Interval-Valued Fuzzy $BF$-Algebras

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Abstract

In this note the notion of interval-valued fuzzy $BF$-algebras (briefly, i-v fuzzy $BF$-algebras), the level and strong level $BF$-subalgebra is introduced. Then we state and prove some theorems which determine the relationship between these notions and $BF$-subalgebras. The images and inverse images of i-v fuzzy $BF$-subalgebras are defined, and how the homomorphic images and inverse images of i-v fuzzy $BF$-subalgebra becomes i-v fuzzy $BF$-algebras are studied.

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1 Introduction

In 1966, Y. Imai and K. Iseki [5] introduced two classes of abstract algebras: $BCK$-algebras and $BCI$-algebras. It is known that the class of $BCK$-algebras is a proper subclass of the class of $BCI$-algebras. In [4] Q. P. Hu and X. Li introduced a wide class of abstract algebras: $BCH$-algebra. They shown that the class of $BCI$-algebras is a proper subclass of the class of $BCH$-algebras. Y. Imai and K. Iseki [4] introduced two classes of abstract algebras: $BCK$-algebras and $BCI$-algebras. It is known that the class of $BCK$-algebras is a proper subclass of the class of $BCI$-algebras. In [8], J. Neggers and H. S. Kim introduced the notion of $B$-algebras, which is a generalization of $BCK$-algebra. In [7], Y. B. Jun , E. H. Roh , and H. S. Kim introduced $BH$-algebras, which
are a generalization of $BCK/BCI/B$-algebras. Recently, Andrzej Walendziak defined a $BF$-algebra [10].

In [11], Zadeh made an extension of the concept of a fuzzy set by an interval-valued fuzzy set (i.e., a fuzzy set with an interval-valued membership function). This interval-valued fuzzy set is referred to as an i-v fuzzy set, also he constructed a method of approximate inference using his i-v fuzzy sets. Biswas [1], defined interval-valued fuzzy subgroups and S. M. Hong et. al. applied the notion of interval-valued fuzzy to $BCI$-algebras [3].

In the present paper, we using the notion of interval-valued fuzzy set and introduced the concept of interval-valued fuzzy $Q$-subalgebras (briefly i-v fuzzy $BF$-subalgebras) of a $BF$-algebra, and study some of their properties. We prove that every $BF$-subalgebra of a $BF$-algebra $X$ can be realized as an i-v level $BF$-subalgebra of an i-v fuzzy $BF$-subalgebra of $X$, then we obtain some related results which have been mentioned in the abstract.

2 Preliminary Notes

Definition 2.1. [10] A $BF$-algebra is a non-empty set $X$ with a consonant 0 and a binary operation $*$ satisfying the following axioms:

(I) $x * x = 0$,
(II) $x * 0 = x$,
(III) $0 * (x * y) = (y * x)$,
for all $x, y \in X$.

Example 2.2. [10] (a) Let $R$ be the set of real numbers and let $A = (R; *, 0)$ be the algebra with the operation $*$ defined by

$$x * y = \begin{cases} x & \text{if } y = 0, \\ y & \text{if } x = 0, \\ 0 & \text{otherwise} \end{cases}$$

Then $A$ is a $BF$-algebra.

(b) Let $A = [0; \infty)$. Define the binary operation $*$ on $A$ as follows: $x * y = |x - y|$, for all $x, y \in A$. Then $(A; *, 0)$ is a $BF$-algebra.

Definition 2.3. [10] Let $X$ be a $BF$-algebra. Then for any $x$ and $y$ in $X$, the following hold:

(a) $0 * (0 * x) = x$ for all $x \in A$;
(b) if $0 * x = 0 * y$, then $x = y$ for any $x, y \in A$;
(c) if $x * y = 0$, then $y * x = 0$ for any $x, y \in A$. 
Definition 2.4. [10] A non-empty subset $S$ of a BF-algebra $X$ is called a subalgebra of $X$ if $x \ast y \in S$ for any $x, y \in S$.

