Fixed Points for Biased Maps on Metric Space

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Abstract: In this paper, we characterize weakly biased concept with weakly compatible maps and establish a common fixed point theorem on metric space using the idea of weakly biased. Our result improves, extends or generalizes some results of Fisher, Rao & Rao, Jungck, Pathak-Cho & Kang etc.

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1. Introduction and Preliminaries:

In 1986, Jungck ([7]) introduced the concept of compatible maps which is a generalization of commuting maps and used to extend a theorem of Park-Bae([12]). Generalizing the concept of compatible maps, Jungck & Pathak ([5]) introduced the concepts of biased (weakly biased) maps and proved some fixed point theorems of Meir-Keeler type and also extended a theorem of Kang & Rhoades([9]). In [6] Jungck, Murthy & Cho introduced the concept of compatible of type (A) and obtained some fixed point theorems (also, see [2]). Further, Pathak, Cho & Kang ([11]) introduced the concepts of biased (weakly biased) maps of type (A) as a generalization of compatible of type (A) and proved some fixed point theorems and showed the existence of solutions of non linear integral equations. The aim of this paper is to characterize biased concepts (viz.
weakly biased, weakly biased of type (A)) with weakly compatible maps and to give two counter examples on the Prop.2.1 ([5]) and Prop. 2.3 ([11]) respectively. Further, we establish a common fixed point theorem using the concepts of property (E.A) and weakly biased as well. In the sequel, we need the following definitions and results.

**Definition 1.1** ([7]): A pair of self maps \( \{A, S\} \) of a metric space \((X, d)\) is said to be compatible iff \( d(SA_{x_n}, AS_{x_n}) \to 0 \) whenever \( \{x_n\} \) is a sequence in \( X \) such that \( Ax_n, Sx_n \to t \), for some \( t \in X \).

It has to be noted that, \( A \) and \( S \) are non-compatible if there exists at least one sequence \( \{x_n\} \) in \( X \) such that \( \lim_{n \to \infty} Ax_n = \lim_{n \to \infty} Sx_n = t \), for some \( t \in X \) but \( \lim_{n \to \infty} d(SAx_n, ASx_n) \) is either non-zero or non-existence (see also [1], [13],[17] etc.)

**Definition 1.2** ([1]): Let \( A \) and \( S \) be two self maps of a metric space \((X, d)\), we say that \( A \) and \( S \) satisfy property (E. A), if there exists a sequence \( \{x_n\} \) in \( X \) such that \( \lim_{n \to \infty} Ax_n = \lim_{n \to \infty} Sx_n = t \) for some \( t \in X \).

It is easy to see that compatible and non-compatible classes of maps of a metric space \((X, d)\) are necessarily satisfied property (E. A).

**Definition 1.3** ([5]): A pair of self maps \( \{A, S\} \) of a metric space \((X, d)\) is said to be S-biased iff whenever \( \{x_n\} \) is a sequence in \( X \) such that \( Ax_n, Sx_n \to t \in X \), then \( \alpha \inf d(SSx_n, Ax_n) \leq \alpha \sup d(ASx_n, Sx_n) \leq \alpha \) whenever \( \{x_n\} \) is a sequence in \( X \) such that \( \lim_{n \to \infty} Ax_n = \lim_{n \to \infty} Sx_n = t \) for some \( t \in X \).

We remind the reader that \( \alpha = \lim \inf x_n = \sup \{x_n : n \in I^+ \} \) where \( I^+ \) is the set of positive integers and \( \lim \inf = \lim \sup = \lim \), if the limit exits. The pair \( \{A, S\} \) is A-biased if by interchanging the role of \( A \) and \( S \) in the above Definition 1.3. If the pair \( \{A, S\} \) is compatible maps, then it is both S- and A- biased but the converse is not true (see for details Remark 1.1 and Ex. 1.2 of [5]).

**Definition 1.4** ([5]): A pair of self maps \( \{A, S\} \) of a metric space \((X, d)\) is said to be weakly S-biased iff \( \alpha = \lim \inf \{x_n, n \in I^+ \} \) is the set of positive integers and \( \lim \inf = \lim \sup = \lim \), if the limit exits. The pair \( \{A, S\} \) is A-biased if by interchanging the role of \( A \) and \( S \) in the above Definition 1.3. If the pair \( \{A, S\} \) is compatible maps, then it is both S- and A- biased but the converse is not true (see for details Remark 1.1 and Ex. 1.2 of [5]).

