Composition of Entire Functions
and their Maximum Terms

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Abstract

In the paper we compare the maximum term of composition of two entire functions with their corresponding left and right factors on the basis of a slowly changing function.

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1 Introduction, Definitions and Notations.

Let $f$ be an entire function defined in the open complex plane $\mathbb{C}$. The maximum term $\mu(r,f)$ of $f = \sum_{n=0}^{\infty} a_n z^n$ on $|z| = r$ is defined by $\mu(r,f) = \max_{n \geq 0} (|a_n| r^n)$. To start our paper we just recall the following definitions.

Definition 1. The order $\rho_f$ and lower order $\lambda_f$ of an entire function $f$ is defined as follows:

$$\rho_f = \limsup_{r \to \infty} \frac{\log^{[2]} M(r,f)}{\log r} \quad \text{and} \quad \lambda_f = \liminf_{r \to \infty} \frac{\log^{[2]} M(r,f)}{\log r}$$
where
\[ \log^k x = \log \left( \log^{k-1} x \right) \quad \text{for } k = 1, 2, 3, \ldots \text{ and} \]
\[ \log^0 x = x. \]

**Definition 2.** The hyper order \( \bar{\rho}_f \) and hyper lower order \( \bar{\lambda}_f \) of \( f \) is defined by
\[ \bar{\rho}_f = \limsup_{r \to \infty} \frac{\log^3 M(r, f)}{\log r} \quad \text{and} \quad \bar{\lambda}_f = \liminf_{r \to \infty} \frac{\log^3 M(r, f)}{\log r}. \]

Since for \( 0 \leq r < R \),
\[ \mu(r, f) \leq M(r, f) \leq \frac{R}{R-r} \mu(R, f) \]
it is easy to see that
\[ \rho_f = \limsup_{r \to \infty} \frac{\log^2 \mu(r, f)}{\log r}, \quad \lambda_f = \liminf_{r \to \infty} \frac{\log^2 \mu(r, f)}{\log r} \]
and
\[ \bar{\rho}_f = \limsup_{r \to \infty} \frac{\log^3 \mu(r, f)}{\log r}, \quad \bar{\lambda}_f = \liminf_{r \to \infty} \frac{\log^3 \mu(r, f)}{\log r}. \]

Singh [3] proved some theorems on the comparative growth properties of \( \log^3 \mu(r, f \circ g) \) with respect to \( \log^2 \mu(r^A, f) \) and \( \log^2 \mu(r^A, g) \) for every positive constant \( A \).

Somasundaram and Thamizharasi [2] introduced the notions of \( L \)-order, \( L \)-lower order and \( L \)-type for entire functions where \( L = L(r) \) is a positive continuous function increasing slowly i.e., \( L(ar) \sim L(r) \) as \( r \to \infty \) for every constant \( a \). Their definitions are as follows:

**Definition 3.** [2] The \( L \)-order \( \rho^L_f \) and \( L \)-lower order \( \lambda^L_f \) of an entire function \( f \) are defined as follows:
\[ \rho^L_f = \limsup_{r \to \infty} \frac{\log^2 M(r, f)}{\log [rL(r)]} \quad \text{and} \quad \lambda^L_f = \liminf_{r \to \infty} \frac{\log^2 M(r, f)}{\log [rL(r)]}. \]

When \( f \) is meromorphic, then
\[ \rho^L_f = \limsup_{r \to \infty} \frac{\log T(r, f)}{\log [rL(r)]} \quad \text{and} \quad \lambda^L_f = \liminf_{r \to \infty} \frac{\log T(r, f)}{\log [rL(r)]}. \]
Definition 4. [2] The L-type $\sigma_f^L$ of an entire function $f$ with L-order $\rho_f^L$ is defined as

$$\sigma_f^L = \limsup_{r \to \infty} \frac{\log M(r, f)}{[rL(r)]^{\rho_f^L}}, 0 < \rho_f^L < \infty.$$ 

For meromorphic $f$, the L-type $\sigma_f^L$ becomes

$$\sigma_f^L = \limsup_{r \to \infty} \frac{T(r, f)}{[rL(r)]^{\rho_f^L}}, 0 < \rho_f^L < \infty.$$ 

With the help of the notion of maximum terms of entire functions, Definition 3 and Definition 4 can be alternatively stated as follows:

Definition 5. The L-order $\rho_f^L$ and L-lower order $\lambda_f^L$ of an entire function $f$ are defined as follows:

$$\rho_f^L = \limsup_{r \to \infty} \frac{\log^2 \mu(r, f)}{\log [rL(r)]} \quad \text{and} \quad \lambda_f^L = \liminf_{r \to \infty} \frac{\log^2 \mu(r, f)}{\log [rL(r)]}.$$ 

When $f$ is meromorphic then $\rho_f^L$ and $\lambda_f^L$ cannot be defined in the above way.

