On Certain Class of Harmonic Univalent Functions

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Abstract

Let $S_H$ denote the class of functions $f = h + \overline{g}$ which are harmonic univalent and sense-preserving in the unit disk $U = \{ z : |z| < 1 \}$ where $h(z) = z + \sum_{k=2}^{\infty} a_k z^k$, $g(z) = \sum_{k=1}^{\infty} b_k z^k (|b_1| < 1)$. In [6], the authors introduced the operator $D_{m,\lambda}^n$ which defined by convolution involving the polylogarithms functions. Using this operator, we introduce the class $S_H(n, m, \lambda, \alpha)$ of functions $f = h + \overline{g}$ which are harmonic in $U$. A sufficient coefficient of this class is determined. It is shown that this coefficient bound is also necessary for the class $T_H(n, m, \lambda, \alpha)$ if $f_n(z) = h + \overline{g_n} \in S_H(n, m, \lambda, \alpha)$ where $h(z) = z - \sum_{k=2}^{\infty} |a_k| z^k$, $g_n(z) = (-1)^n \sum_{k=1}^{\infty} |b_k| z^k$ and $n \in \mathbb{N}_0$. We obtain some properties of the class $T_H(n, m, \lambda, \alpha)$.

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1 Introduction

A continuous function $f = u + iv$ is a complex valued harmonic function in a complex domain $\mathbb{C}$ if both $u$ and $v$ are real harmonic in $\mathbb{C}$. In any simply connected domain $D \subset \mathbb{C}$ we can write $f(z) = h + \overline{g}$, where $h$ and $g$ are analytic in $D$. We call $h$ the analytic part and $g$ the co-analytic part of $f$. A necessary
and sufficient condition for $f$ to be locally univalent and sense-preserving in $D$ is that $|h'(z)| > |g'(z)|$ in $D$. See Clunie and Sheil-Small (see [3]).

Denote by $S_h$ the class of functions $f = h + \overline{g}$ that are harmonic univalent and sense-preserving in the unit disk $U = \{ z : |z| < 1 \}$ for which $f(0) = h(0) = f_z(0) - 1 = 0$. For $f = h + \overline{g} \in S_h$ we may express the analytic functions $h$ and $g$ as

$$h(z) = z + \sum_{k=2}^{\infty} a_k z^k, \quad g(z) = \sum_{k=1}^{\infty} b_k z^k \quad |b_1| < 1. \quad (1.1)$$

Observe that $S_h$ reduces to $S$, the class of normalized univalent analytic functions, if the co-analytic part of $f$ is zero.

For $f = h + \overline{g}$ given by (1.1), we define the derivative operator introduced by the authors (see [6]) of $f$ as:

$$D_{n,m,\lambda}^n f(z) = D_{n,m,\lambda}^n h(z) + (-1)^n \overline{D_{m,\lambda}^n g(z)} \quad (n, m \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}, \lambda \geq 0; z \in U), \quad (1.2)$$

where

$$D_{\lambda}^n h(z) = z + \sum_{k=2}^{\infty} [1 + \lambda(k - 1)]^n C(m, k) a_k z^k,$$

$$D_{\lambda}^n g(z) = \sum_{k=1}^{\infty} [1 + \lambda(k - 1)]^n C(m, k) b_k z^k, \quad |b_1| < 1,$$

and $C(m, k) = \binom{k+m-1}{m}$.

We let $S_h(n, m, \lambda, \alpha)$ denote the family of harmonic functions $f$ of the form (1.1) such that:

$$\text{Re} \left\{ \frac{D_{m,\lambda}^{n+1} f(z)}{z} \right\} > \alpha \quad (0 \leq \alpha < 1), \quad (1.3)$$

where $D_{m,\lambda}^n f$ is defined by (1.2).