A mapping $f : X \rightarrow Y$ of BF-algebras is called a BF-homomorphism if $f(x \ast y) = f(x) \ast f(y)$ for all $x, y \in X$.

We now review some fuzzy logic concept (see [10]).

Let $X$ be a set. A fuzzy set $A$ in $X$ is characterized by a membership function $\mu_A : X \rightarrow [0, 1]$. Let $f$ be a mapping from the set $X$ to the set $Y$ and let $BF$ be a fuzzy set in $Y$ with membership function $\mu_B$.

The inverse image of $BF$, denoted $f^{-1}(B)$, is the fuzzy set in $X$ with membership function $\mu_{f^{-1}(B)}$ defined by $\mu_{f^{-1}(B)}(x) = \mu_B(f(x))$ for all $x \in X$.

Conversely, let $A$ be a fuzzy set in $X$ with membership function $\mu_A$ Then the image of $A$, denoted by $f(A)$, is the fuzzy set in $Y$ such that:

$$\mu_{f(A)}(y) = \begin{cases} \sup_{z \in f^{-1}(y)} \mu_A(z) & \text{if } f^{-1}(y) = \{x : f(x) = y\} \neq \emptyset, \\ 0 & \text{otherwise} \end{cases}$$

A fuzzy set $A$ in the BF-algebra $X$ with the membership function $\mu_A$ is said to be have the sup property if for any subset $T \subseteq X$ there exists $x_0 \in T$ such that

$$\mu_A(x_0) = \sup_{t \in T} \mu_A(t)$$

An interval-valued fuzzy set (briefly, i-v fuzzy set) $A$ defined on $X$ is given by

$$A = \{(x, [\mu^L_A(x), \mu^U_A(x)]), \forall x \in X \}.$$  

Briefly, denoted by $A = [\mu^L_A, \mu^U_A]$ where $\mu^L_A$ and $\mu^U_A$ are any two fuzzy sets in $X$ such that $\mu^L_A(x) \leq \mu^U_A(x)$ for all $x \in X$.

Let $\overline{\mu}_A(x) = [\mu^L_A(x), \mu^U_A(x)]$, for all $x \in X$ and let $D[0, 1]$ denotes the family of all closed sub-intervals of $[0, 1]$. It is clear that if $\mu^L_A(x) = \mu^U_A(x) = c$, where $0 \leq c \leq 1$ then $\overline{\mu}_A(x) = [c, c]$ is in $D[0, 1]$. Thus $\overline{\mu}_A(x) \in D[0, 1]$, for all $x \in X$. Therefore the i-v fuzzy set $A$ is given by

$$A = \{(x, \overline{\mu}_A(x))\}, \forall x \in X$$

where

$$\overline{\mu}_A : X \rightarrow D[0, 1]$$

Now we define refined minimum (briefly, rmin) and order ” $\leq$ ” on elements $D_1 = [a_1, b_1]$ and $D_2 = [a_2, b_2]$ of $D[0, 1]$ as:

$$\text{rmin}(D_1, D_2) = [\text{min}\{a_1, a_2\}, \text{min}\{b_1, b_2\}]$$


\[ D_1 \leq D_2 \iff a_1 \leq a_2 \land b_1 \leq b_2 \]

Similarly we can define \( \geq \) and \( = \).

**Definition 2.3.** [2] Let \( \mu \) be a fuzzy set in a BF-algebra. Then \( \mu \) is called a fuzzy BF-subalgebra (BF-algebra) of \( X \) if

\[ \mu(x \ast y) \geq \min\{\mu(x), \mu(y)\} \]

for all \( x, y \in X \).

**Proposition 2.4.** [2] Let \( f \) be a BF-homomorphism from \( X \) into \( Y \) and \( G \) be a fuzzy BF-subalgebra of \( Y \) with the membership function \( \mu_G \). Then the inverse image \( f^{-1}(G) \) of \( G \) is a fuzzy BF-subalgebra of \( X \).

