On Relative Defects of Differential Polynomials

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Abstract

The purpose of this paper is to compare the relative Valiron defect with the relative Nevanlinna defect of differential polynomials generated by a meromorphic function.

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1 Introduction, Definitions and Notations.

Let \( f \) be a meromorphic function defined in the open complex plane \( \mathbb{C} \). For \( a \in \mathbb{C} \cup \{ \infty \} \) we denote by \( n(t, a; f) \) (\( \tilde{n}(t, a; f) \)) the number of \( a \)-points (distinct \( a \)-points) of \( f \) in \( |z| \leq t \), where an \( \infty \) -point is a pole of \( f \). We put

\[
N(r, a; f) = \int_{0}^{r} \frac{n(t, a; f) - n(0, a; f)}{t} dt + \tilde{n}(0, a; f) \log r.
\]
The function \( N(r, a; f) \) are called the counting function of \( a \)-points (distinct \( a \)-points) of \( f \). In many occasions \( N(r, \infty; f) \) and \( \tilde{N}(r, \infty; f) \) are denoted by \( N(r, f) \) and \( \tilde{N}(r, f) \) respectively. We also put

\[
m(r, f) = \frac{1}{2\pi} \int_0^{2\pi} \log^+ |f(re^{i\theta})| \, d\theta,
\]

where

\[
\log^+ x = \log x, \text{ if } x \geq 1
\]

\[
= 0, \text{ if } 0 \leq x < 1.
\]

For \( a \in \mathbb{C} \) we denote by \( m \left( r, \frac{1}{f-a} \right) \) by \( m(r, a; f) \) and we mean by \( m(r, \infty; f) \) the function \( m(r, f) \), which is called the proximity function of \( f \).

The function \( T(r, f) = m(r, f) + N(r, f) \) is called the characteristic function of \( f \). If \( a \in \mathbb{C} \cup \{\infty\} \), the quantity

\[
\delta(a; f) = 1 - \limsup_{r \to \infty} \frac{N(r, a; f)}{T(r, f)} = \liminf_{r \to \infty} \frac{m(r, a; f)}{T(r, f)}
\]

is called the Nevanlinna deficiency of the value \('a'\). Similarly, the Valiron deficiency \( \Delta(a; f) \) of the value \('a'\) is defined as

\[
\Delta(a; f) = 1 - \liminf_{r \to \infty} \frac{N(r, a; f)}{T(r, f)} = \limsup_{r \to \infty} \frac{m(r, a; f)}{T(r, f)}.
\]

Milloux [5] introduced the concept of absolute defect of \('a'\) with respect to \( f' \). Later Xiong [9] extended this definition. He introduced the term

\[
\delta^{(k)}_R(a; f) = 1 - \limsup_{r \to \infty} \frac{N(r, a; f^{(k)})}{T(r, f)}
\]

for \( k = 1, 2, 3, \ldots \) and called it the relative Nevanlinna defect of \('a'\) with respect to \( f^{(k)} \). Xiong [9] has shown various relations between the usual defects and the relative defects for meromorphic functions. Singh [7] introduced the term relative defect for distinct zeros and poles and established various relations between it, relative defects and the usual defects.

Let \( f \) be a non-constant meromorphic function defined in the open complex plane \( \mathbb{C} \). Also let \( n_{0j}, n_{1j}, \ldots, n_{kj}(k \geq 1) \) be non-negative integers such that for each \( j \), \( \sum_{i=0}^{k} n_{ij} \geq 1 \). We call \( M_j[f] = A_{j_1} f^{n_{0j}} (f^{(1)})^{n_{1j}} \ldots (f^{(k)})^{n_{kj}} \)

where \( T(r, A_j) = S(r, f) \) to be a differential monomial generated by \( f \). The numbers \( \gamma_{M_j} = \sum_{i=0}^{k} n_{ij} \) and \( \Gamma_{M_j} = \sum_{i=0}^{k} (i+1) n_{ij} \) are called respectively the
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degree and weight of $M_j[f] \ {[2],[6]}$. The expression $P[f] = \sum_{j=1}^{s} M_j[f]$ is called a differential polynomial generated by $f$. The numbers $\gamma_P = \max_{1<j<s} \gamma_{M_j}$ and $\Gamma_P = \max_{1<j<s} \Gamma_{M_j}$ are called respectively the degree and weight of $P[f] \ {[2],[6]}$. Also we call the numbers $\gamma_P = \min_{1<j<s} \gamma_{M_j}$ and $z$ (the order of the highest derivative of $f$) the lower degree and the order of $P[f]$ respectively. If $\gamma_P = \gamma_P$, $P[f]$ is called a homogeneous differential polynomial.

