Atomic Decomposition for Weighted Bergman Space $L^2_a$ on the Unit Ball of $C$

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Abstract

Throughout this paper by using the frame theory we give a short proof for atomic decomposition for weighted Bergman space. In fact we show that the weighted Bergman space $L^2_a(dA_\alpha)$ admit an atomic decomposition i.e every analytic function in this space can be presented as a linear combination of ”atoms “ defined using the normalized reproducing kernel of this space .

Keywords: Bergman space ,Atomic decomposition ,Frame

1 Introduction

Let $D = \{z \in C : |z| < 1\}$ be the open unit disk in $C$ and $dA(z)$ be the area measure on D normalized so that the area of D is $1$ . i.e

$$dA(z) = \frac{1}{\pi} dxdy = \frac{1}{\pi} r dr d\theta$$

The Bergman space $L^2_a(dA)$ is the subset of $L^2(D, dA)$ consisting of analytic functions $L^2_a(dA)$ is a Hilbert space with reproducing kernel $K(z, w) = \frac{1}{(1-z\overline{w})^2}$ .and normalized kernel

$$k_\lambda(z) = \frac{K(z, \lambda)}{\sqrt{K(\lambda, \lambda)}} = \frac{(1 - |\lambda|^2)}{(1 - \lambda z)^2}$$
where \( \lambda \in D \). For any \( \alpha > -1 \), let \( dA_\alpha \) be the measure on \( D \) defined by 
\[
dA_\alpha(z) = (\alpha + 1)(1 - |z|^2)^\alpha dA(z).
\]

The Weighted Bergman space \( L^2_a(dA_\alpha) \) is the subset of \( L^2(D, dA_\alpha) \) consisting of analytic functions. Then \( L^2_a(dA_\alpha) \) is a Hilbert space with reproducing kernel 
\[
K_\alpha^\lambda(z, w) = \frac{1}{(1 - \bar{z}w)^{2+\alpha}}.
\]

We say that a sequence \( \{\lambda_n\} \) of distinct points in the disk is a sampling sequence for \( L^2_a(dA_\alpha) \) if there exist positive constant \( A \) and \( B \) such that 
\[
A\|f\|^2_{a,2} \leq \sum_{n \in I} (1 - |\lambda_n|^2)^{2+\alpha} |f(\lambda_n)|^2 \leq B\|f\|^2_{a,2}
\]

The Frame theory was introduced by Duffin and Schaeffer (1952) (Daubechies, 1992) in order to establish general conditions under which one can reconstruct perfectly a function \( f \) in a Hilbert space \( H \) from its inner product \( \langle \cdot, \cdot \rangle_H \) with a family of vectors \( \{f_n\}_{n \in I} \) where \( I \) can be a finite or infinite countable index set.

### 1.1 Definition

A set of vectors \( \{f_n\}_{n \in I} \) in a Hilbert space \( H \) is a frame if there exists two constants \( A, B > 0 \) so that
\[
\forall f \in H, \quad A\|f\|^2_H \leq \sum_{n \in I} |\langle f, f_n \rangle_H|^2 \leq B\|f\|^2_H
\]

### 1.2 Theorem[2]

A sequence \( \{f_n\}_{n \in I} \) is a frame for \( H \) if and only if the mapping \( T : a_n \to \sum a_n f_n \) is a well defined and bounded mapping of \( \ell^2 \) onto \( H \).

### 1.3 Theorem[1]

Given sequence \( \{f_n\}_{n \in I} \) in a Hilbert space \( H \), the following two statements are equivalent:

1. \( \{f_n\} \) is a frame with bounds \( A, B \),
2. \( Sf = \sum_{n \in I} < f, f_n > f_n \) is a bounded and invertible linear operator on \( H \), with \( AI \leq S \leq BI \), called the frame operator for \( \{f_n\} \).
2 Atomic decomposition

Atomic decomposition were first studied by Coifman and Roehrborg in the setting of Bergman spaces on the unit disk in 1980. Zhu in 1990 gave a presentation for atomic decomposition in this space [3].

The purpose of this section is to present a very short proof for atomic decomposition for weighted Bergman spaces and show normalized reproducing kernels are building blocks for \( L^2_a(dA_\alpha) \). In some sense, they play the role of an orthonormal basis for this space, although they are clearly not mutually orthogonal. At first we prove following lemma.