**Proposition 2.5.** [2] Let \( f \) be a BF-homomorphism from \( X \) onto \( Y \) and \( D \) be a fuzzy BF-subalgebra of \( X \) with the sup property. Then the image \( f(D) \) of \( D \) is a fuzzy BF-subalgebra of \( Y \).

### 3 Interval-valued Fuzzy BF-algebra

From now on \( X \) is a BF-algebra, unless otherwise is stated.

**Definition 3.1.** An i-v fuzzy set \( A \) in \( X \) is called an interval-valued fuzzy BF-subalgebras (briefly i-v fuzzy BF-subalgebra) of \( X \) if:

\[ \overline{\pi}_A(x \ast y) \geq \text{rmin}\{\overline{\pi}_A(x), \overline{\pi}_A(y)\} \]

for all \( x, y \in X \).

**Example 3.2.** Let \( X = \{0, 1, 2, 3\} \) be a set with the following table:

<table>
<thead>
<tr>
<th>( \ast )</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>2</td>
<td>2</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>3</td>
<td>3</td>
<td>3</td>
<td>3</td>
<td>0</td>
</tr>
</tbody>
</table>

Then \( (X, \ast, 0) \) is a BF-algebra, which is not a BCH/BCI/BCK-algebra.

Define \( \overline{\pi}_A \) as:

\[ \overline{\pi}_A(x) = \begin{cases} [0.3, 0.9] & \text{if } x \in \{0, 2\} \\ [0.1, 0.6] & \text{Otherwise} \end{cases} \]
It is easy to check that $A$ is an i-v fuzzy $BF$-subalgebra of $X$.

**Lemma 3.3.** If $A$ is an i-v fuzzy $BF$-subalgebra of $X$, then for all $x \in X$

$$
\overline{\mu}_A(0) \geq \overline{\mu}_A(x).
$$

**Proof.** For all $x \in X$, we have

\[
\begin{align*}
\overline{\mu}_A(0) &= \overline{\mu}_A(x \ast x) \geq r\min\{\overline{\mu}_A(x), \overline{\mu}_A(x)\} \\
&= r\min\{[\mu^L_A(x), \mu^U_A(x)], [\mu^L_A(x), \mu^U_A(x)]\} \\
&= [\mu^L_A(x), \mu^U_A(x)] = \overline{\mu}_A(x).
\end{align*}
\]

**Proposition 3.4.** Let $A$ be an i-v fuzzy $BF$-subalgebra of $X$, and let $n \in \mathbb{N}$. Then

(i) $\overline{\mu}_A(\prod_{i=1}^{n} x \ast x) \geq \overline{\mu}_A(x)$, for any odd number $n$,

(ii) $\overline{\mu}_A(\prod_{i=1}^{n} x \ast x) \geq \overline{\mu}_A(0)$, for any even number $n$.

**Proof.** Let $x \in X$ and assume that $n$ is odd. Then $n = 2k - 1$ for some positive integer $k$. We prove by induction, definition and above lemma imply that $\overline{\mu}_A(x \ast x) = \overline{\mu}_A(0) \geq \overline{\mu}_A(x)$. Now suppose that $\overline{\mu}_A(\prod_{i=1}^{2k-1} x \ast x) \geq \overline{\mu}_A(x)$.

Then by assumption

\[
\begin{align*}
\overline{\mu}_A(\prod_{i=1}^{2k+1} x \ast x) &= \overline{\mu}_A(\prod_{i=1}^{2k+1} x \ast x) \\
&= \overline{\mu}_A(\prod_{i=1}^{2k-1} x \ast (x \ast (x \ast x))) \\
&= \overline{\mu}_A(\prod_{i=1}^{2k-1} x \ast x) \\
&\geq \overline{\mu}_A(x).
\end{align*}
\]

Which proves (i). Similarly we can prove (ii).

**Theorem 3.5.** Let $A$ be an i-v fuzzy $BF$-subalgebra of $X$. If there exists a sequence $\{x_n\}$ in $X$, such that

$$
\lim_{n \to \infty} \overline{\mu}_A(x_n) = [1, 1]
$$


Then $\overline{\mu}_A(0) = [1, 1]$.