If the co-analytic part of $f = h + \overline{g}$ is identically zero, $n = 0$ and $m = 0$, then the family $S_h(n, m, \lambda, \alpha)$ turns out to be the class $F_\lambda(\alpha)$ introduced by Bhoosnurmath and Swamy [7] for the analytic case.
We let the subclass $T_H(n, m, \lambda, \alpha)$ consist of harmonic functions $f_n = h + \overline{g_n}$ in $S_H(n, m, \lambda, \alpha)$ so $h$ and $g$ are of the form:

$$h(z) = z - \sum_{n=2}^{\infty} |a_n|z^n, \quad g_n(z) = (-1)^n \sum_{n=1}^{\infty} |b_n|z^n \quad |b_1| < 1. \quad (1.4)$$

It is clear that the class $T_H(n, m, \lambda, \alpha)$ includes a variety of well-known subclasses of $S_H$. For example, $T_H(0, 0, 1, \alpha) \equiv HP(\alpha)$ was introduced and studied by Karpuzogullar et.al. [9].

In 1984 Clunie and Sheil-Small [3] investigated the class $S_H$ as well as its geometric subclasses and obtained some coefficient bounds. Since then, there have been several related papers on $S_H$ and its subclasses such that Silverman [1], Silverman and Silvia [2], and Jahangiri [4]-[5]. Karpuzogullar et.al. [9], prove the following theorem:

**Theorem 1.1** Let $f = h + \overline{g}$ given by (1.1) and $a_1 = 1$. If

$$\sum_{k=1}^{\infty} k(|a_k| + |b_k|) \leq 2 - \alpha, \quad (0 \leq \alpha < 1), \quad (1.5)$$

then $f$ is sense-preserving, harmonic, and univalent in $U$ and $f \in HP(\alpha) \equiv S_H(0, 0, 1, \alpha)$. The condition (1.5) is also necessary if $f \in HP^*(\alpha)$.

In this paper, we will give the sufficient condition for functions $f = h + \overline{g}$ where $h$ and $g$ given by (1.1) to be in the class $S_H(n, m, \lambda, \alpha)$ and it is shown that this coefficient condition is also necessary for functions in the class $T_H(n, m, \lambda, \alpha)$. Distortion bounds, extreme points, convolution conditions, convex combination, extreme points and distortion theorem for fractional calculus of this class are also obtained.

### 2 Coefficient Bounds

Firstly, we introduce a sufficient coefficient condition for functions in $S_H(n, m, \lambda, \alpha)$.

**Theorem 2.1** Let $f = h + \overline{g}$ be given by (1.1). If

$$\sum_{k=1}^{\infty} [1 + \lambda(k - 1)]^{n+1} C(m, k)(|a_k| + |b_k|) \leq 2 - \alpha, \quad (2.1)$$

where $a_1 = 1$, $n, m \in \mathbb{N}_0$, $\lambda > 0$, $C(m, k) = \binom{k+m-1}{m}$ and $0 \leq \alpha < 1$, then $f$ is sense-preserving, harmonic univalent in $U$, and $f \in S_H(n, m, \lambda, \alpha)$.
Proof. If \( z_1 \neq z_2 \), then

\[
\left| \frac{f(z_1) - f(z_2)}{h(z_1) - h(z_2)} \right| \geq 1 - \left| \frac{g(z_1) - g(z_2)}{h(z_1) - h(z_2)} \right| = 1 - \left| \frac{\sum_{k=1}^\infty b_k (z_1^k - z_2^k)}{(z_1 - z_2) + \sum_{k=2}^\infty a_k (z_1^k - z_2^k)} \right|
\]

\[
> 1 - \frac{\sum_{k=1}^\infty k b_k}{1 - \sum_{k=2}^\infty k a_k} \geq 1 - \frac{\sum_{k=1}^\infty \frac{[1+\lambda(k-1)]^{n+1} C(m,k)}{1-\alpha} |b_k|}{1 - \sum_{k=2}^\infty \frac{[1+\lambda(k-1)]^{n+1} C(m,k)}{1-\alpha} |a_k|} \geq 0,
\]

which proves univalence. Note that \( f \) is sense-preserving in \( \mathbb{U} \). This is because

\[
|h'(z)| \geq 1 - \sum_{k=2}^\infty k |a_k| |z|^{k-1} > 1 - \sum_{k=2}^\infty [1 + \lambda(k-1)]^{n+1} C(m,k) |a_k| 
\]

\[
\geq \alpha + \sum_{k=1}^\infty [1 + \lambda(k-1)]^{n+1} C(m,k) |b_k| 
\]

\[
> \sum_{k=1}^\infty [1 + \lambda(k-1)]^{n+1} C(m,k) |b_k| |z|^{k-1} \geq |g'(z)|.
\]

