|x_n\| = \{0,0,0,\ldots,1,0,\ldots\} \). As this converges to zero, \( s^{(k)} \in \ell_s \). Hence

\[ \|s^{(k)}\| = \begin{cases} 0 \vline & |0 + 0| + |0 + 0 + 0| + \ldots \\ \|0 + 0 + \ldots + 1\| + |0 + 0 + \ldots + 1 - 1| + \ldots & \end{cases} \]
\[ \Rightarrow \|s^{(k)}\| = 1. \]

But

(2.4) \[ |y_n| = |f(s^{(k)})| \leq \|f\|\|s^{(k)}\| \leq \|f\| \cdot 1 = \|f\|. \]

Thus \( \{y_n\} \) is a bounded sequence. Further, as \( y \) is arbitrary in \( (\ell_s)^\beta \).

(2.5) \[ (\ell_s)^\beta \subset \ell_\infty. \]

From (2.3) and (2.4) we get \( (\ell_s)^\beta = \ell_\infty \). This completes the proof.

**Proposition-3**: \( \ell_s \) is solid.
Proof: Let $|x_k| \leq |y_k|$ with $y = (y_k) \in \ell_s$. So $|\xi_k| \leq |\eta_k|$ with $\eta = (y_k) \in \ell$. But $\ell$ is solid. Hence $\xi = (\xi_k) \in \ell$. Therefore $x = (x_k) \in \ell$. Hence $\ell_s$ is solid. This completes the proof.

Corollary-2: In $\ell_s$, weak convergence does not imply strong convergence.

Proof: Assume that weak convergence implies strong convergence in $\ell_s$. Then we would have $(\ell_s)^\beta = \ell_s$ [see (1)]. But $(\ell_s)^\beta = (\ell_\infty)^\beta = \ell$. By Proposition 2, $\ell_s$ is a proper subspace of $\ell$. Thus $(\ell_s)^\beta \neq \ell_s$. Hence weak convergence does not imply strong convergence in $\ell_s$.

This completes the proof.

Corollary-3: $(\ell_s)^\mu = \ell_\infty$ where $\mu = \alpha, \beta, \gamma, f$.

Proof: $\ell_s$ has AK property, by theorem- 1. Hence by Theorem- 7.3.9 in [3] we get $(\ell_s)^\beta = (\ell_s)^\gamma = \ell_\infty$. Hence $(\ell_s)^\beta = \ell_\infty$.

(2.6) $(\ell_s)^\gamma = \ell_\infty$.

Since AK $\Rightarrow$ AD, from [3] we get $(\ell_s)^\beta = (\ell_s)^\gamma$. Therefore

(2.7) $(\ell_s)^\beta = (\ell_s)^\gamma = \ell_\infty$.

By proposition-3, we have $\ell_s$ is solid. Hence by Theorem 7.3.9 in [3], We get

(2.8) $(\ell_s)^\beta = (\ell_s)^\gamma = \ell_\infty$.

From (2.6), (2.7) and (2.8), we have $(\ell_s)^\alpha = (\ell_s)^\beta = (\ell_s)^\gamma = (\ell_s)^\gamma = \ell_\infty$.

This completes the proof.

Theorem -4: Let $Y$ be any FK-space $\supset \Phi$. Then $Y \supset \ell_s$ if and only if $\{\phi^{(k)}\}$ is weakly bounded.

Proof: In order to establish the result it is enough to establish the following result:

$Y \supset \ell_s \iff Y^f \subset (\ell_s)^f$.

Since $\ell_s$ has AD and $(\ell_s)^f = \ell_\infty$, by using Theorem 8.6.1 in [3] we have

$Y^f \subset \ell_\infty$. 

Subset of absolutely convergent sequence space

\[\iff f \in Y', \text{ the topological dual of } Y \iff f(\delta^{(k)}) \in \ell_{\infty}\]
\[\iff f(\delta^{(k)}) \text{ is bounded} \iff \text{Thesequence } \{\delta^{(k)}\} \text{ is weakly bounded}.

