On Relative Defects of Meromorphic Functions

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Abstract
The aim of this paper is to establish some results on relative defects of meromorphic functions by means of their proximate deficiency.

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1 Introduction, Definitions and Notations.

Let \( f \) be a non-constant meromorphic function defined in the open complex plane \( \mathbb{C} \). For \( a \in \mathbb{C} \cup \{\infty\} \) let \( n(t, a; f) \) denotes the number of roots of \( f = a \) in \( |z| \leq t \), the multiple roots being counted according to their multiplicities and \( N(r, a; f) \) is defined in the usual way in terms of \( n(t, a; f) \). Similarly \( \bar{n}(t, a; f) \) denotes the number of distinct roots of \( f = a \) in \( |z| \leq t \) and \( \bar{N}(r, a; f) \) is also defined in the usual way in terms of \( \bar{n}(t, a; f) \). We do not explain the standard definitions and notations of Nevanlinna theory as those are available in [1].

The Nevanlinna defect \( \delta(a; f) \) and the Valiron defect \( \Delta(a; f) \) of \( a \) are respectively defined in the following manner:

\[
\delta(a; f) = 1 - \limsup_{r \to \infty} \frac{N(r, a; f)}{T(r, f)} = \liminf_{r \to \infty} \frac{m(r, a; f)}{T(r, f)}
\]
and
\[ \Delta(a; f) = 1 - \liminf_{r \to \infty} \frac{N(r, a; f)}{T(r, f)} = \limsup_{r \to \infty} \frac{m(r, a; f)}{T(r, f)}, \]

where \( T(r, f) \) is the Nevanlinna’s characteristic function of \( f \). Milloux [3] introduced the concept of absolute defect of \( a \) with respect to the derivative \( f' \). Later Xiong [4] extended this definition to \( f^{(k)} \). He [4] introduced the notion of
\[ \delta_R^{(k)}(a; f) = 1 - \limsup_{r \to \infty} \frac{N(r, a; f^{(k)})}{T(r, f)}, \]

for \( k = 1, 2, 3, \ldots \), and call it the relative Nevanlinna defect of \( a \) with respect to \( f^{(k)} \). In [4] he has shown various relations between the usual defects and the relative defects for meromorphic functions.

The following definition is well known.

**Definition 1.** The order \( \rho_f \) of a meromorphic function \( f \) is defined as follows:

\[ \rho_f = \limsup_{r \to \infty} \frac{\log T(r, f)}{\log r}. \]

For a meromorphic function \( f \) of finite order \( \rho_f \), a function \( \rho_f(r) \) is called a proximate order of \( f \) if the following conditions hold:

(i) \( \rho_f(r) \) is non-negative and continuous for \( r > r_0 \), say.
(ii) \( \rho_f(r) \) is differentiable for \( r \geq r_0 \) except possibly at isolated points at which \( \rho'_f(r + 0) \) and \( \rho'_f(r - 0) \) exist,
(iii) \( \lim_{r \to \infty} \rho_f(r) = \rho_f \),
(iv) \( \lim_{r \to \infty} r \rho'_f(r) \log r = 0 \) and
(v) \( \limsup_{r \to \infty} \frac{T(r, f)}{r \rho_f(r)} = 1 \).

The existence of such a proximate order is proved by Lahiri [2]. In the value distribution theory the proximate deficiency of the value \( a \) is defined as
\[ \delta_{\rho_f}(a; f) = 1 - \limsup_{r \to \infty} \frac{N(r, a; f)}{r \rho_f(r)}. \]

Using property (v) one can easily verify that \( \delta(a; f) \leq \delta_{\rho_f}(a; f) \) for every complex number \( a \).

Likewise one may define
\[ \Theta_{\rho_f}(a; f) = 1 - \limsup_{r \to \infty} \frac{\bar{N}(r, a; f)}{r \rho_f(r)}, \]

where \( \bar{N}(r, a; f) \) is the upper density of \( f \).
and call this proximate deficiency of the value \( a \) for distinct \( a \)-points. Clearly for every complex number \( a \), \( \Theta(a; f) \leq \Theta_{\rho_f}(a; f) \).