Using the fact that \( \text{Re} \ w > \alpha \) if and only if \( |1 - \alpha + w| \geq |1 + \alpha - w| \), it suffices to show that

\[
\left| (1 - \alpha) z + \mathfrak{D}^n_{m,\lambda} f(z) \right| - \left| (1 + \alpha) z - \mathfrak{D}^n_{m,\lambda} f(z) \right| \geq 0, \quad (2.2)
\]
Substituting for $D_{m,\lambda}^n f(z)$ in (2.2) yields, by (2.1) we obtain

$$\left| (1 - \alpha)z + D_{m,\lambda}^n f(z) \right| - \left| (1 + \alpha)z - D_{m,\lambda}^n f(z) \right| = \left| (2 - \alpha)z + \sum_{k=2}^{\infty} \frac{1}{1 + \lambda(k - 1)} [1 + \lambda(k - 1)]^{n+1} C(m, k) a_k z^k + (-1)^n \sum_{k=1}^{\infty} [1 + \lambda(k - 1)]^{n+1} C(m, k) b_k z^k \right|$$

$$- \left| \alpha z - \sum_{k=2}^{\infty} \frac{1}{1 + \lambda(k - 1)} [1 + \lambda(k - 1)]^{n+1} C(m, k) a_k z^k - (-1)^n \sum_{k=1}^{\infty} [1 + \lambda(k - 1)]^{n+1} C(m, k) b_k z^k \right| \geq 2 |z| \left\{ (1 - \alpha) - \sum_{k=2}^{\infty} \frac{1}{1 + \lambda(k - 1)} [1 + \lambda(k - 1)]^{n+1} C(m, k) |a_k| z^{|k-1|} \right. $$

$$\left. - \sum_{k=1}^{\infty} [1 + \lambda(k - 1)]^{n+1} C(m, k) |b_k| z^{|k-1|} \right\} \right.$$  

$$> 2 \left\{ (1 - \alpha) - \sum_{k=2}^{\infty} [1 + \lambda(k - 1)]^{n+1} C(m, k) |a_k| \right.$$  

$$- \sum_{k=1}^{\infty} [1 + \lambda(k - 1)]^{n+1} C(m, k) |b_k| \right\}.$$  

This last expression is non-negative by (2.1), and so the proof is complete.

The harmonic function

$$f(z) = z + \sum_{k=2}^{\infty} \frac{1 - \alpha}{1 + \lambda(k - 1)} [1 + \lambda(k - 1)]^{n+1} C(m, k) x_k z^k + \sum_{k=1}^{\infty} \frac{1 - \alpha}{1 + \lambda(k - 1)} [1 + \lambda(k - 1)]^{n+1} C(m, k) y_k z^k$$

(2.3)

where $n, m \in \mathbb{N}_0$, $\lambda > 0$ and $\sum_{k=2}^{\infty} |x_k| + \sum_{k=1}^{\infty} |y_k| = 1$, show that the coefficient bound given by (2.1) is sharp. The functions of the form (2.3) are in $\mathcal{SH}(n, m, \lambda, \alpha)$ because

$$\sum_{k=1}^{\infty} [1 + \lambda(k - 1)]^{n+1} C(m, k) (|a_k| + |b_k|)$$

$$= 1 + (1 - \alpha) \left( \sum_{k=2}^{\infty} |x_k| + \sum_{k=1}^{\infty} |y_k| \right)$$

$$= 2 - \alpha.$$  

In the following theorem, it is shown that the condition (2.1) is also necessary for functions $f_n = h + g_n$ where $h$ and $g_n$ are of the form (1.4).
**Theorem 2.2** Let \( f_n = h + g_n \) be given by (1.4). Then \( f_n \in \mathcal{T}_H(n, m, \lambda, \alpha) \), if and only if inequality (2.1) holds for the coefficients \( f_n = h + g_n \).