**Definition 1.5** ([6]): A pair of self maps \( \{A, S\} \) of a metric space \((X, d)\) is said to be compatible of type (A) if \( \lim_{n \to \infty} d(SAx_n, AAx_n) = 0 \) and \( \lim_{n \to \infty} d(SSx_n, Sx_n) = 0 \) whenever \( \{x_n\} \) is a sequence in \( X \) such that \( \lim_{n \to \infty} Ax_n = \lim_{n \to \infty} Sx_n = t \), for some \( t \in X \).

The concepts of compatible maps and compatible maps of type (A) are equivalent under the continuity conditions (see [2]).

**Definition 1.6** ([11]): A pair of self maps \( \{A, S\} \) of a metric space \((X, d)\) is said to be S-biased of type (A) if \( \inf d(SSx_n, Ax_n) \leq \sup d(ASx_n, Sx_n) \) whenever \( \{x_n\} \) is a sequence in \( X \) such that \( Ax_n, Sx_n \to t \in X \), where \( \alpha = \lim \inf \) and \( \alpha = \lim \sup \).

Note that there exists a pair \( \{A, S\} \) of maps which is S-biased but not S-biased of type (A) and conversely (see for details Ex. 2.11 and Ex.2.12 of [11]).

**Definition 1.7** ([11]): A pair of self maps \( \{A, S\} \) of a metric space \((X, d)\) is said to be weakly S-biased of type (A) if \( \alpha = \lim \inf \) and \( \alpha = \lim \sup \).
If the pair \{A, S\} is compatible of type (A), then it is both S-biased and A-biased of type (A). However, the converse is not necessarily true (see Remark 2.3 and Ex. 2.4 of [11]).

**Definition 1.8** ([4]): A pair of self maps \{A, S\} of a metric space \((X, d)\) is said to be weakly compatible if they commute at their coincidence points, i.e. \(A t = S t\) for some \(t \in X\), then \(S A t = A S t\).

Throughout this paper, we shall denote \(\emptyset = N_0 \cup \{0\}\), where \(N_0\) is the set of natural numbers and \((X, d)\), a metric space.

### 2. Main Results:

Before we arrive our results, it may be mentioned that the notions of weakly biased ([5]) and weakly biased of type (A)([11]) are invariant. Hence, we need to demonstrate that the concept of weakly biased of type (A) is indeed not distinct from the original concept of weakly biased maps. For, suppose that \{A, S\} is weakly S-biased then \(A t = S t\) implies \(d(S A t, S t) \leq d(A S t, A t)\). Also \(A t = S t\) implies \(A A t = A S t\) and \(S A t = S S t\). Therefore, \(d(S S t, A t) = d(S A t, S t) \leq d(A S t, A t) = d(A S t, S t)\) and hence \{A, S\} is a weakly S-biased maps of type (A). Similarly, one can see that weakly A-biased and weakly A-biased of type (A) are invariant.

In [5], Jungck & Pathak proved the following Proposition 2.1 for S-biased maps.

**Proposition 2.1** (cf. from Prop. 1.1(a) of [5]): Let A and S be self maps of a metric space \((X, d)\). If the pair \{A, S\} is S-biased and \(A t = S t\) then \(d(S A t, S t) \leq d(A S t, A t)\).

In view of the following example the above Prop. 2.1 may not be true.

**Example 2.2**: Let \(X = [0, 1]\) with usual metric \(d(x, y) = |x - y|\).

Define \(A = \begin{cases} 0 & 0 \leq x < 1/2 \\ x & x = 1/2 \\ 1 & x > 1/2 \end{cases}\) and \(S = \begin{cases} 0 & 0 \leq x < 1/2 \\ 1-x & x = 1/2 \\ 1-x & 1/2 < x \leq 1 \end{cases}\).

Since A and S are not continuous at \(x = 1/2\). Now, we show that A and S are compatible maps. Suppose that \(\{x_n\} \subset X\) and \(A x_n, S x_n \rightarrow t\), for some \(t \in X\).