**Proof.** By above lemma we have $\overline{\mu}_A(0) \geq \overline{\mu}_A(x)$, for all $x \in X$, thus $\overline{\mu}_A(0) \geq \overline{\mu}_A(x_n)$, for every positive integer $n$. Consider

$$[1, 1] \geq \overline{\mu}_A(0) \geq \lim_{n \to \infty} \overline{\mu}_A(x_n) = [1, 1].$$

Hence $\overline{\mu}_A(0) = [1, 1]$.

**Theorem 3.6.** An i-v fuzzy set $A = [\mu^L_A, \mu^U_A]$ in $X$ is an i-v fuzzy BF-subalgebra of $X$ if and only if $\mu^L_A$ and $\mu^U_A$ are fuzzy BF-subalgebra of $X$.

**Proof.** Let $\mu^L_A$ and $\mu^U_A$ are fuzzy BF-subalgebra of $X$ and $x, y \in X$, consider

$$\overline{\mu}_A(x \ast y) = [\overline{\mu}_A(x \ast y), \overline{\mu}_A(x \ast y)]$$

$$\geq \min\{\mu^L_A(x), \mu^L_A(y)\}, \min\{\mu^U_A(x), \mu^U_A(y)\}$$

$$= r\min\{[\mu^L_A(x), \mu^U_A(x)], [\mu^L_A(y), \mu^U_A(y)]\}$$

$$= r\min\{\overline{\mu}_A(x), \overline{\mu}_A(y)\}.$$

This completes the proof.

Conversely, suppose that $A$ is an i-v fuzzy BF-subalgebras of $X$. For any $x, y \in X$ we have

$$[\mu^L_A(x \ast y), \mu^U_A(x \ast y)] = \overline{\mu}_A(x \ast y)$$

$$\geq r\min\{\overline{\mu}_A(x), \overline{\mu}_A(y)\}$$

$$= r\min\{[\mu^L_A(x), \mu^U_A(x)], [\mu^L_A(y), \mu^U_A(y)]\}$$

$$= \min\{\mu^L_A(x), \mu^L_A(y)\}, \min\{\mu^U_A(x), \mu^U_A(y)\}.$$
Let \( \{ A_i \mid i \in A \} \) be a family of i-v fuzzy \( BF \)-subalgebras of \( X \). Then \( \bigcap_{i \in A} A_i \) is also an i-v fuzzy \( BF \)-subalgebra of \( X \).

**Definition 3.9.** Let \( A \) be an i-v fuzzy set in \( X \) and \( [\delta_1, \delta_2] \in D[0, 1] \). Then the i-v level \( BF \)-subalgebra \( U(A; [\delta_1, \delta_2]) \) of \( A \) and strong i-v \( BF \)-subalgebra \( U(A; >, [\delta_1, \delta_2]) \) of \( X \) are defined as following:

\[
U(A; [\delta_1, \delta_2]) := \{ x \in X \mid \mu_A(x) \geq [\delta_1, \delta_2] \},
\]

\[
U(A; >, [\delta_1, \delta_2]) := \{ x \in X \mid \mu_A(x) > [\delta_1, \delta_2] \}.
\]

**Theorem 3.10.** Let \( A \) be an i-v fuzzy \( BF \)-subalgebra of \( X \) and \( BF \) be closure of image of \( \mu_A \). Then the following condition are equivalent:

(i) \( A \) is an i-v fuzzy \( BF \)-subalgebra of \( X \).

(ii) For all \([\delta_1, \delta_2] \in Im(\mu_A)\), the nonempty level subset \( U(A; [\delta_1, \delta_2]) \) of \( A \) is a \( BF \)-subalgebra of \( X \).

(iii) For all \([\delta_1, \delta_2] \in Im(\mu_A) \setminus B \), the nonempty strong level subset \( U(A; >, [\delta_1, \delta_2]) \) of \( A \) is a \( BF \)-subalgebra of \( X \).