In this paper we call the terms

$$\delta_P^A(a;f) = 1 - \limsup_{r \to \infty} \frac{N(r,a;P[f])}{T(r,P[f])} = \liminf_{r \to \infty} \frac{m(r,a;P[f])}{T(r,P[f])},$$

the usual Nevanlinna defect or the absolute Nevanlinna defect of the value $'a'$ with respect to $P[f]$,

$$\Delta_P^A(a;f) = 1 - \liminf_{r \to \infty} \frac{N(r,a;P[f])}{T(r,P[f])} = \limsup_{r \to \infty} \frac{m(r,a;P[f])}{T(r,P[f])},$$

the usual Valiron defect or the absolute Valiron defect of the value $'a'$ with respect to $P[f]$,

$$\delta_P^R(a;f) = 1 - \limsup_{r \to \infty} \frac{N(r,a;P[f])}{T(r,f)},$$

the relative Nevanlinna defect of $'a'$ with respect to $P[f]$ and

$$\Delta_P^R(a;f) = 1 - \liminf_{r \to \infty} \frac{N(r,a;P[f])}{T(r,f)},$$

the relative Valiron defect of $'a'$ with respect to $P[f]$ and prove various relations among them.

The term $S(r,f)$ denotes any quantity satisfying $S(r,f) = o\{T(r,f)\}$ as $r \to \infty$ through all values of $r$ if $f$ is of finite order and except possibly for a set of $r$ of finite linear measure otherwise. We do not explain the standard notations and definitions on the theory of entire and meromorphic functions because those are available in [8] and [3]. Throughout the paper we consider only the non-constant differential polynomials and we denote by $P_0[f]$ a differential polynomial not containing $f$ i.e. for which $n_{0j} = 0$ for $j = 1, 2, \ldots, s$. We consider only those $P[f], P_0[f]$ singularities of whose individual terms do not cancel each other.

The following definitions are well known.

**Definition 1.** The quantity $\Theta(a;f)$ of a meromorphic function $f$ is defined as follows

$$\Theta(a;f) = 1 - \limsup_{r \to \infty} \frac{N(r,a;f)}{T(r,f)}.$$
Definition 2. [4] For $a \in \mathbb{C} \cup \{\infty\}$, let $n_p(r, a; f)$ denotes the number of zeros of $f - a$ in $|z| \leq r$, where a zero of multiplicity $< p$ is counted according to its multiplicity and a zero of multiplicity $\geq p$ is counted exactly $p$ times; and $N_p(r, a; f)$ is defined in terms of $n_p(r, a; f)$ in the usual way. We define

$$\delta_p(a; f) = 1 - \limsup_{r \to \infty} \frac{N_p(r, a; f)}{T(r, f)}.$$ 

Definition 3. [1] $P[f]$ is said to be admissible if

(i) $P[f]$ is homogeneous, or

(ii) $P[f]$ is non homogeneous and $m(r, f) = S(r, f)$.

Definition 4. The order $\rho_f$ and lower order $\lambda_f$ of a meromorphic function $f$ are defined as

$$\rho_f = \limsup_{r \to \infty} \frac{\log T(r, f)}{\log r} \quad \text{and} \quad \lambda_f = \liminf_{r \to \infty} \frac{\log T(r, f)}{\log r}.$$ 

If $f$ is entire, one can easily verify that

$$\rho_f = \limsup_{r \to \infty} \frac{\log |f|r^{\rho} M(r, f)}{\log r} \quad \text{and} \quad \lambda_f = \liminf_{r \to \infty} \frac{\log |f|r^{\lambda} M(r, f)}{\log r},$$

where $\log^k x = \log \left( \log^{k-1} x \right)$ for $k = 1, 2, 3, \ldots$ and $\log^0 x = x$.

2 Lemmas.

In this section we present some lemmas which will be needed in the sequel.

**Lemma 1.** [1] Let $P_0[f]$ be admissible. If $f$ is of finite order or of non zero lower order and $\sum_{a \neq \infty} \Theta(a; f) = 2$ then

$$\lim_{r \to \infty} \frac{T(r, P_0[f])}{T(r, f)} = \Gamma_{P_0}.$$

**Lemma 2.** [1] Let $f$ be either of finite order or of non-zero lower order such that $\Theta(\infty; f) = \sum_{a \neq \infty} \delta_p(a; f) = 1$. Then for homogeneous $P_0[f],$

$$\lim_{r \to \infty} \frac{T(r, P_0[f])}{T(r, f)} = \gamma_{P_0}.$$
Lemma 3. Let $f$ be a meromorphic function of finite order or of non zero lower order. If $\sum_{a \neq \infty} \Theta(a; f) = 2$, then for any $\alpha$,

\[
\Delta_{R}^{P_0}(\alpha; f) = (1 - \Gamma_{P_0}) + \limsup_{r \to \infty} \frac{m(r, \alpha; P_0[f])}{T(r, f)}
\]

and \(\delta_{R}^{P_0}(\alpha; f) = (1 - \Gamma_{P_0}) + \liminf_{r \to \infty} \frac{m(r, \alpha; P_0[f])}{T(r, f)}\).