2.1 Lemma

\( \lambda_n \) is a sampling sequence for \( L^2_a(dA_\alpha) \) if and only if the set of normalized reproducing kernels \( \{ k^\alpha_{\lambda_n} \} \) is a frame for \( L^2_a(dA_\alpha) \).

**Proof.** For \( f \in L^2_a(dA_\alpha) \)

\[
\sum | <f, k^\alpha_{\lambda_n}> |^2 = \sum | \int_D f(z) k^\alpha_{\lambda_n}(z) dA_\alpha(z) |^2
\]

\[
= \sum | \int_D f(z) \frac{(1-|\lambda_n|^2)^{1+\alpha}}{(1-\lambda_n^2)^{2+\alpha}} dA_\alpha(z) |^2
\]

\[
= \sum (1-|\lambda_n|^2)^{2+\alpha} |f(\lambda_n)|^2.
\]

Let \( \{\lambda_n\} \) be a sampling sequence for \( L^2_a(dA_\alpha) \), we define two operators

\[
T^\alpha : \ell^2 \rightarrow L^2_a(dA_\alpha)
\]

\[
T^\alpha (\{ a_n \})(z) = \sum a_n \frac{(1-|\lambda_n|^2)^{1+\alpha}}{(1-\lambda_n^2)^{2+\alpha}}
\]

\[
S^\alpha : L^2_a(dA_\alpha) \rightarrow L^2_a(dA_\alpha)
\]

\[
S^\alpha (f(z)) = \sum (1-|\lambda_n|^2)^{2+\alpha} \frac{f(\lambda_n)}{(1-\lambda_n^2)^{2+\alpha}}
\]

2.2 Theorem

\( T^\alpha \) is bounded and onto.

**Proof.** Since \( \{ k^\alpha_{\lambda_n} \} \) is a frame for \( L^2_a(dA_\alpha) \), so

\[
T^\alpha (\{ a_n \})(z) = \sum a_n k^\alpha_{\lambda_n}(z) = \sum a_n \frac{(1-|\lambda_n|^2)^{1+\alpha}}{(1-\lambda_n^2)^{2+\alpha}}
\]

thus by theorem (1.2) \( T^\alpha \) is bounded and onto.
2.3 Theorem

$S^\alpha$ is bounded and invertible.

Proof. Since $\{k_{\lambda_n}^\alpha\}$ is a frame for $L_a^2(dA_\alpha)$,

$$S^\alpha(f(z)) = \sum <f, k_{\lambda_n}^\alpha> k_{\lambda_n}^\alpha(z)$$

$$= \sum f_D f(z)k_{\lambda_n}^\alpha(z)dA_\alpha(z)k_{\lambda_n}^\alpha(z)$$

$$= \sum f_D f(z)\frac{(1-|\lambda_n|^2)^{1+\frac{\alpha}{2}}}{(1-\lambda_n z)^{2+\alpha}}dA_\alpha(z)k_{\lambda_n}^\alpha(z)$$

$$= \sum (1-|\lambda_n|^2)^{2+\alpha} \frac{f(\lambda_n)}{(1-\lambda_n z)^{2+\alpha}}.$$ 

Thus by theorem (1.3) $S^\alpha$ is bounded and invertible.

Finally we state the atomic decomposition theorem for the weighted Bergman space. The proof follows from (2.2) and (2.3) theorems.

2.4 Theorem

There exists a sequence $\{\lambda_n\}$ in D and constant $C > 0$ with the following properties:

1. For any $\{a_n\}$ in $\ell^2$, the function

$$f(z) = \sum a_n \frac{(1-|\lambda_n|^2)^{1+\frac{\alpha}{2}}}{(1-\lambda_n z)^{2+\alpha}}$$

is in $L_a^2(dA_\alpha)$ with

$$\|f\|_{L_a^2(dA_\alpha)} \leq C\|a_n\|_{\ell^2}$$

2. For any $f \in L_a^2$, there is $\{a_n\} \in \ell^2$ such that

$$f(z) = \sum a_n \frac{(1-|\lambda_n|^2)^{1+\frac{\alpha}{2}}}{(1-\lambda_n z)^{2+\alpha}}$$

References


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