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Schröder Paths and Pattern Avoiding Partitions

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Abstract

In this paper, we show that both 12312-avoiding partitions and 12321-avoiding partitions of the set $[n+1]$ are in one-to-one correspondence with Schröder paths of semilength n without peaks at even level. As a consequence, the refined enumeration of 12312-avoiding (resp. 12321-avoiding) partitions according to the number of blocks can be reduced to the enumeration of certain Schröder paths according to the number of peaks. Furthermore, we get the enumeration of irreducible 12312-avoiding (resp. 12321-avoiding) partitions, which are closely related to skew Dyck paths.

Mathematics Subject Classification: 05A15, 05A19

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1. Introduction and notations

A *Schröder path* of semilength n is a lattice path on the plane from $(0,0)$ to $(2n,0)$ that does not go below the x-axis and consists of up steps $U=(1,1)$, down steps $D=(1,-1)$ and horizontal steps $H=(2,0)$. They are counted by the larger Schröder numbers (A006318 in [7]). A *UH-free* Schröder path is a Schröder path without up steps followed immediately by horizontal steps. A UH-free Schröder path of semilength 12 is illustrated as Figure 1.

An up step followed by a down step in a path is called a *peak*. The *level* of an up step (a horizontal step) is defined as the larger y coordinate of the step. The *level* of a peak is defined as the level of the up step in the peak. Denote by $\mathcal{SE}_n$ and $\mathcal{SH}_n$ the set of Schröder paths of semilength n without peaks at even level and the set of UH-free Schröder paths of semilength n, respectively.

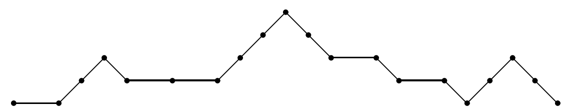


Figure 1: A UH-free Schröder path

A partition π of the set $[n] = \{1, 2, \ldots, n\}$ is a collection $B_1, B_2, \ldots, B_k$ of nonempty disjoint subsets of $[n]$. The elements of a partition are called blocks. We assume that $B_1, B_2, \ldots, B_k$ are listed in the increasing order of their minimum elements, that is $minB_1 < minB_2 < \ldots < minB_k$. A partition π of $[n]$ with k blocks can also be represented by a sequence $\pi_1\pi_2\ldots\pi_n$ on the set $\{1, 2, \ldots, k\}$ such that $\pi_i = j$ if and only if $i \in B_j$. Such a representation is called the *Davenport-Schinzel sequence* or the *canonical sequential form*. In this paper, we will always represent a partition by its canonical sequential form.

In the terminology of canonical sequential forms, we say that a partition π *avoids* a partition τ, or it is τ-*avoiding*, if there is no subsequence which is order-isomorphic to τ in π. In such context, τ is usually called a pattern. The set of τ-avoiding partitions of $[n]$ is denoted $\mathcal{P}_n(\tau)$. The enumeration on pattern avoiding partitions has received extensive attention from several authors, see [1, 2, 3, 5, 6] and references therein.

By using kernel method, Mansour and Severini [5] deduced that the number of 12312-avoiding partitions of $[n+1]$ is equal to the number of Schröder paths of semilength n without peaks at even level (A007317 in [7]). Recently, Jelinek and Mansour[3] proved that the cardinality of $\mathcal{P}_n(12312)$ is equal to that of $\mathcal{P}_n(12321)$. In this paper, we will provide a bijection between the set of 12312-avoiding partitions of $[n+1]$ and the set of UH-free Schröder paths of semilength n. By making a simple variation of this bijection, we get a bijection between the set of 12321-avoiding partitions of $[n+1]$ and the set of UH-free Schröder paths of semilength n. A bijection between the set of UH-free Schröder paths of semilength n and the set of Schröder paths of semilength n without peaks at even level is also provided, which leads to a bijection between 12312-avoiding (resp. 12321-avoiding) partitions of of $[n+1]$ and the set of Schröder paths of semilength n without peaks at even level. As a consequence, the refined enumeration of 12312-avoiding (resp. 12321-avoiding) partitions according to the number of blocks can be reduced to the enumeration of certain Schröder paths according to the number of peaks. Furthermore, we also get the enumeration of irreducible 12312-avoiding (resp. 12321-avoiding) partitions, which are closely related to skew Dyck paths.

2. Bijection between $\mathcal{P}_{n+1}(12312)$ and $\mathcal{SE}_n$

In this section, we will provide a bijection between the set of 12312-avoiding (resp. 12321-avoiding)partitions of $[n+1]$ and the set of UH-free Schröder paths of semilength n. A bijection between the set of UH-free Schröder paths of semilength n and the set of Schröder paths of semilength n without peaks at even level is also given, which leads to a bijection between the set of 12312-avoiding (resp. 12321-avoiding)partitions of $[n+1]$ and the set of Schröder paths of semilength n without peaks at even level.