Definition 6. The L-type $\sigma_f^L$ of an entire function $f$ with L-order $\rho_f^L$ is defined as

$$\sigma_f^L = \limsup_{r \to \infty} \frac{\log \mu(r, f)}{[rL(r)]^{\rho_f^L}}, 0 < \rho_f^L < \infty.$$ 

For meromorphic $f$, the L-type $\sigma_f^L$ cannot be defined in the above way.

The more generalised concept of L-order and L-type of entire and meromorphic functions are $L^*$-order and $L^*$-type. Their definitions are as follows:

Definition 7. The $L^*$-order, $L^*$-lower order and $L^*$-type of a meromorphic function are defined by

$$\rho_f^{L^*} = \limsup_{r \to \infty} \frac{\log T(r, f)}{\log [reL(r)]}, \quad \lambda_f^{L^*} = \liminf_{r \to \infty} \frac{\log T(r, f)}{\log [reL(r)]}$$

and

$$\sigma_f^{L^*} = \limsup_{r \to \infty} \frac{T(r, f)}{[reL(r)]^{\rho_f^{L^*}}}, 0 < \rho_f^{L^*} < \infty.$$ 

When $f$ is entire, one can easily verify that

$$\rho_f^{L^*} = \limsup_{r \to \infty} \frac{\log^2 M(r, f)}{\log [reL(r)]^2}, \quad \lambda_f^{L^*} = \liminf_{r \to \infty} \frac{\log^2 M(r, f)}{\log [reL(r)]^2}.$$
and

\[ \sigma_f^* = \limsup_{r \to \infty} \frac{\log M(r, f)}{[re^{L(r)}]^{\rho_f^*}}, 0 < \rho_f^* < \infty. \]

In view of the notion of maximum term of entire functions one can restate Definition 7 in the following way.

**Definition 8.** The \( L^* \)-order, \( L^* \)-lower order and \( L^* \)-type of an entire function \( f \) are defined as

\[ \rho_f^{L^*} = \limsup_{r \to \infty} \frac{\log \mu(r, f)}{\log [re^{L(r)}]}, \lambda_f^{L^*} = \liminf_{r \to \infty} \frac{\log \mu(r, f)}{\log [re^{L(r)}]} \]

and

\[ \sigma_f^{L^*} = \limsup_{r \to \infty} \frac{\log \mu(r, f)}{[re^{L(r)}]^{\rho_f^*}}, 0 < \rho_f^* < \infty. \]

Definition 8 fails if \( f \) is meromorphic.

Singh [3] proved some theorems on the comparative growth properties of \( \log^{[2]} \mu(r, f \circ g) \) with respect to \( \log^{[2]} \mu(r^A, f) \) for every positive constant \( A \). In the paper we further investigate the comparative growths of maximum term of two entire functions with their corresponding left and right factors on the basis of \( L \)-order and \( L \)-lower order. We do not explain the standard notations and definitions on the theory of entire and meromorphic functions because those are available in [4] and [1].

## 2 Theorems.

In this section we present the main results of the paper.

**Theorem 1.** Let \( f \) and \( g \) be two entire functions such that \( 0 < \lambda_f^{L^*} \leq \rho_f^L < \infty \) and \( 0 < \rho_g^L < \infty \). Then for any integer \( A \),

\[ (i) \liminf_{r \to \infty} \frac{\log^{[2]} \mu(r, f \circ g)}{\log^{[2]} \mu(r^A, g)} \leq \frac{\rho_{f \circ g}^{L^*}}{A \rho_g^L} \leq \limsup_{r \to \infty} \frac{\log^{[2]} \mu(r, f \circ g)}{\log^{[2]} \mu(r^A, g)}. \]