This completes the proof.

**Theorem-5:** In \(\ell_s\), weakly convergent sequences are norm convergent.

**Proof:** Let \(a = \{a_1, a_1 + a_2, a_1 + a_2 + a_3, \ldots\}\) be weakly convergent and let \(A = (a_{nk})\) be an infinite matrix. Let us assume that \(A\) is coercive. Since \((\ell_s)' = \ell_{\infty}\), it is a conservative matrix. So the column exists by, Theorem 1.3.6 in [3]. By using Theorem 1.3.7 in [3]

\[\left| a_1, a_1 + a_2, a_1 + a_2 + a_3, \ldots \right| = \lim_{n \to \infty} \left\{ \frac{|a_{n1}| + |a_{n1} + a_{n2}| + \ldots}{|a_{n1} + a_{n2} + a_{n3}| + \ldots} \right\} \leq \|A\|.

Since bounded monotonic sequence converges, \(a \in \ell_s\).

This completes the proof.

**Proposition-4:** \(\ell_s\) is not perfect.

**Proof:** We know that \((\ell_s)' = \ell_{\infty}\). Hence \((\ell_s)' = (\ell_{\infty})'\). But as \((\ell_{\infty})' = \ell\), \((\ell_s)' = \ell\). Hence \(\ell_s\) is not perfect. This completes the proof.

**Proposition-5:** The space \(\ell_s\) is separable.

**Proof:** By Proposition 1, we have \(\ell_s\) has Schauder basis \(\{e_1, e_2, \ldots, e_n, \ldots\}\). Also \(\ell_s\) is a Banach space. Hence, by the Lemma-2, it follows that \(\ell_s\) is separable. This completes the proof.

**Proposition-6:** The space \(\ell_s\) is not separable.

**Proof:** By Theorem 1.3.9 in [2].

**Proposition-7:** The space \(\ell_s\) is not reflexive.

**Proof:** By Proposition-5, we have \(\ell_s\) is separable. But, by Proposition 2 \((\ell_s)' = \ell_{\infty}\).

Since \(\ell_{\infty}\) is not separable by Proposition-6, \(\ell_s\) is not reflexive. This completes the proof.
Theorem-6: The space $\ell_\infty$ is an inner product space but not a Hilbert space.

**Proof.** The proof will be established by showing that the norm satisfies the law of parallelogram. Let us take

$$x = \{1,-1,0,\ldots\} \in \ell_\infty \quad \text{and} \quad y = \{1,-1,0,\ldots\} \in \ell_\infty.$$  

Then

$$\|x\|_\infty = \{\|x_1\| + \|x_1 + x_2\| + \|x_1 + x_2 + x_3\| + \ldots\}$$

$$= \{1 + 1 + 1 + 0 + \ldots\} = 1$$

Similarly, 

$$\|y\|_\infty = \{1 + 1 + 1 + 1 + 0 + \ldots\} = 1$$

Consider,

$$\|x + y\|_\infty = \{\|x_1 + y_1\| + \|(x_1 + y_1) + (x_2 + y_2)\| + \ldots\} = 2$$

Similarly,

$$\|x - y\|_\infty = \{\|x_1 - y_1\| + \|(x_1 - y_1) + (x_2 - y_2)\| + \ldots\} = 0.$$

Now

$$2^2 + 0 = 2\{1^2 + 1^2\} \implies 4 = 4.$$  

Thus parallelogram law is satisfied. Therefore $\ell_\infty$ is an inner product space.

For the proof of the second part let us suppose that $\ell_\infty$ is a Hilbert space. Then by [2] (Theorem 4.6.6) $\ell_\infty$ would satisfy reflexivity condition. This contradicts Proposition-7. Hence $\ell_\infty$ is not a Hilbert space. This completes the proof.

**REFERENCES**


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