Dirac Operator in Heisenberg Group $Heis^3$

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Abstract

We give a setting for constructing a Weierstrass representation formula in the Heisenberg group $Heis^3$. Using the spectral properties of the corresponding Dirac operator.

Mathematics Subject Classifications: 53C41, 53A10

Keywords: Weierstrass representation, Heisenberg group, Dirac operator

1. Introduction

Recently surfaces in three-dimensional homogeneous spaces which differ from the space forms attract a lot of attention. Mainly for ambient spaces there are taken three-dimensional spaces with the Thurston geometries or simply connected spaces with a four-dimensional isometry group.

The immersion geometry is currently studied in the various fields, e.g., soliton theory, the differential geometry, the harmonic map theory, string theory and so on. In the soliton theory, the question what is the integrable is the most important theme and one of its answers might be found in the immersed geometry.

It is well-known that the classical Weierstrass-Enneper representation formula describes minimal surfaces in Euclidean 3-space $\mathbb{R}^3$ in terms of their Gauss maps and auxiliary holomorphic functions (see [11]). More generally, a remarkable representation formula has been discovered by Kenmotsu (see [2]) for arbitrary surfaces in $\mathbb{R}^3$ with nonvanishing mean curvature, which describes these surfaces in terms of their Gauss maps and mean curvature functions. On the other hand, Kobayashi (see [3,4]) proved the Lorentzian version of the
classical Weierstrass-Enneper representation formula for maximal surfaces in Minkowski 3-space $L^3$ (see also McNertney [7]) and applied it to the study of maximal surfaces with conelike singularities.

D. A. Berdinski and I. A. Taimanov (see [1]) gave a representation formula for minimal surfaces in 3-dimensional Lie groups in terms of spinors and Dirac operators.

2. Preliminaries

Let $(Heis^3, g)$ be an 3-dimensional Riemannian Heisenberg group, $\Sigma$ a Riemann surface and $f : \Sigma \to Heis^3$ a smooth map. The pull-back bundle $f^*(THeis^3)$ has a (fiber) metric and a compatible connection, the pull-back connection, induced by the Riemannian metric and the Levi–Civita connection of $Heis^3$. Consider the complexified bundle $E = f^*(THeis^3) \otimes \mathbb{C}$. The metric $g$ may be extended to $E$ as:

- a complex bilinear form $(., .) : E \times E \to \mathbb{C}$;
- a hermitian metric $\langle\langle ., . \rangle\rangle : E \times E \to \mathbb{C}$;

and the two extensions are related by: $\langle\langle V, W \rangle\rangle = (V, W)$. Let $(u, v)$ be local coordinates on $\Sigma$.

and $z = u + iv$ the (local) complex parameter and set, as usual,

$$\frac{\partial}{\partial z} = \frac{1}{2} \left( \frac{\partial}{\partial u} - i \frac{\partial}{\partial v} \right); \quad \frac{\partial}{\partial \bar{z}} = \frac{1}{2} \left( \frac{\partial}{\partial u} + i \frac{\partial}{\partial v} \right)$$

The pull-back connection extends to a complex connection on $E$, hermitian with respect to $\langle\langle ., . \rangle\rangle$, and it is well known that $E$ has a unique holomorphic structure such that a section $W : \Sigma \to E$ is holomorphic if and only if:

$$\nabla_\phi W = 0$$

where $\nabla$ is the pull-back connection. Setting

$$\left. \frac{\partial f}{\partial u} \right|_p = f_{vp} \left( \frac{\partial}{\partial u} \right|_p, \quad \left. \frac{\partial f}{\partial v} \right|_p = f_{vp} \left( \frac{\partial}{\partial v} \right|_p$$

We can regard

$$\phi = \frac{\partial f}{\partial z} = \frac{1}{2} \left( \frac{\partial f}{\partial u} - i \frac{\partial f}{\partial v} \right)$$

as a section of $E$ and we have the following properties:

- The map $f$ is an immersion if and only if $\langle\langle \phi, \phi \rangle\rangle \neq 0$;
- If $f$ is an immersion then $f$ is conformal if and only if $\langle\langle \phi, \phi \rangle\rangle = 0$. 
3. Heisenberg Group $Heis^3$

This group is formed by all matrices of the form

$$Heis^3 = \left\{ \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} : x, y, z \in \mathbb{R} \right\}$$

with the group multiplication induced by the standard matrix product. The Riemannian metric $g$ given by

$$g = dx^2 + dy^2 + (dz + xdy)^2 \quad (3.1)$$

The Lie algebra of $Heis^3$ has an orthonormal basis

$$e_1 = \frac{\partial}{\partial x}, \quad e_2 = \frac{\partial}{\partial y} + x \frac{\partial}{\partial z}, \quad e_3 = \frac{\partial}{\partial z}$$