Further if \( \lambda_g^L > 0 \) then

\[ (ii) \frac{\lambda_{f \circ g}^L}{A \rho_g^L} \leq \liminf_{r \to \infty} \frac{\log^{[2]} \mu(r, f \circ g)}{\log^{[2]} \mu(r^A, g)} \leq \frac{\lambda_{f \circ g}^L}{A \lambda_g^L} \leq \limsup_{r \to \infty} \frac{\log^{[2]} \mu(r, f \circ g)}{\log^{[2]} \mu(r^A, g)} \leq \frac{\rho_{f \circ g}^{L^*}}{A \lambda_g^L}. \]
and

\[
(iii) \liminf_{r \to \infty} \frac{\log^2 \mu (r, f \circ g)}{\log^2 \mu (r^A, g)} \leq \min \left\{ \frac{\lambda_{f \circ g}^L}{A \lambda_g^L}, \frac{\rho_{f \circ g}^L}{A \rho_g^L} \right\} \leq \max \left\{ \frac{\lambda_{f \circ g}^L}{A \lambda_g^L}, \frac{\rho_{f \circ g}^L}{A \rho_g^L} \right\} \leq \limsup_{r \to \infty} \frac{\log^2 \mu (r, f \circ g)}{\log^2 \mu (r^A, g)}.
\]

**Proof.** (i) From the definition of $L$-order we have for arbitrary positive $\varepsilon$ and for all large values of $r$,

\[
\log^2 \mu (r, f \circ g) \leq \left( \rho_{f \circ g}^L + \varepsilon \right) \log \left[ r L (r) \right]
\]

and for a sequence of values of $r$ tending to infinity,

\[
\log^2 \mu (r^A, g) \geq A \left( \rho_g^L - \varepsilon \right) \log \left[ r L (r) \right].
\]

Now from (1) and (2) it follows for a sequence of values of $r$ tending to infinity,

\[
\frac{\log^2 \mu (r, f \circ g)}{\log^2 \mu (r^A, g)} \leq \frac{\rho_{f \circ g}^L + \varepsilon}{A \left( \rho_g^L - \varepsilon \right)}.
\]

As $\varepsilon (> 0)$ is arbitrary we obtain that

\[
\liminf_{r \to \infty} \frac{\log^2 \mu (r, f \circ g)}{\log^2 \mu (r^A, g)} \leq \frac{\rho_{f \circ g}^L}{A \rho_g^L}.
\]

(3)

Again for a sequence of values of $r$ tending to infinity,

\[
\log^2 \mu (r, f \circ g) \geq \left( \rho_{f \circ g}^L - \varepsilon \right) \log \left[ r L (r) \right].
\]

(4)

Also for all sufficiently large values of $r$,

\[
\log^2 \mu (r^A, g) \leq A \left( \rho_g^L + \varepsilon \right) \log \left[ r L (r) \right].
\]

(5)

So combining (4) and (5) we get for a sequence of values of $r$ tending to infinity,

\[
\frac{\log^2 \mu (r, f \circ g)}{\log^2 \mu (r^A, g)} \geq \frac{\rho_{f \circ g}^L - \varepsilon}{A \left( \rho_g^L + \varepsilon \right)}.
\]

Since $\varepsilon (> 0)$ is arbitrary it follows that

\[
\limsup_{r \to \infty} \frac{\log^2 \mu (r, f \circ g)}{\log^2 \mu (r^A, g)} \geq \frac{\rho_{f \circ g}^L}{A \rho_g^L}.
\]

(6)

Thus the first part of Theorem 1 follows from (3) and (6).
(ii) From the definition of $L$-lower order we have for arbitrary positive $\varepsilon$ and for all large values of $r$,

$$\log^{[2]} \mu (r, f \circ g) \geq (\lambda^L_{f \circ g} - \varepsilon) \log \left[ r L (r) \right].$$

(7)

Now from (5) and (7) it follows for all large values of $r$,

$$\frac{\log^{[2]} \mu (r, f \circ g)}{\log^{[2]} \mu (r^A, g)} \geq \frac{\lambda^L_{f \circ g} - \varepsilon}{A (\rho^L_{g} + \varepsilon)}.$$

As $\varepsilon (> 0)$ is arbitrary we obtain that

$$\liminf_{r \to \infty} \frac{\log^{[2]} \mu (r, f \circ g)}{\log^{[2]} \mu (r^A, g)} \geq \frac{\lambda^L_{f \circ g}}{A \rho^L_{g}}. \tag{8}$$

Again for a sequence of values of $r$ tending to infinity,

$$\log^{[2]} \mu (r, f \circ g) \leq (\lambda^L_{f \circ g} + \varepsilon) \log \left[ r L (r) \right]$$