For this, let \(x_n \rightarrow 1/2\) and \(x_n > 1/2\), for all \(n \in \mathbb{N}\). Then, we have \(A x_n = x_n \rightarrow 1/2 = t\) and \(S x_n = 1-x_n \rightarrow 1/2 = t\). Since \(1-x_n < 1/2\), for all \(n \in \mathbb{N}\) and \(x_n > 1/2\). Now, we have \(S A x_n = S(x_n) = 1-x_n \rightarrow 1/2\) and \(A S x_n = A(1-x_n) = 1/2 \rightarrow 1/2\). It follows that \(d(S A x_n, A S x_n) \rightarrow 0\) as \(d(A x_n, S x_n) \rightarrow 0\), where \(x_n \rightarrow 1/2\). Therefore, A and S are compatible maps. By remark 2.3 of [5], \{A, S\} is both S- and A-biased maps. Further, we have \(A t = A(1/2) = 0 = S(1/2) = S t\). Also, \(A S t = A(1/2) = A(0) = 1/2, S A t = S(1/2) = S(0) = 1\). Then, we have \(d(S A t, S t) = |1-0| = 1 \neq d(A S t, A t) = |1/2-0| = 1/2\).
Thus, \( S\)-biased of \( \{A, S\} \) with \( At = St \) need not imply \( d(SAt, At) \leq d(ASSt, St) \). This completes our assertion.

In [11], Pathak-Cho & Kang proved the following Prop. 2.3 for \( S\)-biased of type (A) which is analogous to Prop. 2.1 for \( S\)-biased maps.

**Proposition 2.3** (cf. Prop. 2.5 of [11]): Let \( A \) and \( S \) be self maps of a metric space \( (X, d) \). If the pair \( \{A, S\} \) is \( S\)-biased of type (A) and \( At = St \), then \( d(SSSt, At) \leq d(ASSt, St) \).

The following example shows that the above Prop. 2.3 may not be true.

**Example 2.4:** Let \( A, S : X \rightarrow X \), where \( X = [0, 1] \) with usual metric \( d(x, y) = |x - y| \).

Define
\[
A_x = \begin{cases} 
1/3 & 0 \leq x < 1/2 \\
2/3 & x = 1/2 \\
1/2 & 1/2 < x \leq 1
\end{cases} \quad \text{and} \quad S_x = \begin{cases} 
1/3 & 0 \leq x < 1/2 \\
2/3 & x = 1/2 \\
1- & 1/2 < x \leq 1
\end{cases} ;
\]

First, we show that \( \{A, S\} \) is compatible maps of type (A). For this, let \( x_n \rightarrow 1/2 \) and \( x_n > 1/2 \) for all \( n \in \mathbb{N} \). Then, we have \( Ax_n = x_n \rightarrow 1/2 = t \) and \( Sx_n = 1-x_n \rightarrow 1/2 = t \).

Clearly, \( d(SA_n, A_n) \rightarrow 0 \) and \( d(AS_n, SS_n) \rightarrow 0 \) as \( A_n, S_n \rightarrow 1/2 \), where \( x_n \rightarrow 1/2 \). Therefore, \( \{A, S\} \) is compatible of type (A). By remark 2.3 of [11] \( \{A, S\} \) is both \( S\)- and \( A\)-biased of type (A). Further, we have \( At = A(1/2) = 2/3 = S(1/2) = St \).

Also, \( SAt = SSSt = 1/3, ASt = AAAt = 2/3 \), where \( t = 1/2 \). Then we have
\[
d(SAt, St) = |2/3 - 2/3| = 0 \leq d(ASAt, At) = |1/3 - 2/3| = 1/3 \quad \text{and} \quad d(SSSt, At) = |1/3 - 2/3| = 1/3 \leq d(ASSt, St) = |2/3 - 2/3| = 0 .
\]

Thus, \( S\)-biased of type (A) with \( At = St \) need not imply \( d(SSSt, At) \leq d(ASSt, St) \). This completes our assertion.

From the above examples (see Ex. 2.2 and 2.4), it may be noted that there exists a pair \( \{A, S\} \) of maps which is \( S\)-biased (\( S\)-biased of type (A)) but not weakly \( S\)-biased (weakly \( S\)-biased of type (A)).