From the second fundamental theorem it is not difficult to prove that the set \( \{ a : \Theta_{\rho_f}(a; f) > 0 \} \) is countable and \( \sum_a \Theta_{\rho_f}(a; f) \leq 2 \). Also for a complex number \( a \), we put

\[
\Delta_{\rho_f}(a; f) = 1 - \liminf_{r \to \infty} \frac{N(r, a; f)}{r^{\rho_f(r)}}.
\]

In the paper we use the term

\[
R^\delta_{\rho_f}^{(k)}(a; f) = 1 - \limsup_{r \to \infty} \frac{N(r, a; f^{(k)})}{r^{\rho_f(r)}},
\]

for relative proximate deficiency of \( a \) with respect to \( f^{(k)} \) and

\[
R^{\Theta_{\rho_f}^{(k)}}(a; f) = 1 - \limsup_{r \to \infty} \frac{\bar{N}(r, a; f^{(k)})}{r^{\rho_f(r)}},
\]

for relative proximate deficiency of \( a \) with respect to \( f^{(k)} \) for distinct \( a \)-points, and establish various relations between \( \delta_{\rho_f}(a; f) \), \( \Theta_{\rho_f}(a; f) \), \( R^\delta_{\rho_f}^{(k)}(a; f) \) and \( R^{\Theta_{\rho_f}^{(k)}}(a; f) \), where \( k = 1, 2, 3, \ldots \). The term \( S(r, f) \) denotes any quantity satisfying \( S(r, f) = o\{T(r, f)\} \) i.e. \( \frac{S(r, f)}{T(r, f)} \to 0 \) as \( r \to \infty \) through all values of \( r \) if \( f \) is of finite order and except possibly for a set of \( r \) of finite linear measure otherwise.

2 Lemma.

In this section we present a lemma which will be needed in the sequel.

**Lemma 1.** Let \( f \) be a meromorphic function of finite order. Then

\[
\lim_{r \to \infty} \frac{S(r, f)}{r^{\rho_f(r)}} = 0.
\]

**Proof.** Since \( f \) is of finite order then

\[
S(r, f) = o\{T(r, f)\}.
\]

i.e., \( \lim_{r \to \infty} \frac{S(r, f)}{T(r, f)} = 0. \)
So
\[ \limsup_{r \to \infty} \frac{S(r, f)}{r^{\rho_f(r)}} = \limsup_{r \to \infty} \frac{T(r, f)}{r^{\rho_f(r)}} = 0. \]

i.e., \( \lim_{r \to \infty} \frac{S(r, f)}{r^{\rho_f(r)}} = 0. \)

This proves the lemma.

3 Theorems.

In this section we present the main results of the paper.

**Theorem 1.** Let \( f \) be a meromorphic function of finite order \( \rho_f \). Also let \( a_i (i = 1, 2, \ldots, p) \) be finite, distinct, non-zero complex numbers. Then for \( p \geq 1 \),

\[
p_{\rho_f}(0; f) + \Theta_{\rho_f}(\infty; f) + \sum_{i=1}^{p} R_{\rho_f}(a_i; f) \leq p + 1.
\]

**Proof.** From Nevanlinna’s first fundamental theorem and in view of Milloux’s theorem \( \{p.55, [1]\} \), it follows that

\[
T(r, f) \leq N \left( r, \frac{1}{f^{(k)}} \right) + m \left( r, \frac{f^{(k)}}{f} \right) + m \left( r, \frac{1}{f^{(k)}} \right) + S(r, f)
\]
i.e., \( T(r, f) \leq N \left( r, \frac{1}{f^{(k)}} \right) + m \left( r, \frac{1}{f^{(k)}} \right) + S(r, f) \)
i.e., \( T(r, f) \leq N \left( r, \frac{1}{f^{(k)}} \right) + T \left( r, \frac{1}{f^{(k)}} \right) - N \left( r, \frac{1}{f^{(k)}} \right) + S(r, f) \)
i.e., \( pT(r, f) \leq pN \left( r, \frac{1}{f^{(k)}} \right) + pT \left( r, \frac{1}{f^{(k)}} \right) - pN \left( r, \frac{1}{f^{(k)}} \right) + S(r, f) \)

\[ (1) \]