**Proof.** Since \( \mathcal{T}_H(n, m, \lambda, \alpha) \subset \mathcal{S}_H(n, m, \lambda, \alpha) \), we only need to prove the "only if" part of the theorem. To this end, for functions \( f_n \) of the form (1.4), we notice that the condition (1.3) is equivalent to

\[
\Re\left\{ \frac{(1 - \alpha)z - \sum_{k=2}^{\infty} [1 + \lambda(k - 1)]^{n+1}C(m, k)|a_k|z^k}{z^2} \right\} \geq 0.
\]

(2.4)

The above required condition (2.4) must hold for all values of \( z \) in \( \mathbb{U} \). Upon choosing the values of \( z \) on the positive real axis where \( 0 \leq z = r < 1 \), we must have

\[
\frac{(1 - \alpha) - \sum_{k=2}^{\infty} [1 + \lambda(k - 1)]^{n+1}C(m, k)|a_k|r^{k-1}}{1 - \sum_{k=1}^{\infty} [1 + \lambda(k - 1)]^{n+1}C(m, k)|b_k|r^{k-1}} \geq 0.
\]

(2.5)

If the condition (2.1) does not hold, then the numerator in (2.5) is negative for \( r \) sufficiently close to 1. Hence there exist \( z_0 = r_0 \) in \( (0, 1) \) for which the quotient in (2.1) is negative. This contradicts the required condition for \( f_n \in \mathcal{T}_H(n, m, \lambda, \alpha) \) and so the proof is complete.

### 3 Distortion Bounds and Extreme Points.

In this section, first we shall obtain distortion bounds for functions in \( \mathcal{T}_H(n, m, \lambda, \alpha) \).

**Theorem 3.1** Let \( f_n \in \mathcal{T}_H(n, m, \lambda, \alpha) \). Then for \( |z| = r < 1 \) we have

\[
|f_n(z)| \leq (1 + |b_1|)r + \frac{1 - \alpha - |b_1|}{(\lambda + 1)^{n+1}(m + 1)}r^2,
\]

and

\[
|f_n(z)| \geq (1 - |b_1|)r - \frac{1 - \alpha - |b_1|}{(\lambda + 1)^{n+1}(m + 1)}r^2.
\]
Proof. We only prove the second inequality. The proof for the first inequality is similar and will be omitted. Let \( f_n \in \mathcal{T}_H(n, m, \lambda, \alpha) \). Taking the absolute value of \( f_n(z) \) we obtain

\[
|f_n(z)| \geq \left| z - \sum_{k=2}^{\infty} a_k z^k - (-1)^n \sum_{k=1}^{\infty} b_k z^k \right|
\]

\[
\geq (1 - |b_1|) r - \sum_{k=2}^{\infty} (|a_k| + |b_k|) r^k
\]

\[
\geq (1 - |b_1|) r - r^2 \sum_{k=2}^{\infty} (|a_k| + |b_k|)
\]

\[
\geq (1 - |b_1|) r - \frac{1}{(\lambda + 1)^{n+1}(m + 1)} \left( \sum_{k=2}^{\infty} (\lambda + 1)^{n+1}(m + 1)(|a_k| + |b_k|) \right) r^2
\]

\[
\geq (1 - |b_1|) r - \frac{1}{(\lambda + 1)^{n+1}(m + 1)} \left( 1 + \lambda(k - 1) \right)^{n+1} C(m, k)(|a_k| + |b_k|) \right) r^2
\]

\[
\geq (1 - |b_1|) r - \frac{1}{(\lambda + 1)^{n+1}(m + 1)} \left( 1 - \alpha - |b_1| \right) r^2.
\]

The bounds given in Theorem 3.1 for the functions \( f_n = h + g_n \) of the form (1.4) also hold for functions of the form (1.1) if the coefficient condition (2.1) is satisfied. The functions

\[
f_n(z) = z + |b_1| z + \frac{1 - \alpha - |b_1|}{(\lambda + 1)^{n+1}(m + 1)} z^2
\]

and

\[
f_n(z) = (1 - |b_1|) z - \frac{1 - \alpha - |b_1|}{(\lambda + 1)^{n+1}(m + 1)} z^2
\]

for \( |b_1| < 1 - \alpha \) show that the bounds given Theorem 3.1 are sharp.

The following covering result follows from the second inequality in Theorem 3.1.