**Proposition 2.5:** Let \( A, S : (X, d) \rightarrow (X, d) \) be continuous maps.

i) If \( \{A, S\} \) is \( S\)-biased and \( At = St \), for some \( t \in X \), then \( d(SAt, St) \leq d(ASSt, At) \).

ii) If \( \{A, S\} \) is \( S\)-biased of type (A) and \( At = St \), for some \( t \in X \), then \( d(SSSt, At) \leq d(ASSt, St) \).

**Proof:** i) By definition of \( S\)-biased, it is guaranteed to choose a sequence \( \{x_n\} \) in \( X \) such that \( x_n \rightarrow p \) and \( Ax_n, Sx_n \rightarrow Ap, Sp \), for some \( p \in X \). Since \( A \) and \( S \) are continuous maps. Then, we obtain \( d(SAp, Sp) = \lim_\infty d(SAx_n, Sx_n) \leq \lim_\infty d(ASx_n, Ax_n) = d(ASp, Ap) \). This completes the proof of (i). Similarly, one can proof (ii). □

Analogous results for \( A\)-biased (\( A\)-biased of type (A)) by interchanging the role of \( A \) and \( S \) may be obtained. It may be noted from Prop. 2.5 that \( S\)-biased (\( S\)-biased of type (A)) for a pair of continuous maps implies weakly \( S\)-biased (weakly \( S\)-biased of type (A)) (see Ex. 2.10). We recall the following properties for compatible maps and compatible maps of type (A).
Proposition 2.6 (cf. [7]): Let \( A, S: (X,d) \to (X,d) \) be compatible. If \( At = St \) then \( ASt = SAT \).

Proposition 2.7(cf. [6]): Let \( A, S: (X,d) \to (X,d) \) be maps. If the pair \( \{A, S\} \) is compatible of type (A) and \( At = St \), for some \( t \in X \). Then \( ASt = SSSt = SAT = AAt \).

Remark 2.8: In [18], the present authors had already shown that compatible maps (compatible maps of type (A), compatible maps of type (P)) need not be weakly compatible and also converse may not be true (see for details [10], [14], [15], [18] etc.). The interested reader can also check with Ex. 2.2, 2.4 that Prop. 2.6 and Prop. 2.7 are not true. On the other hand, in [11], it has mentioned that “the equalities amongst \( ASt, SSSt, SAT \) and \( AAt \) of Prop. 2.7 imply the conclusion of Prop. 2.3 in the strong sense ‘\( = \)’ but not in the weak sense ‘\( \leq \)’ (see the line 7(\( \downarrow \)) page no. 684 of [11])” ; however, the observation proposed by the authors may not be true. Similarly, the implication of Prop. 2.6 to the conclusion of Prop. 2.1 may not be true (see Ex. 2.2 and 2.4).

Proposition 2.9: Let \( \{A, S\} \) be a pair of self maps of a metric space \((X, d)\). If the pair \( \{A, S\} \) is weakly compatible then it is both weakly S- and weakly A-biased.

Proof: Suppose that \( \{A, S\} \) is weakly compatible, then there exists a point \( t \in X \) such that \( At = St \) implies \( ASt = SAT \). Now, we have

\[
d(SAT, St) \leq d(SAT, ASt) + d(ASt, At) + d(At, St)
\]

Therefore, \( d(SAT, St) \leq d(ASt, At) \), since \( At = St \). Thus, weakly compatibility of \( \{A, S\} \) implies weakly S- biased. Similarly, if \( \{A, S\} \) is weakly compatible then it is weakly A- biased. Hence, weakly compatibility of \( \{A, S\} \) implies both weakly S- and A-biased maps.

The conclusion of the above proposition is in the strong sense “\( = \)” but not in the weak sense “\( \leq \)”. The converse of the above Prop. 2.9 may not be necessarily true as shown in Ex. 2.10, where the pair \( \{A, S\} \) is not commuting at the coincidence point but weakly S- and A-biased respectively.