Since \( S(r, f^{(k)}) = S(r, f) \), by Nevanlinna’s second fundamental theorem we obtain that

\[
pT(r, f^{(k)}) \leq N(r, f^{(k)}) + \frac{p}{N} \sum_{i=1}^{p} \frac{1}{f^{(k)} - a_i} + S(r, f).
\]

\[ (2) \]
As $\bar{N}(r, f^{(k)}) = \bar{N}(r, f)$, we get from (1) and (2) that

$$pT(r, f) \leq p\bar{N}\left(r, \frac{1}{f}\right) + \bar{N}(r, f) + \bar{N}\left(r, \frac{1}{f^{(k)}}\right) + \sum_{i=1}^{p} \bar{N}\left(r, \frac{1}{f^{(k)} - a_i}\right)$$

$$-p\bar{N}\left(r, \frac{1}{f^{(k)}}\right) + S(r, f). \quad (3)$$

Since $\bar{N}\left(r, \frac{1}{f^{(k)}}\right) - p\bar{N}\left(r, \frac{1}{f^{(k)}}\right) \leq 0$, it follows from (3) that

$$pT(r, f) \leq p\bar{N}\left(r, \frac{1}{f}\right) + \bar{N}(r, f) + \sum_{i=1}^{p} \bar{N}\left(r, \frac{1}{f^{(k)} - a_i}\right) + S(r, f). \quad (4)$$

Dividing both sides of (4) by $r^{\rho_f(r)}$ and taking limit superior we get in view of Lemma 1 that

$$\limsup_{r \to \infty} \frac{T(r, f)}{r^{\rho_f(r)}} \leq \limsup_{r \to \infty} \frac{\bar{N}(r, f)}{r^{\rho_f(r)}} + \sum_{i=1}^{p} \limsup_{r \to \infty} \frac{\bar{N}\left(r, \frac{1}{f^{(k)} - a_i}\right)}{r^{\rho_f(r)}}$$

i.e., $p \leq p\left\{1 - \delta_{\rho_f}(0; f)\right\} + \left\{1 - \Theta_{\rho_f}(\infty; f)\right\} + \left\{p - \sum_{i=1}^{p} R\Theta_{\rho_f}^{(k)}(a_i; f)\right\}$

i.e., $p\delta_{\rho_f}(0; f) + \Theta_{\rho_f}(\infty; f) + \sum_{i=1}^{p} R\Theta_{\rho_f}^{(k)}(a_i; f) \leq p + 1$.

This proves the theorem.

**Theorem 2.** If $a \neq 0, \infty$ and $b \neq 0, \infty$ be any two complex numbers, then for any meromorphic function $f$ of finite order $\rho_f$,

$$R\Theta_{\rho_f}^{(k)}(a; f) + R\Theta_{\rho_f}^{(k)}(b; f) + \delta_{\rho_f}(0; f) \leq 2.$$

**Proof.** In view of Milloux’s theorem [p.55, [1]], we get that

$$m\left(r, \frac{1}{f}\right) \leq m\left(r, \frac{f^{(k)}}{f}\right) + m\left(r, \frac{1}{f^{(k)}}\right)$$

i.e., $m\left(r, \frac{1}{f}\right) \leq m\left(r, \frac{1}{f^{(k)}}\right) + S(r, f)$

i.e., $m\left(r, \frac{1}{f}\right) \leq T\left(r, \frac{1}{f^{(k)}}\right) - N\left(r, \frac{1}{f^{(k)}}\right) + S(r, f). \quad (5)$
Now by Nevanlinna’s first and second fundamental theorem we obtain from (5), on using

\[ N\left(r, \frac{1}{f(k)}\right) - N\left(r, \frac{1}{f(k)}\right) \leq 0 \]

\[ m\left(r, \frac{1}{f}\right) \]
\[ \leq \bar{N}\left(r, \frac{1}{f(k)}\right) + \bar{N}\left(r, \frac{1}{f(k) - a}\right) + \bar{N}\left(r, \frac{1}{f(k) - b}\right) \]
\[ -N\left(r, \frac{1}{f(k)}\right) + S(r, f) \]
\[ \leq \bar{N}\left(r, \frac{1}{f(k) - a}\right) + \bar{N}\left(r, \frac{1}{f(k) - b}\right) + S(r, f) \]

i.e., \( T(r, f) \leq \bar{N}\left(r, \frac{1}{f(k) - a}\right) + \bar{N}\left(r, \frac{1}{f(k) - b}\right) + N\left(r, \frac{1}{f}\right) + S(r, f). \) (6)