Now let $C^k_{ij}$ be the structure’s constants of the Lie algebra $g$ of $G$ (see [8]), that is,

$$[e_i, e_j] = C^k_{ij} e_k.$$  

The corresponding Lie brackets are

$$[e_1, e_2] = e_3, \quad [e_1, e_3] = [e_2, e_3] = 0$$

The Coshul formula for the Levi–Civita connection is:

$$2g(\nabla e_i e_j, e_k) = C^k_{ij} - C^i_{jk} + C^j_{ki} := L^k_{ij}$$

where the non zero $L^k_{ij}$’s are

$$L^3_{12} = 1, \quad L^3_{21} = -1, \quad L^2_{13} = -1, \quad L^2_{31} = -1, \quad L^1_{23} = 1, \quad L^1_{32} = 1 \quad (3.2)$$

4. The Weierstrass representation of surfaces in $Heis^3$

The Weierstrass representation for the case when the ambient space is a three-dimensional Heisenberg group $Heis^3$ with a left-invariant metric (3.1) we have,

$$\frac{\partial}{\partial z} \in \mathbb{C}^3 \longrightarrow \Psi = f^{-1} \frac{\partial f}{\partial z} \in g \otimes \mathbb{C}$$

where $f : M \rightarrow Heis^3$ is an immersion of a surface and $z$ is a conformal parameter on $M$. 
Let us expand Ψ and Ψ* with respect to this basis to obtain

$$\Psi = \sum_{k=1}^{3} Z_k e_k, \quad \Psi^* = \sum_{k=1}^{3} \overline{Z_k} e_k$$

Since the parameter $z$ is conformal, we have

$$\langle \Psi, \Psi \rangle = \langle \Psi^*, \Psi^* \rangle = 0, \quad \langle \Psi, \Psi^* \rangle = \frac{1}{2} e^{2z}$$

Using the expression of Ψ, Ψ*, respectively, we get

$$Z_1^2 + Z_2^2 + Z_3^2 = 0, \quad |Z_1|^2 + |Z_2|^2 + |Z_3|^2 = \frac{1}{2} e^{2z}$$

The Dirac equation:

$$D\psi = \left[ \begin{pmatrix} 0 & \partial \\ -\overline{\partial} & 0 \end{pmatrix} + \begin{pmatrix} U & V \\ V & -U \end{pmatrix} \right] \psi = 0 \quad (4.1)$$

which is satisfied by

$$\psi = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}$$

The first equality implies that the vector $Z$ can be parametrized in the form

$$Z_1 = \frac{i}{2}(\overline{\psi_2} + \psi_1^2), \quad Z_1 = \frac{1}{2}(\overline{\psi_2} - \psi_1^2), \quad Z_3 = \psi_1 \overline{\psi_2} \quad (4.2)$$

From [1] we have

$$\partial Z_1 - \overline{\partial} Z_1 = 0$$
$$\partial Z_2 - \overline{\partial} Z_2 = 0$$
$$\partial Z_3 - \overline{\partial} Z_3 - (\overline{Z_1} Z_2 - Z_1 \overline{Z_2}) = 0$$
$$\partial Z_1 + \overline{\partial} Z_1 + (\overline{Z_2} Z_3 + Z_2 \overline{Z_3}) = 2iH(\overline{Z_3} Z_1 - Z_3 \overline{Z_1})$$
$$\partial Z_2 + \overline{\partial} Z_2 - (\overline{Z_3} Z_1 + Z_3 \overline{Z_1}) = 2iH(\overline{Z_1} Z_2 - Z_1 \overline{Z_2})$$
$$\partial Z_3 + \overline{\partial} Z_3 = 2iH(\overline{Z_3} Z_2 - Z_3 \overline{Z_2})$$

Lemma 4.1

$$\partial \psi_2^2 + \overline{\partial} \psi_1^2 = 0$$
$$\partial \psi_2^2 + \overline{\partial} \psi_1^2 + i\psi_1 \psi_2 (|\psi_2|^2 - |\psi_1|^2) = 2iH \left(|\psi_1|^2 + |\psi_2|^2\right).$$

The system of these two equations at the points where $\psi_1 \psi_2 \neq 0$, are written in the form of the Dirac equation which is satisfied by continuity on the whole surface: where $H$ is the mean curvature of the surface.
Theorem 4.2 If (4.1) Dirac equation which is satisfied by continuity on the whole surface, then

\[ U = \frac{1}{2}(\psi_1^2 - \psi_2^2)V. \]  

(4.3)

Proof. From (4.1), we have

\[ V = H \frac{\psi_1 \psi_2}{\psi_1^2 + \psi_2^2} e^\alpha + i \frac{\psi_1 \psi_2}{\psi_1^2 + \psi_2^2} \left( |\psi_1|^2 - |\psi_2|^2 \right) \]

Using Lemma 4.1, we get

\[ 2U \psi_1 \psi_2 = V(\psi_1^2 - \psi_2^2) \]

The theorem is proved.

Theorem 4.3 For a surface in $Heis^3$ its generating spinor $\psi$ satisfies the Dirac equation (4.1).

References


Received: November, 2008