Example 2.10: Let \( A, S: X \to X \), where \( X = \mathbb{R} \), the set of real numbers with usual metric \( d(x,y) = |x-y| \). Define \( Ax = 1-2x \) and \( Sx = 2x \). Clearly, \( \{A, S\} \) is continuous and non commuting. Now, we show that \( \{A, S\} \) is both S- and A-biased (S-and A-biased of type (A)) but not compatible (compatible of type (A)). For this, let \( \{x_n\} \) be a sequence in \( X \) such that \( x_n \to 1/4 \), then \( Ax_n = 1-2x_n \), \( Sx_n = 2x_n \) so, \( Ax_n, Sx_n \to 1/2 \). Also, we have

\[
\lim_{n \to \infty} d(SAx_n, Sx_n) = 1/2 \leq \lim_{n \to \infty} d(ASx_n, Ax_n) = 1/2.
\]

\[
\lim_{n \to \infty} d(SSx_n, Ax_n) = 1/2 \leq \lim_{n \to \infty} d(SSx_n, Sx_n) = 1/2
\]

\[
\lim_{n \to \infty} d(AAx_n, Sx_n) = 1/2 \leq \lim_{n \to \infty} d(AAx_n, Ax_n) = 1/2.
\]

The pair \( \{A, S\} \) is S- and A-biased (S-and A-biased of type (A)) but not compatible (compatible of type (A)). Further \( At = St = 1/2 \), where \( t = 1/4 \), then \( AAt = ASt = 0 \) and \( SAT = SSSt = 1 \) but \( ASt \neq SAT \). Since, \( \{A, S\} \) is S-and A-biased (S-and A-biased of type (A)) and \( At = St \), where \( t = 1/4 \). Also, we have
\[ |SA_t - ST| = 1/2 \leq |AS_t - AT| = 1/2, \quad |AS_t - AT| = 1/2 \leq |SA_t - ST| = 1/2, \quad |SA_t - ST| = 1/2 \leq |AT - AT| = 1/2 \leq |SA_t - ST| = 1/2. \]

Therefore, S-and A biased (S-and A-biased of type (A)) for a continuous pair of maps \{A, S\} implies weakly S-and A-biased (weakly S-and A-biased of type (A)) but the pair \{A, S\} is not weakly compatible.

**Theorem 2.11**: Let A, B, S and T be self maps on a metric space \((X, d)\) into itself satisfying the following conditions

i) \(X(T) \subseteq X(A)\) and \(X(S) \subseteq X(B)\);

ii) \(d(Ax, By) \leq \alpha d^p(Sx, Ty) + \beta \max\{d^p(Sx, Ax), d^p(Ty, By)\}\)

for any \(x, y \in X\), \(p \geq 1\), \(0 < \alpha + \beta < 1\);

iii) \{A, S\} or \{B, T\} satisfy the property (E. A);

iv) \{A, S\} and \{B, T\} are weakly S- and T-biased maps respectively;

If one range of the maps A, B, S and T is closed subspace of X, then A, B, S and T have a unique common fixed point.

**Proof**: Suppose that \{B, T\} satisfy the property (E. A). Then there exists a sequence \(\{x_n\}\) in X such that \(\lim_{n \to \infty} Bx_n = \lim_{n \to \infty} Tx_n = t\), for some \(t \in X\). Since \(B(X) \subseteq S(X)\), then there exists a sequence \(\{y_n\}\) in X such that \(Bx_n = Sx_n\). Hence, \(\lim_{n \to \infty} Sy_n = t\). Now, we show that \(Ay_n \to t\). Since \(d(Ay_n, t) \leq d(Ay_n, Bx_n) + d(Bx_n, t)\), so it is sufficient to show that \(\lim d(Ay_n, Bx_n) = 0\). Suppose not, then there exists a subsequence \(\{y_{n_k}\}\) of \(\{y_n\}\) in X and real number \(\varepsilon > 0\) such that for some positive integer \(k \geq n\), \(\lim_{k \to \infty} d(Ay_{n_k}, Bx_{n_k}) \geq \varepsilon\). Using (ii), we obtain

\[
d(Ay_{n_k}, Bx_{n_k}) \leq [\alpha d^p(Sy_{n_k}, Tx_{n_k}) + \beta \max\{d^p(Sy_{n_k}, Ay_{n_k}), d^p(Tx_{n_k}, Bx_{n_k})\}^{1/p}
\]

\[
= [\alpha d^p(Bx_{n_k}, Tx_{n_k}) + \beta \max\{d^p(Sy_{n_k}, Ay_{n_k}), d^p(Bx_{n_k}, Bx_{n_k})\}^{1/2p}
\]