Now dividing both sides of (6) by \( r^{\rho_f(r)} \) and taking limit superior it follows from Lemma 1 that

\[ \limsup_{r \to \infty} \frac{T(r, f)}{r^{\rho_f(r)}} \]
\[ \leq \limsup_{r \to \infty} \frac{\bar{N}\left(r, \frac{1}{f(k) - a}\right)}{r^{\rho_f(r)}} + \limsup_{r \to \infty} \frac{\bar{N}\left(r, \frac{1}{f(k) - b}\right)}{r^{\rho_f(r)}} + \limsup_{r \to \infty} \frac{N\left(r, \frac{1}{f}\right)}{r^{\rho_f(r)}} \]

i.e., \( 1 \leq \{1 - R\Theta^{(k)}_{\rho_f}(a; f)\} + \{1 - R\Theta^{(k)}_{\rho_f}(b; f)\} + \{1 - \delta_{\rho_f}(0; f)\} \)

i.e., \( R\Theta^{(k)}_{\rho_f}(a; f) + R\Theta^{(k)}_{\rho_f}(b; f) + \delta_{\rho_f}(0; f) \leq 2. \)

Thus the theorem is established.

**Theorem 3.** Let \( f \) be a meromorphic function of finite order \( \rho_f \) and

\( T(r, f^{(k)}) \sim aT(r, f) \) i.e., \( T(r, f^{(k)}) = aT(r, f) + S(r, f) \) where \( a \geq 1. \) Then

\( k\Theta_{\rho_f}(\infty; f) + a \leq k + 1. \)
Proof. Since \( f \) is of finite order it follows that
\[
m(r, f^{(k)}) \leq m(r, f) + S(r, f)
\]
and therefore
\[
T(r, f^{(k)}) = N(r, f^{(k)}) + m(r, f^{(k)})
\]
i.e.,
\[
T(r, f^{(k)}) \leq N(r, f) + k\bar{N}(r, f) + m(r, f) + S(r, f)
\]
i.e.,
\[
T(r, f^{(k)}) \leq T(r, f) + k\bar{N}(r, f) + S(r, f).
\] (7)
But by hypothesis \( T(r, f^{(k)}) \sim aT(r, f) \) and we get from (7) that
\[
aT(r, f) \leq T(r, f) + k\bar{N}(r, f) + S(r, f)
\]
i.e.,
\[
(a - 1)T(r, f) \leq k\bar{N}(r, f) + S(r, f).
\] (8)
Now dividing both sides of (8) by \( r^{\rho}(r) \) and taking limit superior it follows that
\[
(a - 1)\limsup_{r \to \infty} \frac{T(r, f)}{r^{\rho}(r)} \leq k\limsup_{r \to \infty} \frac{\bar{N}(r, f)}{r^{\rho}(r)}
\]
i.e.,
\[
(a - 1) \leq k\{1 - \Theta_{\rho}(\infty; f)\}
\]
i.e.,
\[
k\Theta_{\rho}(\infty; f) + a \leq k + 1.
\]
This proves the theorem.

Theorem 4. Let \( a \neq 0, \infty \) be any complex number. Then for any meromorphic function \( f \) with finite order \( \rho \),
\[
\Theta_{\rho}(0; f) + \Theta_{\rho}(a; f) + R_{\rho}(k)(\infty; f) \leq 2.
\]
Proof. The following identity is well known \{cf. p. 43, [1]\}
\[
T(r, f) \leq \bar{N}(r, f) + \bar{N}\left(r, \frac{1}{f}\right) + \bar{N}\left(r, \frac{1}{f - a}\right) + S(r, f).
\] (9)
Now from (9) we have for all non-negative integer \( k \),
\[
T(r, f) \leq k\bar{N}(r, f) + N(r, f) + \bar{N}\left(r, \frac{1}{f}\right) + \bar{N}\left(r, \frac{1}{f - a}\right) + S(r, f)
\]
\[ T(r, f) \leq N(r, f^{(k)}) + \tilde{N} \left( r, \frac{1}{f} \right) + \tilde{N} \left( r, \frac{1}{f - a} \right) + S(r, f). \]  \hspace{1cm} (10)