(iv) For all \([\delta_1, \delta_2] \in D[0, 1] \), the nonempty strong level subset \( U(A; >, [\delta_1, \delta_2]) \) of \( A \) is a \( BF \)-subalgebra of \( X \).

(v) For all \([\delta_1, \delta_2] \in D[0, 1] \), the nonempty level subset \( U(A; [\delta_1, \delta_2]) \) of \( A \) is a \( BF \)-subalgebra of \( X \).

**Proof.** (i \( \rightarrow \) iv) Let \( A \) be an i-v fuzzy \( BF \)-subalgebra of \( X \), \([\delta_1, \delta_2] \in D[0, 1] \) and \( x, y \in U(A; <, [\delta_1, \delta_2]) \), then we have

\[
\overline{\mu}_A(x \ast y) \geq \text{rmin}\{\overline{\mu}_A(x), \overline{\mu}_A(y)\} > \text{rmin}\{[\delta_1, \delta_2], [\delta_1, \delta_2]\} = [\delta_1, \delta_2]
\]

thus \( x \ast y \in U(A; >, [\delta_1, \delta_2]) \). Hence \( U(A; >, [\delta_1, \delta_2]) \) is a \( BF \)-subalgebra of \( X \).

(iv \( \rightarrow \) iii) It is clear.

(iii \( \rightarrow \) ii) Let \([\delta_1, \delta_2] \in Im(\mu_A)\). Then \( U(A; [\delta_1, \delta_2]) \) is a nonempty. Since

\[
U(A; [\delta_1, \delta_2]) = \bigcap_{[\delta_1, \delta_2] > [\alpha_1, \alpha_2]} U(A; >, [\delta_1, \delta_2]),
\]

where \([\alpha_1, \alpha_2] \in Im(\mu_A) \setminus B \). Then by (iii) and Corollary 3.7, \( U(A; [\delta_1, \delta_2]) \) is a \( BF \)-subalgebra of \( X \).
(ii → v) Let \([\delta_1, \delta_2] \in D[0, 1]\) and \(U(A; [\delta_1, \delta_2])\) be nonempty. Suppose \(x, y \in U(A; [\delta_1, \delta_2])\). Let \([\beta_1, \beta_2] = \min\{\mu_A(x), \mu_A(y)\}\), it is clear that \([\beta_1, \beta_2] = \min\{\mu_A(x), \mu_A(y)\} \geq ([\delta_1, \delta_2], [\delta_1, \delta_2]) = [\delta_1, \delta_2]\). Thus \(x, y \in U(A; [\beta_1, \beta_2])\) and \([\beta_1, \beta_2] \in \text{Im}(\mu_A)\), by (ii) \(U(A; [\beta_1, \beta_2])\) is a BF-subalgebra of \(X\), hence \(x * y \in U(A; [\beta_1, \beta_2])\). Then we have

\[
\overline{\mu}_A(x * y) \geq rmin\{\mu_A(x), \mu_A(y)\} \geq ([\beta_1, \beta_2], [\beta_1, \beta_2]) = [\beta_1, \beta_2] \geq [\delta_1, \delta_2].
\]

Therefore \(x * y \in U(A; [\delta_1, \delta_2])\). Then \(U(A; [\delta_1, \delta_2])\) is a BF-subalgebra of \(X\).

(v → i) Assume that the nonempty set \(U(A; [\delta_1, \delta_2])\) is a BF-subalgebra of \(X\), for every \([\delta_1, \delta_2] \in D[0, 1]\). In contrary, let \(x_0, y_0 \in X\) be such that

\[
\overline{\mu}_A(x_0 * y_0) < rmin\{\overline{\mu}_A(x_0), \overline{\mu}_A(y_0)\}.
\]

Let \(\overline{\mu}_A(x_0) = [\gamma_1, \gamma_2]\), \(\overline{\mu}_A(y_0) = [\gamma_3, \gamma_4]\) and \(\overline{\mu}_A(x_0 * y_0) = [\delta_1, \delta_2]\). Then

\[
[\delta_1, \delta_2] < rmin\{[\gamma_1, \gamma_2], [\gamma_3, \gamma_4]\} = [\min\{\gamma_1, \gamma_3\}, \min\{\gamma_2, \gamma_4\}].
\]

So \(\delta_1 < \min\{\gamma_1, \gamma_3\}\) and \(\delta_2 < \min\{\gamma_2, \gamma_4\}\).