(9)

and for all large values of $r$,

$$\log^{[2]} \mu (r^A, g) \geq A (\lambda^L_{g} - \varepsilon) \log \left[ r L (r) \right]. \tag{10}$$

So combining (9) and (10) we get for a sequence of values of $r$ tending to infinity,

$$\frac{\log^{[2]} \mu (r, f \circ g)}{\log^{[2]} \mu (r^A, g)} \leq \frac{\lambda^L_{f \circ g} + \varepsilon}{A (\lambda^L_{g} - \varepsilon)}.$$

Since $\varepsilon (> 0)$ is arbitrary it follows that

$$\liminf_{r \to \infty} \frac{\log^{[2]} \mu (r, f \circ g)}{\log^{[2]} \mu (r^A, g)} \leq \frac{\lambda^L_{f \circ g}}{A \lambda^L_{g}}. \tag{11}$$

Also for a sequence of values of $r$ tending to infinity,

$$\log^{[2]} \mu (r^A, g) \leq A (\lambda^L_{g} + \varepsilon) \log \left[ r L (r) \right]. \tag{12}$$

Now from (7) and (12) we obtain for a sequence of values of $r$ tending to infinity,

$$\frac{\log^{[2]} \mu (r, f \circ g)}{\log^{[2]} \mu (r^A, g)} \geq \frac{\lambda^L_{f \circ g} - \varepsilon}{A (\lambda^L_{g} + \varepsilon)}.$$

As $\varepsilon (> 0)$ is arbitrary we get that

$$\limsup_{r \to \infty} \frac{\log^{[2]} \mu (r, f \circ g)}{\log^{[2]} \mu (r^A, g)} \geq \frac{\lambda^L_{f \circ g}}{A \lambda^L_{g}}.$$
Again from (1) and (10) it follows for all large values of \( r \),
\[
\frac{\log^2 \mu (r, f \circ g)}{\log^2 \mu (r^A, g)} \leq \frac{\rho_{fog}^L}{A (\lambda_g^L - \varepsilon)}.
\]

As \( \varepsilon (> 0) \) is arbitrary we obtain that
\[
\limsup_{r \to \infty} \frac{\log^2 \mu (r, f \circ g)}{\log^2 \mu (r^A, g)} \leq \frac{\rho_{fog}^L}{A \lambda_g^L}.
\]

Thus the second part of Theorem 1 follows from (8), (11), (13) and (14).

(iii) Combining (i) and (ii) of Theorem 1, (iii) follows.

In the line of Theorem 1 we may prove the following theorem.

**Theorem 2.** Let \( f \) and \( g \) be two entire functions such that \( 0 < \bar{\lambda}_{fog}^L \leq \bar{\rho}_{fog}^L < \infty \) and \( 0 < \bar{\rho}_g^L < \infty \). Then for any positive number \( A \),
\[
(i) \liminf_{r \to \infty} \frac{\log^3 \mu (r, f \circ g)}{\log^3 \mu (r^A, g)} \leq \frac{\bar{\rho}_{fog}^L}{A \bar{\rho}_g^L} \leq \limsup_{r \to \infty} \frac{\log^3 \mu (r, f \circ g)}{\log^3 \mu (r^A, g)} \leq \frac{\bar{\lambda}_{fog}^L}{A \bar{\lambda}_g^L}
\]

Further if \( \bar{\lambda}_g^L > 0 \) then
\[
(ii) \frac{\bar{\lambda}_{fog}^L}{A \bar{\rho}_g^L} \leq \liminf_{r \to \infty} \frac{\log^3 \mu (r, f \circ g)}{\log^3 \mu (r^A, g)} \leq \frac{\bar{\lambda}_{fog}^L}{A \bar{\lambda}_g^L} \leq \limsup_{r \to \infty} \frac{\log^3 \mu (r, f \circ g)}{\log^3 \mu (r^A, g)} \leq \frac{\bar{\rho}_{fog}^L}{A \bar{\rho}_g^L}
\]

and
\[
(iii) \liminf_{r \to \infty} \frac{\log^3 \mu (r, f \circ g)}{\log^3 \mu (r^A, g)} \leq \min \left\{ \frac{\bar{\lambda}_{fog}^L}{A \bar{\lambda}_g^L}, \frac{\bar{\rho}_{fog}^L}{A \bar{\rho}_g^L} \right\} \leq \max \left\{ \frac{\bar{\lambda}_{fog}^L}{A \bar{\lambda}_g^L}, \frac{\bar{\rho}_{fog}^L}{A \bar{\rho}_g^L} \right\} \leq \limsup_{r \to \infty} \frac{\log^3 \mu (r, f \circ g)}{\log^3 \mu (r^A, g)}.
\]