Dividing both sides of (10) by \( r^{\rho_f(r)} \) and taking limit superior it follows in view of Lemma 1 that

\[ \limsup_{r \to \infty} \frac{T(r, f)}{r^{\rho_f(r)}} \leq \limsup_{r \to \infty} \frac{N(r, f^{(k)})}{r^{\rho_f(r)}} + \limsup_{r \to \infty} \frac{\tilde{N} \left( r, \frac{1}{f} \right)}{r^{\rho_f(r)}} + \limsup_{r \to \infty} \frac{\tilde{N} \left( r, \frac{1}{f - a} \right)}{r^{\rho_f(r)}} \]

i.e.,

\[ 1 \leq \left\{ 1 - R \delta^{(k)}_{\rho_f}(\infty; f) \right\} + \left\{ 1 - \Theta_{\rho_f}(0; f) \right\} + \left\{ 1 - \Theta_{\rho_f}(a; f) \right\} \]

i.e.,

\[ R \delta^{(k)}_{\rho_f}(\infty; f) + \Theta_{\rho_f}(0; f) + \Theta_{\rho_f}(a; f) \leq 2. \]

Thus the theorem is proved.

**Theorem 5.** If \( f \) be a meromorphic function of finite order and \( a, b, c \) be three distinct finite complex numbers with \( b \neq 0, c \neq 0 \), then

\[ \delta_{\rho_f}(a; f) + R \Theta^{(k)}_{\rho_f}(b; f) + R \Theta^{(k)}_{\rho_f}(c; f) \leq 2. \]

**Proof.** By Milloux’s theorem {p.55, [1]}, we get that

\[ m \left( r, \frac{1}{f - a} \right) \leq m \left( r, \frac{1}{f^{(k)}} \right) + S(r, f). \]  \hspace{1cm} (11)

So by Nevanlinna’s first fundamental theorem it follows from (11) that

\[ m \left( r, \frac{1}{f - a} \right) \leq T(r, f^{(k)}) - N \left( r, \frac{1}{f^{(k)}} \right) + S(r, f). \]  \hspace{1cm} (12)

Again in view of Nevanlinna’s second fundamental theorem and \( S(r, f^{(k)}) = S(r, f) \) we get that

\[ T(r, f^{(k)}) \leq \tilde{N} \left( r, \frac{1}{f^{(k)}} \right) + \tilde{N} \left( r, \frac{1}{f^{(k)} - b} \right) + \tilde{N} \left( r, \frac{1}{f^{(k)} - c} \right) + S(r, f). \]  \hspace{1cm} (13)

Combining (12) and (13) we obtain that

\[ m \left( r, \frac{1}{f - a} \right) \]

\[ \leq \tilde{N} \left( r, \frac{1}{f^{(k)}} \right) + \tilde{N} \left( r, \frac{1}{f^{(k)} - b} \right) + \tilde{N} \left( r, \frac{1}{f^{(k)} - c} \right) \\
- N \left( r, \frac{1}{f^{(k)}} \right) + S(r, f). \]
On relative defects of meromorphic functions

Since $N\left(\frac{1}{f(x)}\right) - N\left(\frac{1}{f(x)}\right) \leq 0$, it follows from above in view of $T\left(\frac{1}{f(x)}\right) = T\left(\frac{1}{f(x)}\right) + O(1)$, 

$$T(r, f) \leq N\left(\frac{1}{f(x)} - b\right) + N\left(\frac{1}{f(x)} - c\right) + N\left(\frac{1}{f(x)} - a\right) + S(r, f).$$

(14)

Dividing both sides of (14) by $r^{\rho_f(r)}$ and taking limit superior we get by Lemma 1 that

$$\limsup_{r \to \infty} \frac{T(r, f)}{r^{\rho_f(r)}} \leq \limsup_{r \to \infty} N\left(\frac{1}{f(x)} - a\right) + \limsup_{r \to \infty} \frac{N\left(\frac{1}{f(x)} - b\right)}{r^{\rho_f(r)}} + \limsup_{r \to \infty} \frac{N\left(\frac{1}{f(x)} - c\right)}{r^{\rho_f(r)}}$$

i.e., $1 \leq \left\{1 - \delta_{\rho_f(a; f)}\right\} + \left\{1 - R\Theta^{(k)}(b; f)\right\} + \left\{1 - R\Theta^{(k)}(c; f)\right\}$

i.e., $\delta_{\rho_f(a; f)} + R\Theta^{(k)}(b; f) + R\Theta^{(k)}(c; f) \leq 2$.