As \(k \to \infty\), it follows that

\[
\varepsilon \leq \left(\frac{\beta}{2^p} \max\{0, \varepsilon^{2p}, 0\}\right)^{1/2p}
\]
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\[ \left( \frac{\beta}{2^p} \right)^{1/2p} \leq \varepsilon, \quad 0 < \frac{\beta}{2^p} < 1 \]

Therefore, \( \lim_{k \to \infty} d(Ay, Bx) = 0 \) and hence \( \lim_{k \to \infty} Ay = t \). Thus, we obtain

\[ \lim_{n \to \infty} Bx_n = \lim_{n \to \infty} Tx_n = \lim_{n \to \infty} Sy_n = \lim_{n \to \infty} Ay = t. \]

Suppose that \( S(X) \) is closed subspace of \( X \), then there exists \( u \in X \) such that \( t = Su \). We show that \( Su = Au \). Using (ii), we obtain

\[
d(Au_Bx_n) \leq \left[ \alpha d^2(Su, Tx_n) + \frac{\beta}{2^p} \max \{d^p(Su, Au) d^p(Tx_n, Bx_n), \right.
\]

\[
\left. d^p(Su, Bx_n) d^p(Tx_n, Au), d^p(Su, Au) d^p(Tx_n, Au), \right]
\]

\[
\left. d^p(Su, Bx_n) d^p(Tx_n, Bx_n) \right]^{1/2p}
\]

As \( n \to \infty \), it follows that

\[ d(Au, Su) \leq \left( \frac{\beta}{2^p} \right)^{1/2p} \max \{0, 0, d^2(Su, Au), 0 \}^{1/2p} \]

\[ \leq \left( \frac{\beta}{2^p} d^2(Su, Au) \right)^{1/2p} < d(Su, Au), \quad \text{since} \quad 0 < \frac{\beta}{2^p} < 1, \quad \text{a contradiction if} \quad Su \neq Au. \]

It follows that \( Au = Su \). Thus, \( \{A, S\} \) have a coincidence point. Since, \( A(X) \subset T(X) \) there exists a point \( v \in X \) such that \( Au = Tv \). If \( Tv \neq Bv \), then by (iii), we obtain

\[ d(Au, Bv) \leq \left[ \alpha d^2(Su, Tv) + \frac{\beta}{2^p} \max \{d^p(Su, Au) d^p(Tv, Bv), \right. \]

\[
\left. d^p(Su, Bv) d^p(Tv, Au), d^p(Su, Au) d^p(Tv, Au), \right]
\]

\[
\left. d^p(Su, Bv) d^p(Tv, Bv) \right]^{1/2p}
\]

\[ = \left( \frac{\beta}{2^p} \max \{0, 0, d^2(Su, Bv), d^p(Tv, Bv) \} \right)^{1/2p} \]

\[ \leq \left( \frac{\beta}{2^p} d^2(Au, Bv) \right)^{1/2p} < d(Au, Bv), \quad \text{a contradiction.} \]

It follows that \( Au = Bv \) and hence \( Tv = Bv \). Therefore, \( \{B, T\} \) have a coincidence point. Since, \( Au = Su \) and \( Tv = Bv \). Therefore, weakly \( S \)-biased of \( \{A, S\} \) implies \( d(SAu, Su) \leq d(ASu, Au) \). Similarly, weakly \( T \)-biased of \( \{B, T\} \) implies \( d(TBV, TV) \leq d(BTV, BV) \). On the other hand, we obtain \( Su = Au \) implies \( ASu = AAu, SSu = SAu, \) and \( Tv = Bv \) implies \( BTv = BBv, TTv = TBv. \) Now, we show that \( Au \) is a common fixed point of \( A, B, S \) and \( T \). Using (ii), we obtain
\[ d(AAu, Au) = d(AAu, Bv) \]
\[ \leq \left[ \alpha d^{2p}(SAu, Tv) + \frac{\beta}{2^p} \max \{d^p(SAu, AAAu), d^p(Tv, Bv), d^p(SAu, Bv), d^p(Tv, Bv) \} \right]^{1/2p} \]
\[ = \left[ \alpha d^{2p}(SAu, Su) + \frac{\beta}{2^p} \max \{0, d^p(SAu, Su), d^p(Au, AAAu), d^p(SAu, AAu), d^p(Au, AAAu), d^p(Su, AAu), d^p(Au, AAAu), 0 \} \right]^{1/2p} \]
\[ \leq \left[ \alpha d^{2p}(AAu, Au) + \frac{\beta}{2^p} \max \{0, d^p(AAu, Au), d^p(AAu, Au), d^p(AAu, Au), 0 \} \right]^{1/2p} \]
\[ \leq \left[ \alpha d^{2p}(AAu, Au) + \frac{\beta}{2^p} \cdot 2^p d^{2p}(AAu, Au) \right]^{1/2p} \]
\[ \leq \left[ (\alpha + \beta) d^{2p}(AAu, Au) \right]^{1/2p}, \ 0 < \alpha + \beta < 1. \]
\[ < d(AAu, Au), \text{a contradiction if } AAu \neq Au. \]