**Theorem 3.** If \( f \) and \( g \) be two entire functions with \( \rho_g^L < \infty \) and \( \rho_{fog}^L = \infty \), then for every positive number \( A \),
\[
\limsup_{r \to \infty} \frac{\log^2 \mu (r, f \circ g)}{\log^2 \mu (r^A, g)} = \infty.
\]

**Proof.** Let us assume that the conclusion of Theorem 3 does not hold. Then there exists a constant \( B > 0 \) such that for all sufficiently large values of \( r \),
\[
\log^2 \mu (r, f \circ g) \leq B \log^2 \mu (r^A, g).
\]
Again from the definition of \( \rho_g \) it follows that
\[
\log^{[2]} \mu (r^A, g) \leq (\rho_g + \varepsilon) A \log [r L (r)] .
\] (16)
holds for all large values of \( r \).
So from (15) and (16) we obtain for all sufficiently large values of \( r \),
\[
\log^{[2]} \mu (r, f \circ g) \leq (\rho_g + \varepsilon) AB \log [r L (r)] .
\] (17)
From (17) it follows that \( \rho_{f \circ g}^L < \infty \).
So we arrive at a contradiction.
This proves the theorem.

**Remark 1.** If we take \( \rho_f^L < \infty \) instead of \( \rho_g^L < \infty \) in Theorem 3 and the other conditions remain the same then the theorem remains valid with \( g \) replaced by \( f \) in the denominator.

In the following theorems we establish the comparative growths of maximum term of two entire functions with their corresponding left and right factors on the basis of \( L^*\)-order and \( L^*\)-lower order.

**Theorem 4.** Let \( f \) and \( g \) be two entire functions such that \( 0 < \lambda_g^{L^*} \leq \rho_g^{L^*} < \infty \) and \( 0 < \rho_g^{L^*} < \infty \). Then for any positive integer \( A \)
\[
(i) \liminf_{r \to \infty} \frac{\log^{[2]} \mu (r, f \circ g)}{\log^{[2]} \mu (r^A, g)} \leq \frac{\rho_{f \circ g}^{L^*}}{A \rho_g^{L^*}} \leq \limsup_{r \to \infty} \frac{\log^{[2]} \mu (r, f \circ g)}{\log^{[2]} \mu (r^A, g)} .
\]
Further if \( \lambda_g^{L^*} > 0 \) then
\[
(ii) \frac{\lambda_{f \circ g}^{L^*}}{A \rho_g^{L^*}} \leq \liminf_{r \to \infty} \frac{\log^{[2]} \mu (r, f \circ g)}{\log^{[2]} \mu (r^A, g)} \leq \frac{\lambda_{f \circ g}^{L^*}}{A \rho_g^{L^*}} \leq \limsup_{r \to \infty} \frac{\log^{[2]} \mu (r, f \circ g)}{\log^{[2]} \mu (r^A, g)} \leq \frac{\rho_{f \circ g}^{L^*}}{A \lambda_g^{L^*}}
\]
and
\[
(iii) \liminf_{r \to \infty} \frac{\log^{[2]} \mu (r, f \circ g)}{\log^{[2]} \mu (r^A, g)} \leq \min \left\{ \frac{\lambda_{f \circ g}^{L^*}}{A \lambda_g^{L^*}}, \frac{\rho_{f \circ g}^{L^*}}{A \rho_g^{L^*}} \right\} \leq \max \left\{ \frac{\lambda_{f \circ g}^{L^*}}{A \lambda_g^{L^*}}, \frac{\rho_{f \circ g}^{L^*}}{A \rho_g^{L^*}} \right\} \leq \limsup_{r \to \infty} \frac{\log^{[2]} \mu (r, f \circ g)}{\log^{[2]} \mu (r^A, g)} .
\]

**Proof.** (i) From the definition of \( L^*\)-order we have for arbitrary positive \( \varepsilon \) and for all large values of \( r \),
\[
\log^{[2]} \mu (r, f \circ g) \leq (\rho_{f \circ g}^{L^*} + \varepsilon) \log [r e^{L(r)}] \] (18)
and for a sequence of values of $r$ tending to infinity,
\[
\log^{[2]} \mu (r^A, g) \geq A \left( \rho^L_g - \varepsilon \right) \log \left[ r e^{L(r)} \right]. \tag{19}
\]