This proves the theorem.

**Theorem 6.** Let $f$ be a meromorphic function of finite order $\rho_f$. Also let almost all zeros of $f$ has multiplicity $\geq n$. Then for all non-negative integer $k$,

$$(k + 1)\delta_{\rho_f(0; f)} + nR\Theta^{(k)}(1; f) + n\Theta(\infty; f) \leq n + k + 1.$$

**Proof.** Let us consider the identity

$$\frac{1}{f} = \left(\frac{f^{(k)}}{f} - \frac{f^{(k)} - 1}{f^{(k+1)}} \cdot \frac{f^{(k+1)}}{f}\right).$$

Then by Milloux’s theorem {p.55, [1]}, we get

$$m\left(r, \frac{1}{f}\right) \leq m\left(r, \frac{f^{(k)} - 1}{f^{(k+1)}}\right) + S(r, f).$$

(15)

Using Nevanlinna’s first fundamental theorem and in view of {p.34, [1]} we obtain that

$$m\left(r, \frac{1}{f}\right) \leq T\left(r, \frac{f^{(k)} - 1}{f^{(k+1)}}\right) - N\left(r, \frac{f^{(k)} - 1}{f^{(k+1)}}\right) + S(r, f)$$
i.e., \( m\left(r, \frac{1}{f}\right) \leq T\left(r, \frac{f^{(k+1)}}{f^{(k)} - 1}\right) - N\left(r, \frac{f^{(k)}}{f^{(k+1)}} - 1\right) + S(r, f) \)

i.e., \( m\left(r, \frac{1}{f}\right) \leq N\left(r, \frac{f^{(k+1)}}{f^{(k)} - 1}\right) - N\left(r, \frac{f^{(k)}}{f^{(k+1)}} - 1\right) + S(r, f) \)

i.e., \[
m\left(r, \frac{1}{f}\right) \\
\leq N\left(r, \frac{f^{(k+1)}}{f^{(k)} - 1}\right) - N\left(r, \frac{f^{(k)}}{f^{(k+1)}} - 1\right) + S(r, f) \]

i.e., \( T(r, f) \)
\[
\leq N\left(r, \frac{f^{(k+1)}}{f^{(k)} - 1}\right) - N\left(r, \frac{f^{(k)}}{f^{(k+1)}} - 1\right) + S(r, f). \tag{16} \]

Let \( N_a\left(r, \frac{1}{f^{(k+1)}}\right) \) denotes the zeros of \( f^{(k+1)} \) which are not the zeros of \( (f^{(k)} - 1) \). So from (16) we get that

\[
T(r, f) \\
\leq N\left(r, \frac{1}{f}\right) + \left\{ N\left(r, \frac{f^{(k+1)}}{f^{(k)} - 1}\right) - N\left(r, \frac{f^{(k)}}{f^{(k+1)}} - 1\right) \right\} \\
\quad + \left\{ \bar{N}\left(r, \frac{1}{f^{(k)}}\right) - N_a\left(r, \frac{1}{f^{(k+1)}}\right) \right\} + S(r, f) \]

i.e., \( T(r, f) \)
\[
\leq N\left(r, \frac{1}{f}\right) + \bar{N}\left(r, f^{(k)}\right) + \bar{N}\left(r, \frac{1}{f^{(k)} - 1}\right) \\
\quad - N_a\left(r, \frac{1}{f^{(k+1)}}\right) + S(r, f). \tag{17} \]