**Corollary 3.2** If function \( f_n = h + g_n \), where \( h \) and \( g_n \) given by (1.4) be in \( \mathcal{T}_H(n, m, \lambda, \alpha) \). Then

\[
\left\{ w : |w| < 1 - \frac{1 - \alpha - [1 - (\lambda + 1)^{n+1}(m + 1)]|b_1|}{(\lambda + 1)^{n+1}(m + 1)} \right\} \subset f_n(\mathbb{U}).
\]

Next we determine the extreme points of closed convex hulls of \( \mathcal{T}_H(n, m, \lambda, \alpha) \) denoted by \( \text{clco} \mathcal{T}_H(n, m, \lambda, \alpha) \).
Theorem 3.3 Let $f_n$ be given by (1.4). Then $f_n \in \mathcal{T}_H(n, m, \lambda, \alpha)$ if and only if

$$f_n(z) = \sum_{k=1}^{\infty} (X_k h_k(z) + Y_k g_{n_k}(z))$$  \hspace{1cm} (3.1)

where $h_1(z) = z$, $h_k(z) = z - \frac{1-\alpha}{[1+\lambda(k-1)]^{n+1}C(m,k)} z^k \ (k = 2, 3, ...)$, $g_{n_k}(z) = z + (-1)^n \frac{1-\alpha}{[1+\lambda(k-1)]^{n+1}C(m,k)} z^k \ (k = 1, 2, 3, ...)$, $\sum_{k=1}^{\infty} (X_k + Y_k) = 1$, $X_k \geq 0$, $Y_k \geq 0$.

In particular, the extreme points of $\mathcal{T}_H(n, m, \lambda, \alpha)$ are $\{h_k\}$ and $\{g_{m_k}\}$.

Proof. For functions $f_n$ of the form (3.1) we have

$$f_n(z) = \sum_{k=1}^{\infty} (X_k h_k(z) + Y_k g_{n_k}(z))$$

$$= \sum_{k=1}^{\infty} (X_k + Y_k) z - \sum_{k=2}^{\infty} \frac{1-\alpha}{[1+\lambda(k-1)]^{n+1}C(m,k)} X_k z^k$$

$$+ (-1)^n \sum_{k=1}^{\infty} \frac{1-\alpha}{[1+\lambda(k-1)]^{n+1}C(m,k)} Y_k z^k$$

Then

$$\sum_{k=2}^{\infty} \frac{[1+\lambda(k-1)]^{n+1}C(m,k)}{1-\alpha} |a_k| + \sum_{k=1}^{\infty} \frac{[1+\lambda(k-1)]^{n+1}C(m,k)}{1-\alpha} |b_k|$$

$$= \sum_{k=2}^{\infty} X_k + \sum_{k=1}^{\infty} Y_k = 1 - X_1 \leq 1,$$

and so $f_n \in clco \mathcal{T}_H(n, m, \lambda, \alpha)$.

Conversely, suppose that $f_n \in clco \mathcal{T}_H(n, m, \lambda, \alpha)$. Setting

$$X_k = \frac{[1+\lambda(k-1)]^{n+1}C(m,k)}{1-\alpha} |a_k|, \ 0 \leq X_k \leq 1 \ (k = 2, 3, ...),$$

$$Y_k = \frac{[1+\lambda(k-1)]^{n+1}C(m,k)}{1-\alpha} |b_k|, \ 0 \leq Y_k \leq 1 \ (k = 1, 2, 3, ...),$$
and \( X_1 = 1 - \sum_{k=2}^{\infty} X_k - \sum_{k=1}^{\infty} Y_k \). Therefore, \( f \) can be written as

\[
f_n(z) = z - \sum_{k=2}^{\infty} |a_k| z^k + (-1)^{n} \sum_{k=1}^{\infty} |b_k| z^k
\]

\[
= z - \sum_{k=2}^{\infty} \frac{(1 - \alpha)X_k}{[1 + \lambda(k - 1)]^{n+1} C(m, k)} z^k + (-1)^{n} \sum_{k=1}^{\infty} \frac{(1 - \alpha)Y_k}{[1 + \lambda(k - 1)]^{n+1} C(m, k)} z^k
\]

\[
= z + \sum_{k=2}^{\infty} (h_k(z) - z)X_k + \sum_{k=1}^{\infty} (g_n(z) - z)Y_k
\]

\[
= \sum_{k=2}^{\infty} \sum_{k=1}^{\infty} h_k(z)X_k + \sum_{k=1}^{\infty} g_n(z)Y_k + z \left(1 - \sum_{k=2}^{\infty} X_k - \sum_{k=1}^{\infty} Y_k \right)
\]

\[
= \sum_{k=1}^{\infty} (h_k(z)X_k + g_n(z)Y_k), \text{ as required.}
\]

4 \ Convolution and Convex Combination.