Now from (18) and (19) it follows for a sequence of values of $r$ tending to infinity,
\[
\frac{\log^{[2]} \mu (r, f \circ g)}{\log^{[2]} \mu (r^A, g)} \leq \frac{\rho^L_{f \circ g} + \varepsilon}{A \left( \rho^L_g - \varepsilon \right)}.
\]

As $\varepsilon (> 0)$ is arbitrary we obtain that
\[
\liminf_{r \to \infty} \frac{\log^{[2]} \mu (r, f \circ g)}{\log^{[2]} \mu (r^A, g)} \leq \frac{\rho^L_{f \circ g}}{A \rho^L_g}.
\tag{20}
\]

Again for a sequence of values of $r$ tending to infinity,
\[
\log^{[2]} \mu (r^A, g) \leq \rho^L_{f \circ g} + \varepsilon. \tag{22}
\]

So combining (21) and (22) we get for a sequence of values of $r$ tending to infinity,
\[
\frac{\log^{[2]} \mu (r, f \circ g)}{\log^{[2]} \mu (r^A, g)} \geq \frac{\rho^L_{f \circ g} - \varepsilon}{A \left( \rho^L_g + \varepsilon \right)}.
\]

Since $\varepsilon (> 0)$ is arbitrary it follows that
\[
\limsup_{r \to \infty} \frac{\log^{[2]} \mu (r, f \circ g)}{\log^{[2]} \mu (r^A, g)} \geq \frac{\rho^L_{f \circ g}}{A \rho^L_g}.
\tag{23}
\]

Thus the first part of Theorem 4 follows from (20) and (23).

(ii) From the definition of $L^*$-lower order we have for arbitrary positive $\varepsilon$ and for all large values of $r$,
\[
\log^{[2]} \mu (r, f \circ g) \geq \left( \lambda^L_{f \circ g} - \varepsilon \right) \log \left[ r e^{L(r)} \right]. \tag{24}
\]

Now from (22) and (24) it follows for all large values of $r$,
\[
\frac{\log^{[2]} \mu (r, f \circ g)}{\log^{[2]} \mu (r^A, g)} \geq \frac{\lambda^L_{f \circ g} - \varepsilon}{A \left( \rho^L_g + \varepsilon \right)}.
\]
As $\varepsilon (> 0)$ is arbitrary we obtain that

$$\lim \inf_{r \to \infty} \frac{\log^2 \mu (r, f \circ g)}{\log^2 \mu (r^A, g)} \geq \frac{\lambda^*_f g}{A \rho^*_g}.$$  \hfill (25)

Again for a sequence of values of $r$ tending to infinity,

$$\log^2 \mu (r, f \circ g) \leq (\lambda^*_f g + \varepsilon) \log [re^{L(r)}]$$  \hfill (26)

and for all large values of $r$,

$$\log^2 \mu (r^A, g) \geq A (\lambda^*_g - \varepsilon) \log [re^{L(r)}].$$  \hfill (27)

So combining (26) and (27) we get for a sequence of values of $r$ tending to infinity,

$$\frac{\log^2 \mu (r, f \circ g)}{\log^2 \mu (r^A, g)} \leq \frac{\lambda^*_f g + \varepsilon}{A (\lambda^*_g - \varepsilon)}.$$  \hfill (28)

Since $\varepsilon (> 0)$ is arbitrary it follows that

$$\lim \inf_{r \to \infty} \frac{\log^2 \mu (r, f \circ g)}{\log^2 \mu (r^A, g)} \leq \frac{\lambda^*_f g}{A \lambda^*_g}.$$  \hfill (29)

Also for a sequence of values of $r$ tending to infinity,

$$\log^2 \mu (r^A, g) \leq A (\lambda^*_g + \varepsilon) \log [re^{L(r)}].$$  \hfill (29)

Now from (24) and (29) we obtain for a sequence of values of $r$ tending to infinity,

$$\frac{\log^2 \mu (r, f \circ g)}{\log^2 \mu (r^A, g)} \geq \frac{\lambda^*_f g - \varepsilon}{A (\lambda^*_g + \varepsilon)}.$$  \hfill (30)