Let \( N_0\left(r, \frac{1}{f}\right) \) denotes all zeros of \( f \) taken with proper multiplicities if the multiplicity is \( \leq k + 1 \) and each zero of multiplicity \( \geq k + 2 \) being counted \( (k + 1) \) times only. Also
\[
\bar{N}\left(r, f^{(k)}\right) = N\left(r, f\right). \]
Therefore from (17) we obtain that
\[
T(r, f) \leq N\left(r, \frac{1}{f}\right) + \tilde{N}\left(r, \frac{1}{f^{(k+1)}}\right) + \tilde{N}\left(r, \frac{1}{f^{(k)} - 1}\right) + S(r, f)
\]
i.e.,
\[
T(r, f) \leq N_0\left(r, \frac{1}{f}\right) + \tilde{N}\left(r, \frac{1}{f^{(k)}}\right) + \tilde{N}(r, f) + S(r, f). \tag{18}
\]
Again we may write
\[
nN_0\left(r, \frac{1}{f}\right) \leq (k + 1)N\left(r, \frac{1}{f}\right). \tag{19}
\]
Since on the left hand side of (19) each zero is counted at least \(n(k + 1)\) times as it is given that each zero of \(f\) is of order \(\geq n\). So from (18) and (19) it follows that
\[
T(r, f) \leq \left(\frac{k + 1}{n}\right)N\left(r, \frac{1}{f}\right) + \tilde{N}\left(r, \frac{1}{f^{(k)}}\right) + \tilde{N}(r, f) + S(r, f). \tag{20}
\]
Dividing both sides of (20) by \(r^{\rho_f(r)}\) and taking limit superior we get in view of Lemma 1 that
\[
\limsup_{r \to \infty} \frac{T(r, f)}{r^{\rho_f(r)}} \leq \left(\frac{k + 1}{n}\right) \limsup_{r \to \infty} \frac{N\left(r, \frac{1}{f}\right)}{r^{\rho_f(r)}} + \limsup_{r \to \infty} \frac{\tilde{N}\left(r, \frac{1}{f^{(k)}}\right)}{r^{\rho_f(r)}} + \limsup_{r \to \infty} \frac{\tilde{N}(r, f)}{r^{\rho_f(r)}} + \limsup_{r \to \infty} \frac{S(r, f)}{r^{\rho_f(r)}}
\]
i.e.,
\[
1 \leq \left(\frac{k + 1}{n}\right) \left\{1 - \delta_{\rho_f}(0; f)\right\} + \left\{1 - \frac{R^{(k)}}{\Theta_{\rho_f}}(1; f)\right\} + \left\{1 - \Theta_{\rho_f}(\infty; f)\right\}
\]
i.e.,
\[
(k + 1)\delta_{\rho_f}(0; f) + n R^{(k)}(1; f) + n \Theta_{\rho_f}(\infty; f) \leq n + k + 1.
\]
Thus the theorem follows.

We now give a lower bound for relative proximate defect of any complex number \(a\) with respect to the derivative \(f^{(k)}\) in terms of \(k\) and \(\Theta_{\rho_f}(\infty; f)\) where \(k = 1, 2, 3, \ldots\).
Theorem 7. If $f$ be a transcendental meromorphic function of finite order, then for any integer $k \geq 1$,

(i) $R^{k}_{\rho_f}(a; f) + k \geq k \Theta_{\rho_f}(\infty; f)$ for every complex number $a \neq \infty$ and

(ii) $R^{k}_{\rho_f}(a; f) + k \geq k \Theta_{\rho_f}(\infty; f) + \delta_{\rho_f}(\infty; f)$, if $a = \infty$.

Proof. (i) By Nevanlinna’s first fundamental theorem we get that

$$m\left(r, \frac{1}{f}\right) + N\left(r, \frac{1}{f}\right) = m(r, f) + N(r, f) + S(r, f)$$

i.e., $N\left(r, \frac{1}{f}\right) - N(r, f) = m(r, f) - m\left(r, \frac{1}{f}\right) + S(r, f)$

i.e.,

$$N\left(r, \frac{1}{f}\right) - N(r, f)$$

$$= \frac{1}{2\pi} \int_{0}^{2\pi} \log |f| \, d\theta - \frac{1}{2\pi} \int_{0}^{2\pi} \log \left| \frac{1}{f} \right| \, d\theta + S(r, f)$$

$$= \frac{1}{2\pi} \int_{0}^{2\pi} \log |f| \, d\theta + S(r, f). \quad (21)$$

Similarly using (21) we may write

$$N\left(r, f^{(k)} - a\right) - N\left(r, \frac{1}{f^{(k)} - a}\right) = \frac{1}{2\pi} \int_{0}^{2\pi} \log \left| \frac{1}{f^{(k)} - a} \right| \, d\theta + S(r, f) \quad (22)$$

and

$$N\left(r, \frac{f^{(k)} - a}{f}\right) - N\left(r, \frac{f}{f^{(k)} - a}\right) = \frac{1}{2\pi} \int_{0}^{2\pi} \log \left| \frac{f}{f^{(k)} - a} \right| \, d\theta + S(r, f). \quad (23)$$