Consider

\[
[\lambda_1, \lambda_2] = \frac{1}{2}\overline{\mu}_A(x_0 * y_0) + rmin\{\overline{\mu}_A(x_0), \overline{\mu}_A(y_0)\}
\]

We get that

\[
[\lambda_1, \lambda_2] = \frac{1}{2}([\delta_1, \delta_2] + \min\{\gamma_1, \gamma_3\}, \min\{\gamma_2, \gamma_4\})
\]

\[
= \left[\frac{1}{2}(\delta_1 + \min\{\gamma_1, \gamma_3\}), \frac{1}{2}(\delta_2 + \min\{\gamma_2, \gamma_4\})\right]
\]

Therefore

\[
\min\{\gamma_1, \gamma_3\} > \lambda_1 = \frac{1}{2}(\delta_1 + \min\{\gamma_1, \gamma_3\}) > \delta_1
\]

\[
\min\{\gamma_2, \gamma_4\} > \lambda_2 = \frac{1}{2}(\delta_2 + \min\{\gamma_2, \gamma_4\}) > \delta_2
\]

Hence

\[
[\min\{\gamma_1, \gamma_3\}, \min\{\gamma_2, \gamma_4\}] > [\lambda_1, \lambda_2] > [\delta_1, \delta_2] = \overline{\mu}_A(x_0 * y_0)
\]

so that \(x_0 * y_0 \not\in U(A; [\delta_1, \delta_2])\)

which is a contradiction, since

\[
\overline{\mu}_A(x_0) = [\gamma_1, \gamma_2] \geq [\min\{\gamma_1, \gamma_3\}, \min\{\gamma_2, \gamma_4\}] > [\lambda_1, \lambda_2]
\]
Theorem 3.11. Each $BF$-subalgebra of $X$ is an i-v level $BF$-subalgebra of an i-v fuzzy $BF$-subalgebra of $X$.

Proof. Let $Y$ be a $BF$-subalgebra of $X$, and $A$ be an i-v fuzzy set on $X$ defined by

$$\overline{\mu}_A(x) = \begin{cases} [\alpha_1, \alpha_2] & \text{if } x \in Y \\ [0, 0] & \text{Otherwise} \end{cases}$$

where $\alpha_1, \alpha_2 \in [0, 1]$ with $\alpha_1 < \alpha_2$. It is clear that $U(A; [\alpha_1, \alpha_2]) = Y$. Let $x, y \in X$. We consider the following cases:

case 1) If $x, y \in Y$, then $x \ast y \in Y$ therefore

$$\overline{\mu}_A(x \ast y) = [\alpha_1, \alpha_2] = rmin\{[\alpha_1, \alpha_2], [\alpha_1, \alpha_2]\} = rmin\{\overline{\mu}_A(x), \overline{\mu}_A(y)\}.$$ 

case 2) If $x, y \not\in Y$, then $\overline{\mu}_A(x) = [0, 0] = \overline{\mu}_A(y)$ and so

$$\overline{\mu}_A(x \ast y) \geq [0, 0] = rmin\{[0, 0], [0, 0]\} = rmin\{\overline{\mu}_A(x), \overline{\mu}_A(y)\}.$$ 

case 3) If $x \in Y$ and $y \not\in Y$, then $\overline{\mu}_A(x) = [\alpha_1, \alpha_2]$ and $\overline{\mu}_A(y) = [0, 0]$. Thus

$$\overline{\mu}_A(x \ast y) \geq [0, 0] = rmin\{[\alpha_1, \alpha_2], [0, 0]\} = rmin\{\overline{\mu}_A(x), \overline{\mu}_A(y)\}.$$ 

case 4) If $y \in Y$ and $x \not\in Y$, then by the same argument as in case 3, we can conclude that $\overline{\mu}_A(x \ast y) \geq rmin\{\overline{\mu}_A(x), \overline{\mu}_A(y)\}.$

Therefore $A$ is an i-v fuzzy $BF$-subalgebra of $X$.