In this section, we show that the class \( T_H(n, m, \lambda, \alpha) \) is invariant under convolution and convex combination of its member.

For harmonic functions \( f_n(z) = z - \sum_{k=2}^{\infty} a_k z^k + (-1)^{n} \sum_{k=1}^{\infty} b_k z^k \) and \( F_n(z) = \sum_{k=2}^{\infty} A_k z^k + (-1)^{n} \sum_{k=1}^{\infty} B_k z^k \) the convolution of \( f_n \) and \( F_n \) is given by

\[
(f_n * F_n)(z) = f_n(z) * F_n(z) = z - \sum_{k=2}^{\infty} a_k A_k z^k + (-1)^{n} \sum_{k=1}^{\infty} b_k B_k z^k. \tag{4.1}
\]

**Theorem 4.1** For \( 0 \leq \beta \leq \alpha < 1 \), let \( f_n \in T_H(n, m, \lambda, \alpha) \) and \( F_n \in T_H(n, m, \lambda, \beta) \). Then \( f_n * F_n \in T_H(n, m, \lambda, \alpha) \subset T_H(n, m, \lambda, \beta) \).

**Proof.** Since \( f_n \in T_H(n, m, \lambda, \alpha) \) and \( F_n \in T_H(n, m, \lambda, \beta) \), the coefficients of \( f_n * F_n \) must satisfy the required condition given in Theorem 2.2. For \( F_n \in T_H(n, m, \lambda, \beta) \) we note that \( |A_k| \leq 1 \) and \( |B_k| \leq 1 \). Now, for the convolution function \( f_n * F_n \), we obtain

\[
\sum_{k=2}^{\infty} \frac{1 + \lambda(k - 1)]^{n+1} C(m, k)}{1 - \beta} |a_k||A_k| + \sum_{k=1}^{\infty} \frac{1 + \lambda(k - 1)]^{n+1} C(m, k)}{1 - \beta} |b_k||B_k|
\]

\[
\leq \sum_{k=2}^{\infty} \frac{1 + \lambda(k - 1)]^{n+1} C(m, k)}{1 - \beta} |a_k| + \sum_{k=1}^{\infty} \frac{1 + \lambda(k - 1)]^{n+1} C(m, k)}{1 - \beta} |b_k|
\]

\[
\leq \sum_{k=2}^{\infty} \frac{1 + \lambda(k - 1)]^{n+1} C(m, k)}{1 - \alpha} |a_k| + \sum_{k=1}^{\infty} \frac{1 + \lambda(k - 1)]^{n+1} C(m, k)}{1 - \alpha} |b_k| \leq 1,
\]
since $0 \leq \beta \leq \alpha < 1$ and $f_n \in \mathcal{T}_H(n, m, \lambda, \alpha)$. Therefore $f_n \ast F_n \in \mathcal{T}_H(n, m, \lambda, \alpha) \subset \mathcal{T}_H(n, m, \lambda, \beta)$.

We now examine the convex combination of $\mathcal{T}_H(n, m, \lambda, \alpha)$.

Let the functions $f_{n_j}(z)$ be defined, for $j = 1, 2, ...$ by
\[
f_{n_j}(z) = z - \sum_{k=2}^{\infty} |a_{k,j}| z^k + (-1)^n \sum_{k=1}^{\infty} |b_{k,j}| z^k. \tag{4.2}
\]

**Theorem 4.2.** Let the functions $f_{n_j}(z)$ defined by (4.2) be in the class $\mathcal{T}_H(n, m, \lambda, \alpha)$ for every $j = 1, 2, ..., m$. Then the functions $t_j(z)$ defined by
\[
t_j(z) = \sum_{j=1}^{m} c_j f_{n_j}(z), \quad (0 \leq c_j \leq 1) \tag{4.3}
\]
is also in the class $\mathcal{T}_H(n, m, \lambda, \alpha)$ where $\sum_{j=1}^{m} c_j = 1$.