Combining (21), (22) and (23) it follows in view of Nevanlinna’s first fundamental theorem that

$$\left\{ N\left(r, \frac{1}{f}\right) - N(r, f) \right\} + \left\{ N\left(r, f^{(k)} - a\right) - N\left(r, \frac{1}{f^{(k)} - a}\right) \right\}$$

$$= \frac{1}{2\pi} \int_{0}^{2\pi} \log \left| \frac{f}{f^{(k)} - a} \right| \, d\theta + S(r, f)$$

$$= N\left(r, \frac{f^{(k)} - a}{f}\right) - N\left(r, \frac{f}{f^{(k)} - a}\right)$$
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\[ N\left( r, \frac{1}{f^{(k)} - a} \right) \]

\[ = N\left( r, \frac{1}{f} \right) - N(r, f) + N\left( r, f^{(k)} - a \right) - N\left( r, \frac{f^{(k)} - a}{f} \right) \]

\[ + N\left( r, \frac{f}{f^{(k)} - a} \right) + S(r, f) \]

\[ = N\left( r, \frac{1}{f} \right) - N(r, f) + N\left( r, f \right) + kN(r, f) \]

\[ - \left\{ T\left( r, \frac{f^{(k)} - a}{f} \right) - m\left( r, \frac{f^{(k)} - a}{f} \right) \right\} \]

\[ + \left\{ T\left( r, \frac{f}{f^{(k)} - a} \right) - m\left( r, \frac{f}{f^{(k)} - a} \right) \right\} + S(r, f) \]

\[ = N\left( r, \frac{1}{f} \right) + kN(r, f) + m\left( r, \frac{f^{(k)} - a}{f} \right) \]

\[ - m\left( r, \frac{f}{f^{(k)} - a} \right) + S(r, f). \quad (24) \]

We immediately obtain from (24) that

\[ N\left( r, \frac{1}{f^{(k)} - a} \right) \leq N \left( r, \frac{1}{f} \right) + kN(r, f) + m \left( r, \frac{1}{f} \right) + S(r, f) \]

i.e.,

\[ N\left( r, \frac{1}{f^{(k)} - a} \right) \leq T\left( r, \frac{1}{f} \right) + kN(r, f) + S(r, f). \quad (25) \]

Dividing both sides of (25) by \( r^{\rho_f(r)} \) and taking limit superior we obtain that

\[ \limsup_{r \to \infty} \frac{N\left( r, \frac{1}{f^{(k)} - a} \right)}{r^{\rho_f(r)}} \leq \limsup_{r \to \infty} \frac{T\left( r, \frac{1}{f} \right)}{r^{\rho_f(r)}} + \limsup_{r \to \infty} \frac{N(r, f)}{r^{\rho_f(r)}} \]

i.e.,

\[ 1 - R^{\delta_{f^{(k)}}}(a; f) \leq 1 + k \left\{ 1 - \Theta_{f^k}(\infty; f) \right\} \]

i.e.,

\[ R^{\delta_{f^{(k)}}}(a; f) + k \geq 1 - \Theta_{f^k}(\infty; f). \]

This proves the first part of Theorem 7.

(ii) Since \( N(r, f^{(k)}) = N(r, f) + kN(r, f) \), dividing both sides by \( r^{\rho_f(r)} \) and taking limit superior it follows that

\[ \limsup_{r \to \infty} \frac{N\left( r, f^{(k)} \right)}{r^{\rho_f(r)}} \leq \limsup_{r \to \infty} \frac{N(r, f)}{r^{\rho_f(r)}} + \limsup_{r \to \infty} \frac{N(r, f)}{r^{\rho_f(r)}} \]
i.e., \(1 - R_{\rho_f}^{(k)}(\infty; f) \leq \{1 - \delta_{\rho_f}(\infty; f)\} + k \{1 - \Theta_{\rho_f}(\infty; f)\}\)

i.e., \(R_{\rho_f}^{(k)}(\infty; f) + k \geq k \Theta_{\rho_f}(\infty; f) + \delta_{\rho_f}(\infty; f)\).

Thus the second part of Theorem 7 follows.

References


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