As $\varepsilon (> 0)$ is arbitrary we get that

$$\lim \sup_{r \to \infty} \frac{\log^2 \mu (r, f \circ g)}{\log^2 \mu (r^A, g)} \geq \frac{\lambda^*_f g}{A \lambda^*_g}.$$  \hfill (31)

Again from (18) and (27) it follows for all large values of $r$,

$$\frac{\log^2 \mu (r, f \circ g)}{\log^2 \mu (r^A, g)} \leq \frac{\rho^*_f g + \varepsilon}{A (\lambda^*_g - \varepsilon)}.$$  \hfill (32)

As $\varepsilon (> 0)$ is arbitrary we obtain that

$$\lim \sup_{r \to \infty} \frac{\log^2 \mu (r, f \circ g)}{\log^2 \mu (r^A, g)} \leq \frac{\rho^*_f g}{A \lambda^*_g}.$$  \hfill (31)

Thus the second part of Theorem 4 follows from (25), (28), (30) and (31).

(iii) Combining (i) and (ii) of Theorem 4, (iii) follows.

In the line of Theorem 4 we may prove the following theorem.
Theorem 5. Let \( f \) and \( g \) be two entire functions such that \( 0 < \lambda^*_{fog} \leq \bar{\rho}^*_f < \infty \) and \( 0 < \bar{\rho}^*_g < \infty \). Then for any positive number \( A \),

\[
(i) \liminf_{r \to \infty} \frac{\log [3] \mu (r, f \circ g)}{\log [3] \mu (r^A, g)} \leq \frac{\bar{\rho}^*_f g}{A\lambda^*_g} \leq \limsup_{r \to \infty} \frac{\log [3] \mu (r, f \circ g)}{\log [3] \mu (r^A, g)}
\]

Further if \( \lambda^*_g > 0 \) then

\[
(ii) \frac{\lambda^*_{fog}}{A\bar{\rho}^*_g} \leq \liminf_{r \to \infty} \frac{\log [3] \mu (r, f \circ g)}{\log [3] \mu (r^A, g)} \leq \frac{\lambda^*_{fog}}{A\lambda^*_g} \leq \limsup_{r \to \infty} \frac{\log [3] \mu (r, f \circ g)}{\log [3] \mu (r^A, g)} \leq \frac{\bar{\rho}^*_f g}{A\lambda^*_g}
\]

and

\[
(iii) \liminf_{r \to \infty} \frac{\log [3] \mu (r, f \circ g)}{\log [3] \mu (r^A, g)} \leq \min \left\{ \frac{\lambda^*_{fog}}{A\lambda^*_g}, \frac{\bar{\rho}^*_f g}{A\lambda^*_g} \right\} \leq \max \left\{ \frac{\lambda^*_{fog}}{A\lambda^*_g}, \frac{\bar{\rho}^*_f g}{A\lambda^*_g} \right\} \leq \limsup_{r \to \infty} \frac{\log [3] \mu (r, f \circ g)}{\log [3] \mu (r^A, g)}.
\]

Theorem 6. If \( f \) and \( g \) be two entire functions with \( \rho^*_g < \infty \) and \( \rho^*_f = \infty \), then for every positive number \( A \),

\[
\limsup_{r \to \infty} \frac{\log [2] \mu (r, f \circ g)}{\log [2] \mu (r^A, g)} = \infty.
\]

Proof. Let us assume that the conclusion of Theorem 6 does not hold. Then there exists a constant \( B > 0 \) such that for all sufficiently large values of \( r \),

\[
\log [2] \mu (r, f \circ g) \leq B \log [2] \mu (r^A, g). \tag{32}
\]

Again from the definition of \( \rho^*_g \) it follows that

\[
\log [2] \mu (r^A, g) \leq (\rho^*_g + \varepsilon) A \log (re^{L(r)}) \tag{33}
\]

holds for all large values of \( r \).

So from (32) and (33) we obtain for all sufficiently large values of \( r \),

\[
\log [2] \mu (r, f \circ g) \leq (\rho^*_g + \varepsilon) AB \log (re^{L(r)}) \tag{34}
\]

From (34) it follows that \( \rho^*_f g < \infty \).

So we arrive at a contradiction.

This proves the theorem.

Remark 2. If we take \( \rho^*_f < \infty \) instead of \( \rho^*_g < \infty \) in Theorem 6 and the other conditions remain the same then the theorem remains valid with \( g \) replaced by \( f \) in the denominator.
References


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