**Proof.** According to the definition of $t$, we can write
\[
t(z) = z - \sum_{k=2}^{\infty} \left( \sum_{j=1}^{m} c_j a_{k,j} \right) z^k + (-1)^n \sum_{k=1}^{\infty} \left( \sum_{j=1}^{m} c_j b_{k,j} \right) z^k \tag{4.4}
\]
Further, since $f_{n_j}(z)$ are in $\mathcal{T}_H(n, m, \lambda, \alpha)$ for every $(j = 1, 2, ..., m)$. Then by (2.4) we have
\[
\sum_{k=1}^{\infty} \left\{ [1 + \lambda(k-1)]^{n+1} C(m, k) \left( \sum_{j=1}^{m} c_j (|a_{k,j}| + |b_{k,j}|) \right) \right\}
\]
\[
= \sum_{j=1}^{m} c_j \left( \sum_{k=1}^{\infty} [1 + \lambda(k-1)]^{n+1} C(m, k)(|a_{k,j}| + |b_{k,j}|) \right)
\]
\[
\leq \sum_{j=1}^{m} c_j (2 - \alpha) \leq 2 - \alpha.
\]

Hence the theorem follows.

**Corollary 4.2** The class $\mathcal{T}_H(n, m, \lambda, \alpha)$ is close under convex linear combination.
Proof. Let the functions $f_{n_j}(z)$ ($j = 1, 2$) defined by (4.1) be in the class $T_H(n, m, \lambda, \alpha)$. Then the function $\Psi(z)$ defined by

$$\Psi(z) = \mu f_{n_1}(z) + (1 - \mu) f_{n_2}(z) \quad (0 \leq \mu \leq 1)$$

is in the class $T_H(n, m, \lambda, \alpha)$. Also, by taking $m = 2$, $t_1 = \mu$ and $t_2 = (1 - \mu)$ in Theorem 4.1, we have the corollary.

5 An Application of Neighborhood and Fractional Calculus

In this section, first we shall prove that the functions in a neighborhood of $T_H(n, m, \lambda, \alpha)$ are in the class $HP^*(\alpha)$.

Following [8] we defined the $\delta$-neighborhood of a function $f \in HP^*(\alpha)$ by

$$N_\delta(f) = \left\{ F(z) = z - \sum_{k=2}^{\infty} A_k z^k + \sum_{k=1}^{\infty} B_k z^k \right\}$$

and

$$\sum_{k=2}^{\infty} \left\{ \frac{k}{1 - \alpha} \left( |a_k - A_k| + |b_k - B_k| \right) \right\} + \frac{|b_1 - B_1|}{1 - \alpha} \leq \delta,$$

where $\delta > 0$.

Theorem 5.1 Let

$$\delta = \frac{(\lambda + 1)^{n+1}(m+1) - 1}{(\lambda + 1)^{n+1}(m+1)} \left( 1 - \frac{|b_1|}{1 - \alpha} \right).$$

Then, $N_\delta(T_H(n, m, \lambda, \alpha)) \subset HP^*(\alpha)$.

Proof. Suppose $f_n \in T_H(n, m, \lambda, \alpha)$. Let $F_n = H + G_n \in N_\delta(f_n)$ where $H = z - \sum_{k=2}^{\infty} A_k z^k$ and $G_n = (-1)^n \sum_{k=1}^{\infty} B_k z^k$. We need to show that $F_n \in HP^*(\alpha)$. In other words, it suffices to show that $F_n$ satisfies the condition
Now, we define the fractional integral of order $\mu$. Denote the fractional integral of order $\mu$, for a function $f_n = h + g_n$, by

$$D_z^{-\mu} f_n(z) = \frac{1}{\Gamma(\mu)} \left\{ \int_0^z \frac{h(\xi)}{(z - \xi)^{1-\mu}} d\xi + (-1)^n \int_0^z \frac{g_n(\xi)}{(z - \xi)^{1-\mu}} d\xi \right\}, \quad (\mu > 0),$$

where $f_n$ is analytic harmonic function by $h$ and $g_n$ are analytic in a simply-connected region of the $z$-plane containing the origin and the multiplicity of
(z - ξ)^{μ-1} is removed by requiring \(\log(z - ξ)\) to be real when \(z - ξ > 0\).