Hence, Au is a fixed point of A. Further, it is easy to show that Au is also a fixed point of S. Thus, Au is a common fixed point of \{A, S\}. Similarly, one can show that Bv is a common fixed point of \{B, T\}. Since Au = Bv, we therefore conclude that Au is a common fixed point of A, B, S and T. The proof is similar when A(X) (T(X) or B(X)) is closed subspace of X. One can easily check the uniqueness of common fixed point of A, B, S and T. This completes the proof. \(\square\)

The following example shows the validity of Theorem 2.11.

**Example 2.12**: Let \( X = [0,1] \) with usual metric \( d(x, y) = |x - y| \).

Define \( A = B, \ S = T : X \to X \) by

\[
Ax = Bx = \begin{cases} 
(1+x)/2 & 0 \leq x < 1/2 \\
1/2 & x = 1/2 \\
3/4 & 1/2 < x \leq 1
\end{cases}
\]

\[
Sx = Tx = \begin{cases} 
1/2 + x & 0 \leq x < 1/2 \\
x = 1/2 \\
1 & 1/2 < x \leq 1
\end{cases}
\]

Let \( \{x_n\} \subseteq X \) be a sequence such that \( Ax_n, Sx_n \to t \), for some \( t \in X \). For this, let \( x_n \to 0 \) and \( x_n > 0 \), for all \( n \in \mathbb{N} \), then we obtain \( Ax_n = (1 + x_n)/2 \to 1/2 = t \in X \) and \( Sx_n = 1/2 + x_n \to 1/2 = t \in X \) i.e. \( \lim_{n \to \infty} Ax_n = \lim_{n \to \infty} Sx_n = 1/2 = t \in X \).

i) \( \{A, S\} \) satisfies property (E.A);
ii) In particular, we take \( p = 1, \alpha = 1/3 \) and \( \beta = 1/2 \). For all \( x, y \in X \), \( \{A, S\} \) satisfies the inequality (ii) of Theorem 2.11. Since \( A = B, S = T \),

iii) \( \{A, S\} \) is weakly S-biased maps

iv) \( A(X) = B(X) = [1/2, 3/4] \subset S(X) = T(X) = [1/2, 1] \) are closed in \( X \).

Clearly, one can check that \( \{A, S\} \) and \( \{B, T\} \) satisfy all the conditions of Theorem 2.11 and hence, 1/2 is a unique common fixed point of \( A, B, S \) and \( T \).

**Corollary 2.13:** Let \( A, B, S \) and \( T \) be self maps of a metric space \((X,d)\) into itself satisfying the conditions (i) and (ii) of Theorem 2.11. If one of the range of \( A, B, S \) and \( T \) is complete subspace of \( X \), then \( A, B, S \) and \( T \) have a unique common fixed point provided that the pairs \( \{A, S\} \) and \( \{B, T\} \) are weakly compatible maps.

**Proof:** Proof is on line of above Theorem 2.11. □

**Remark 2.14:** Theorem 2.11 improves some results of [3], [8], [11] and [16] etc. Of course, our results do not require the continuities of maps involved and merely required weakly biased in place of commuting maps (compatible maps or compatible maps of type (A)) and also one of the range spaces be closed in place of \( X \) being complete.

### References


[15]. V. Popa, Some fixed point theorems for weakly compatible mappings, Radovi Mat. 10 (2001), 245-252.


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