Let π be a nonempty partition of $[n+1]$ with k blocks. Then π can be uniquely decomposed as

$$1w_1 2w_2 \dots iw_i \dots kw_k, \tag{2.1}$$

where $1, 2, \dots, k$ are the *left-to-right maxima* of π and each w_i is a possibly empty word on $[i]$. For, $1 \le i \le k$, denote by $w_i \setminus \{i\}$ the word obtained from w_i by deleting all the $i's$. The following property of 12312-avoiding (resp. 12321-avoiding) partitions can be verified easily and we omit the proof here.

Lemma 2.1 *A partition π is 12312-avoiding (resp. 12321-avoiding) partition with k blocks if and only if the word $w_1 \setminus \{1\} w_2 \setminus \{2\} \dots w_k \setminus \{k\}$ is in weakly decreasing (increasing) order.*

Now, we proceed to construct a map σ from $\mathcal{P}_{n+1}(12312)$ to $\mathcal{SH}_n$. Given a 12312-avoiding partition π of $[n+1]$ with k blocks, if $\pi = 1$, then let $\sigma(\pi)$ be the empty path. Otherwise, suppose that π is decomposed as (2.1) and for $i = 1, 2, \dots, k-1$, denote by d_i the number of occurrences of i which are right to the first occurrence of $i+1$. We read the decomposition from left to right and generate a path $\sigma(\pi)$ as follows: when a left-to-right maximum i ($i \ge 2$) is read, we adjoin $d_{i-1}+1$ successive up steps followed by one down step; when each element less than i in any word w_i ($1 \le i \le k$) is read, we adjoin one down step; when each element i in any word w_i ($1 \le i \le k$) is read, we adjoin one horizontal step. Lemma 2.1 ensures that the obtained path $\sigma(\pi)$ is a well defined UH-free Schröder path of semilength n. For instance, a 12312-avoiding partition $\pi = 11232343411$ of $[11]$ can be decomposed as $1w_1 2w_2 3w_3 4w_4$, where $w_1 = 1, w_3 = 23, w_4 = 3411, d_1 = 2, d_2 = 1, d_3 = 1$, and w_2 is empty. The corresponding UH-free Schröder path $\sigma(\pi)$ of semilength 10 is illustrated as Figure 2.

Conversely, we can get a 12312-avoiding partition of $[n+1]$ from a UH-free Schröder path P of semilength n. If P is empty, then let $\sigma^{-1}(P) = 1$, otherwise suppose that P has k peaks. Then we can get a word $\sigma^{-1}(P)$ as the following procedure.

Step 1. Firstly, add a peak at the very begining of P and denote by P' the obtained path;

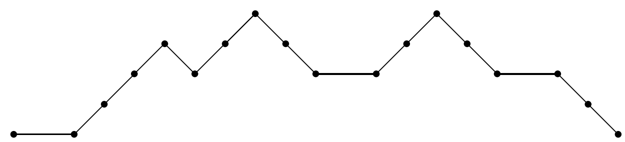


Figure 2: A UH-free Schröder path of semilength 10.

Step 2. Secondly, label all the up steps in peaks of P' with the alphabet $\{1, 2, \ldots, k+1\}$ from left to right and label each remaining up step s and each horizontal step h with the maximum alphabet which are left to the steps s and h, respectively;

Step 3. Thirdly, if a down step s is in a peak, label s with the same label as the label of the up step in the same peak; Otherwise, suppose that L^U (resp. L^D) is the multiset of all the labels of the up (resp. down) steps left to the step s. Then label s with the maximum element of the multiset obtained from L^U by removing all the elements of L^D;

Step 4. Lastly, let $\sigma^{-1}(P)$ be a word obtained by reading the labels of all the down steps and horizontal steps of P' successively.

Obviously, the obtained word $\sigma^{-1}(P)$ is a 12312-avoiding partition of $[n+1]$. An example of the reverse map of σ is shown in Figure 3.