The following theorem gives the distortion bounds for fractional calculus for functions in \(T_\mathcal{H}(n, m, \lambda, \alpha)\) which yields a covering result for this class.

**Theorem 5.2** Let \(f_n \in T_\mathcal{H}(n, m, \lambda, \alpha)\). Then for \(|z| = r < 1\) we have

\[
\frac{|z|^{1+μ}}{Γ(μ + 2)} \left\{ (1 - |b_1|) - \frac{2(1 - α - |b_1|)}{(λ + 1)^{n+1}(m + 1)(μ + 2)}|z| \right\} 
\leq |D_z^{-μ}f_n(z)| \leq 
\frac{|z|^{1+μ}}{Γ(μ + 2)} \left\{ (1 + |b_1|) + \frac{2(1 - α - |b_1|)}{(λ + 1)^{n+1}(m + 1)(μ + 2)}|z| \right\}.
\]

**Proof.** We note that

\[
Γ(μ + 2)z^{-μ}D_z^{-μ}f_n(z) = Γ(μ + 2)z^{-μ}\left\{ D_z^{-μ}h(z) + \frac{(-1)^n D_z^{-μ}g_n(z)}{Γ(k + μ + 1)} \right\}
\]

\[
= z - \sum_{k=2}^{∞} \frac{Γ(k + 1)Γ(μ + 2)}{Γ(k + μ + 1)} |a_k|z^k + (-1)^n \sum_{k=1}^{∞} \frac{Γ(k + 1)Γ(μ + 2)}{Γ(k + μ + 1)} |b_k|z^k
\]

\[
= z + (-1)^n |b_1|z - \sum_{k=2}^{∞} \frac{Γ(k + 1)Γ(μ + 2)}{Γ(k + μ + 1)} [ |a_k|z^k + (-1)^n |b_k|z^k ]
\]

where

\[
Φ(k) = \frac{Γ(k + 1)Γ(μ + 2)}{Γ(k + μ + 1)} \quad (k \geq 2, \ μ > 0).
\]

Noting that \(Φ(k)\) is decreasing functions of \(k\), we have

\[
0 < Φ(k) ≤ Φ(2) = \frac{2}{μ + 2}.
\]

We only prove the second inequality. The proof for the first inequality is similar and will be omitted. Let \(f_n \in T_\mathcal{H}(n, m, λ, α)\). Taking the absolute value of
\[ \Gamma(\mu + 2)z^{-\mu}D_{z}^{-\mu}f_{n}(z) \] we obtain

\[
|\Gamma(\mu + 2)z^{-\mu}D_{z}^{-\mu}f_{n}(z)| \\
\geq (1 - |b_{1}|)r - \sum_{k=2}^{\infty} \Phi(k)(|a_{k}| + |b_{k}|)r^{k} \\
\geq (1 - |b_{1}|)r - \Phi(2)r^{2}\sum_{k=2}^{\infty} (|a_{k}| + |b_{k}|) \\
\geq (1 - |b_{1}|)r - \frac{\Phi(2)}{(\lambda + 1)^{n+1}(m + 1)} \left( \sum_{k=2}^{\infty} (\lambda + 1)^{n+1}(m + 1)(|a_{k}| + |b_{k}|) \right) r^{2} \\
\geq (1 - |b_{1}|)r - \frac{2}{(\lambda + 1)^{n+1}(m + 1)(\mu + 2)} \left( \sum_{k=2}^{\infty} [1 + \lambda(k - 1)]^{n+1}C(m, k)(|a_{k}| + |b_{k}|) \right) r^{2} \\
\geq (1 - |b_{1}|)r - \frac{2}{(\lambda + 1)^{n+1}(m + 1)(\mu + 2)} \left( 1 - (\alpha + |b_{1}|) \right) r^{2}.
\]

Remark 5.3 Letting \( \mu \to 0 \) in Theorem 5.2 we obtain Theorem 3.1.

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References


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