Theorem 3.12. Let $Y$ be a subset of $X$ and $A$ be an i-v fuzzy set on $X$ which is given in the proof of Theorem 3.11. If $A$ is an i-v fuzzy $BF$-subalgebra of $X$, then $Y$ is a $BF$-subalgebra of $X$.

Proof. Let $A$ be an i-v fuzzy $BF$-subalgebra of $X$, and $x, y \in Y$. Then $\overline{\mu}_A(x) = [\alpha_1, \alpha_2] = \overline{\mu}_A(y)$, thus

$$\overline{\mu}_A(x \ast y) \geq rmin\{\overline{\mu}_A(x), \overline{\mu}_A(y)\} = rmin\{[\alpha_1, \alpha_2], [\alpha_1, \alpha_2]\} = [\alpha_1, \alpha_2].$$

which implies that $x \ast y \in Y$. 

Theorem 3.13. If $A$ is an i-v fuzzy $BF$-subalgebra of $X$, then the set

$$X_{\overline{p}_A} := \{ x \in X \mid \overline{p}_A(x) = \overline{p}_A(0) \}$$

is a $BF$-subalgebra of $X$.

Proof. Let $x, y \in X_{\overline{p}_A}$. Then $\overline{p}_A(x) = \overline{p}_A(0) = \overline{p}_A(y)$, and so

$$\overline{p}_A(x \ast y) \geq \text{rmin}\{\overline{p}_A(x), \overline{p}_A(y)\} = \text{rmin}\{\overline{p}_A(0), \overline{p}_A(0)\} = \overline{p}_A(0).$$

by Lemma 3.3, we get that $\overline{p}_A(x \ast y) = \overline{p}_A(0)$ which means that $x \ast y \in X_{\overline{p}_A}$.

Theorem 3.14. Let $N$ be an i-v fuzzy sub set of $X$. Let $N$ be an i-v fuzzy set defined by $\overline{p}_A$ as:

$$\overline{p}_N(x) = \begin{cases} [\alpha_1, \alpha_2] & \text{if } x \in N \\ [\beta_1, \beta_2] & \text{Otherwise} \end{cases}$$

for all $[\alpha_1, \alpha_2], [\beta_1, \beta_2] \in D[0, 1]$ with $[\alpha_1, \alpha_2] \geq [\beta_1, \beta_2]$. Then $N$ is an i-v fuzzy $BF$-subalgebra if and only if $N$ is a $BF$-subalgebra of $X$. Moreover, in this case $X_{\overline{p}_N} = N$.

Proof. Let $N$ be an i-v fuzzy $BF$-subalgebra. Let $x, y \in X$ be such that $x, y \in N$. Then

$$\overline{p}_N(x \ast y) \geq \text{rmin}\{\overline{p}_N(x), \overline{p}_N(y)\} = \text{rmin}\{[\alpha_1, \alpha_2], [\alpha_1, \alpha_2]\} = [\alpha_1, \alpha_2]$$

and so $x \ast y \in N$.

Conversely, suppose that $N$ is a $BF$-subalgebra of $X$, let $x, y \in X$.

(i) If $x, y \in N$ then $x \ast y \in N$, thus

$$\overline{p}_N(x \ast y) = [\alpha_1, \alpha_2] = \text{rmin}\{\overline{p}_N(x), \overline{p}_N(y)\}$$

(ii) If $x \not\in N$ or $y \not\in N$, then

$$\overline{p}_N(x \ast y) \geq [\beta_1, \beta_2] = \text{rmin}\{\overline{p}_N(x), \overline{p}_N(y)\}$$

This show that $N$ is an i-v fuzzy $BF$-subalgebra.