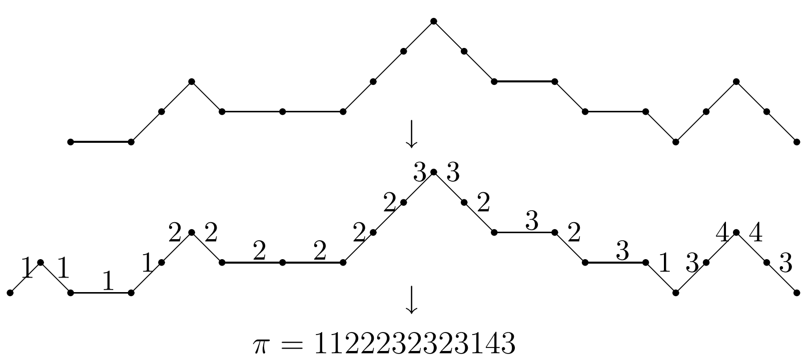
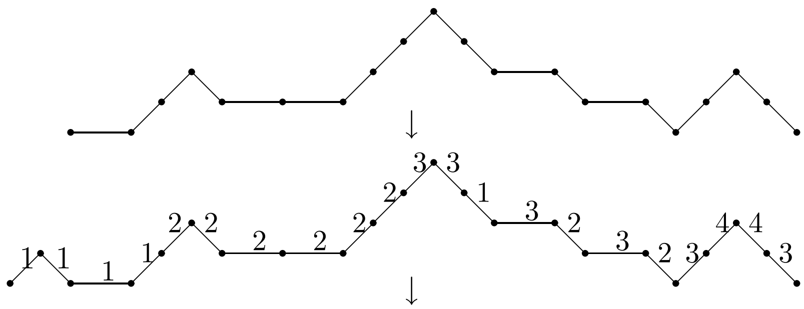


Figure 3: An example of the reverse map of σ.

Theorem 2.2 *The map σ is a bijection between the set of* 12312*-avoiding partitions of* $[n+1]$ *and the set of UH-free Schröder paths of semilength n.*

We define a map ϕ from $\mathcal{P}_{n+1}(12321)$ to $\mathcal{SH}_n$ the same as the map σ and define the reverse of ϕ the same as the reverse of σ except that in Step 3 we label the down step s not in a peak by the minimum element of the multiset obtained from L^U by removing all the elements of L^D. It is easy to check that ϕ is a bijection between the set of 12321-avoiding partitions of $[n+1]$ and the set of UH-free Schröder paths of semilength n. An example of the reverse map of ϕ is illustrated as Figure 4.



$\pi = 1122231323243$

Figure 4: An example of the reverse map of ϕ.

Theorem 2.3 *The map ϕ is a bijection between UH-free Schröder paths of semilength n and $\mathcal{P}_{n+1}(12321)$.*

In order to get a bijection between $\mathcal{P}_{n+1}(12312)$ and $\mathcal{SE}_n$, we should provide a bijection between $\mathcal{SH}_n$ and $\mathcal{SE}_n$. Now we proceed to construct the map ψ from $\mathcal{SH}_n$ and $\mathcal{SE}_n$. Given a UH-free Schröder path $P \in \mathcal{SH}_n$, if it is empty, then let $\psi(P)$ be an empty path. Otherwise, we can get $\psi(P)$ recursively as follows:

- If $P = HP'$, then let $\psi(P) = H\psi(P')$, where P' is a possibly empty UH-free Schröder path;
- If $P = UDP'$, then let $\psi(P) = UD\psi(P')$, where P' is a possibly empty UH-free Schröder path;
- If $P = U^kDP_1DP_2\dots DP_k$, where $k \geq 2$, U^k denotes k consecutive up steps and for $1 \leq i \leq k$, each P_i is a possibly empty UH-free Schröder path, then let $\psi(P) = UP'_1P'_2\dots P'_{k-1}D\psi(P_k)$ such that for $1 \leq i \leq k-1$, each $P'_i = H$ if P_i is empty and $P'_i = U\psi(P_i)D$, otherwise.

Obviously, the obtained path $\psi(P)$ is a Schröder path of semilength n without peaks at even level. It is easy to check that the map ψ is reversible. For the convenience of simplicity, we omit the reverse map of ψ.

Theorem 2.4 *The map ψ is a bijection between the set of UH-free Schröder paths of semilength n and the set of Schröder path of semilength n without peaks at even level.*

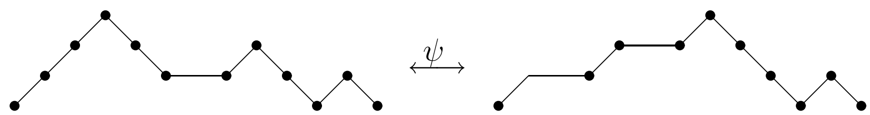


Figure 5: A UH-free Schröder path and its corresponding Schröder path without peaks at even level.

Combining Theorems 2.2, 2.3 and 2.4, we have the following results.

Theorem 2.5 *The map $\psi \cdot \sigma$ (resp. $\psi \cdot \phi$) is a bijection between the set of* 12312*-avoiding (resp.* 12321*-avoiding) partitions of* $[n+1]$ *and the set of Schröder paths of semilength n without peaks at even level.*

3. Refined enumerations

In this section, we aim to get the refined enumeration of 12312-avoiding (resp. 12321-avoidng)partitions according to the number of blocks. By restricting the peaks in Schröder paths, we get the enumeration of irreducible 12321-avoiding (resp. 12321-avoiding) partitions. From the construction of the maps σ and ϕ, we get that each block apart from the first block in a 12312-avoiding (resp. 12321-avoiding) partition π brings up a peak in its corresponding UH-free Schröder path $\sigma(\pi)$ (resp. $\phi(\pi)$). Hence, we get the following result.

Corollary 3.6 *Let π be a 12312-avoiding (resp. 12321-avoiding) partition on $[n+1]$ with $k+1$ blocks, then $\sigma(\pi)$ (resp. $\phi(\pi)$) is a UH-free Schröder path of semilength n with k peaks.*

A *Dyck path* of semilength n is a lattice path on the plane from $(0,0)$ to $(2n,0)$ that does not go below the x-axis and consists of up steps $U=(1,1)$ and down steps $D=(1,-1)$. The number of Dyck paths of semilength n with k peaks is counted by the Narayana number

$$N_{n,k} = \frac{1}{n}\binom{n}{k}\binom{n}{k-1}.$$

Note that any UH-free Schröder path of semilength n with k peaks can be obtained from a Dyck path of semilength j $(0 \le j \le n)$ with k peaks by inserting $n-j$ horizontal steps into the positions after down steps and the position at the very beginning of the Dyck path. The number of such arrangement is equal to $\binom{n}{j}$. Hence, the number of UH-free Schröder paths of semilength n with k peaks is counted by $\sum_{j=k}^{n} \frac{1}{j}\binom{j}{k}\binom{j}{k-1}\binom{n}{j}$. From Corollary 3.6, we get the following result.

Corollary 3.7 *The number of 12312-avoiding (resp. 12321-avoiding) partitions on* $[n+1]$ *with* $k+1$ $(k \geq 1)$ *blocks is equal to*

$$\sum_{j=k}^{n} \frac{1}{j} \binom{j}{k-1} \binom{j}{k} \binom{n}{j}.$$

A partition P of $[n]$ is called an *irreducible partition* if for any $m \in [n-1]$, P can not be reduced to two smaller partitions P_1 and P_2 such that P_1 is a partition of $[m]$ and P_2 is a partition of $\{m+1, m+2, \ldots, n\}$. Irreducible partitions have been studied by Lehner [4]. In fact, a partition π of $[n]$ is irreducible if and only if for any element $i \in [n]$, there is at least one occurrence of an element j which is less than i and right to the first occurrences of i. Hence, by the construction of the maps σ and ϕ, we see that if π is irreducible, then its corresponding Schröder path $\sigma(\pi)$ (resp. $\phi(\pi)$) has no peaks at level one.

Corollary 3.8 *The map* σ *(resp.* ϕ*) is a bijection between the set of irreducible* 12312*-avoiding (resp.* 12321*-avoiding) partitions on* $[n+1]$ *and the set of UH-free Schröder paths of semilength* n *without peaks at level one.*

Denote by $\mathcal{SH'}_n$ the set of UH-free Schröder paths of semilength n without peaks at level one. Let s_n and s'_n the cardinality of $\mathcal{SH}_n$ and $\mathcal{SH'}_n$, respectively. Let $f(x) = \sum_{n=0}^{\infty} s_n x^n$ and $f'(x) = \sum_{n=0}^{\infty} s'_n x^n$ where $s_0 = 1$ and $s'_0 = 1$. Then, it is easy to get the following recurrence relations:

$$f(x) = 1 + 2xf(x) + xf(x)(f(x) - 1 - xf(x)),$$

and

$$f'(x) = 1 + xf'(x) + xf'(x)(f(x) - 1 - xf(x)).$$

Hence, we have

$$f(x) = \frac{1 - x - \sqrt{1 - 6x + 5x^2}}{2(x - x^2)},$$

and

$$f'(x) = \frac{1}{1 - x(1-x)f(x)} = \frac{2}{1 + x + \sqrt{1 - 6x + 5x^2}},$$

which is the generating function for skew Dyck paths of length n ending with a down step, see [7, A033321]. A *skew Dyck path* is a path in the first quadrant which begins at the origin, ends on the x-axis, consists of steps $U = (1, 1)$, $D = (1, -1)$, and $L = (-1, -1)$ so that never lie below the x-axis and up and left steps do not overlap.

Corollary 3.9 *The number of irreducible* 12312*-avoiding (resp.* 12321*-avoiding) partitions of* $[n+1]$ *is equal to the number of skew Dyck paths of semilength* n *ending with a down step.*

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