Moreover, we have

$$X_{\overline{p}_N} := \{ x \in X \mid \overline{p}_N(x) = \overline{p}_N(0) \} = \{ x \in X \mid \overline{p}_N(x) = [\alpha_1, \alpha_2] \} = N.$$

Definition 3.15. [1] Let $f$ be a mapping from the set $X$ into a set $Y$. Let $BF$ be an i-v fuzzy set in $Y$. Then the inverse image of $BF$, denoted by $f^{-1}[B]$,
is the i-v fuzzy set in $X$ with the membership function given by $\overline{\mu}_{f^{-1}[B]}(x) = \overline{\mu}_B(f(x))$, for all $x \in X$.

**Lemma 3.16.** [1] Let $f$ be a mapping from the set $X$ into a set $Y$. Let $m = [m^L, m^U]$ and $n = [n^L, n^U]$ be i-v fuzzy sets in $X$ and $Y$ respectively. Then

1. $f^{-1}(n) = [f^{-1}(n^L), f^{-1}(n^U)]$,
2. $f(m) = [f(m^L), f(m^U)]$.

**Proposition 3.17.** Let $f$ be a $BF$-homomorphism from $X$ into $Y$ and $G$ be an i-v fuzzy $BF$-subalgebra of $Y$ with the membership function $\mu_G$. Then the inverse image $f^{-1}[G]$ of $G$ is an i-v fuzzy $BF$-subalgebra of $X$.

**Proof.** Since $B = [\mu_B^L, \mu_B^U]$ is an i-v fuzzy $BF$-subalgebra of $Y$, by Theorem 3.6, we get that $\mu_B^L$ and $\mu_B^U$ are fuzzy $BF$-subalgebra of $Y$. By Proposition 2.4, $f^{-1}[\mu_B^L]$ and $f^{-1}[\mu_B^U]$ are fuzzy $BF$-subalgebra of $X$, by above lemma and Theorem 3.6, we can conclude that $f^{-1}(B) = [f^{-1}(\mu_B^L), f^{-1}(\mu_B^U)]$ is an i-v fuzzy $BF$-subalgebra of $X$.

**Definition 3.18.** [1] Let $f$ be a mapping from the set $X$ into a set $Y$, and $A$ be an i-v fuzzy set in $X$ with membership function $\mu_A$. Then the image of $A$, denoted by $f[A]$, is the i-v fuzzy set in $Y$ with membership function defined by:

$$\overline{\mu}_{f[A]}(y) = \begin{cases} \rho_{\sup_{z \in f^{-1}(y)} \overline{\mu}_A(z)} & \text{if } f^{-1}(y) \neq \emptyset, \forall y \in Y, \\ [0,0] & \text{otherwise} \end{cases}$$

Where $f^{-1}(y) = \{x \mid f(x) = y\}$.

**Theorem 3.19.** Let $f$ be a $BF$-homomorphism from $X$ onto $Y$. If $A$ is an i-v fuzzy $BF$-subalgebra of $X$, then the image $f[A]$ of $A$ is an i-v fuzzy $BF$-subalgebra of $Y$.

**Proof.** Assume that $A$ is an i-v fuzzy $BF$-subalgebra of $X$, then $A = [\mu_A^L, \mu_A^U]$ is an i-v fuzzy $BF$-subalgebra of $X$ if and only if $\mu_A^L$ and $\mu_A^U$ are fuzzy $BF$-subalgebra of $X$. By Proposition 2.5, $f[\mu_A^L]$ and $f[\mu_A^U]$ are fuzzy $BF$-subalgebra of $Y$, by Lemma 3.16, and Theorem 3.6, we can conclude that $f[A] = [f[\mu_A^L], f[\mu_A^U]]$ is an i-v fuzzy $BF$-subalgebra of $Y$.

**References**


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