Proof. In view of Lemma 1, we obtain that

\[
\Delta_{R}^{P_0}(\alpha; f) = 1 - \liminf_{r \to \infty} \frac{N(r, \alpha; P_0[f])}{T(r, f)}
\]

\[
\Gamma_{P_0} \left\{1 - \liminf_{r \to \infty} \frac{N(r, \alpha; P_0[f])}{T(r, P_0[f])}\right\} + (1 - \Gamma_{P_0})
\]

\[
= \Gamma_{P_0} \limsup_{r \to \infty} \frac{m(r, \alpha; P_0[f])}{T(r, P_0[f])} + (1 - \Gamma_{P_0})
\]

\[
= \Gamma_{P_0} \left\{\limsup_{r \to \infty} \frac{m(r, \alpha; P_0[f])}{T(r, f)} \cdot \liminf_{r \to \infty} \frac{T(r, P_0[f])}{T(r, f)}\right\} + (1 - \Gamma_{P_0})
\]

\[
= (1 - \Gamma_{P_0}) + \limsup_{r \to \infty} \frac{m(r, \alpha; P_0[f])}{T(r, f)}.
\]

This proves the first part of the lemma.

Similarly we can prove the second part of the lemma.

Lemma 4. Let $f$ be a meromorphic function of finite order or of non zero lower order. If $\Theta(\infty; f) = \sum \delta_p(\alpha; f) = 1$, then for any $\alpha$,

\[
\Delta_{R}^{P_0}(\alpha; f) = (1 - \gamma_{P_0}) + \limsup_{r \to \infty} \frac{m(r, \alpha; P_0[f])}{T(r, f)}
\]

and \(\delta_{R}^{P_0}(\alpha; f) = (1 - \gamma_{P_0}) + \liminf_{r \to \infty} \frac{m(r, \alpha; P_0[f])}{T(r, f)}\).

We omit the proof of Lemma 4 because it can be carried out in the line of Lemma 3.

Remark 1. The conclusions of Lemma 3 and Lemma 4 can also be drawn under the hypothesis $\delta(\infty; f) = \sum \delta(\alpha; f) = 1$. 

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Remark 2. Let \( f \) be a meromorphic function of finite order or of non-zero lower order such that \( \delta (\infty; f) = \sum_{a \neq \infty} \delta (a; f) = 1 \). Then the conclusions of Lemma 3 and Lemma 4 can also be drawn under the hypothesis \( \delta (\infty; f) = 1 \).

Lemma 5. [3] Let \( k \) be any positive integer and \( \psi = \sum_{i=0}^{k} a_i f^{(i)} \), where \( a_i \) are meromorphic functions such that \( T (r, a_i) = S (r, f) \), for \( i = 0, 1, 2, \ldots, k \). Then \( m (r, \psi) = S (r, f) \).

3 Theorems.

In this section we present the main results of the paper.

Theorem 1. Let \( f \) be a meromorphic function of finite order \( \rho_f \) and \( a' \) be any non-zero finite complex number. Then
\[
\delta (0; f) + \Delta^{P_0}_R (\infty; f) + \delta (a; f) \leq (2 \gamma P_0 - 1) \Delta (\infty; f) + \Delta^{P_0}_R (0; f).
\]

Proof. Let us consider the following identity
\[
a \frac{f}{f} = 1 - \frac{f - a P_0 [f]}{P_0 [f]}.
\]

Since \( m (r, \frac{1}{f}) \leq m (r, \frac{a}{f}) + O (1) \), in view of Lemma 5 we get from the above identity
\[
m (r, \frac{1}{f}) \leq m \left( r, \frac{f - a P_0 [f]}{P_0 [f]} \right) + m \left( r, \frac{P_0 [f]}{f^{\gamma P_0-1}} \right).
\]
i.e.,
\[
m (r, \frac{1}{f}) \leq m \left( r, \frac{f - a P_0 [f]}{P_0 [f]} \right) + (\gamma P_0 - 1) m (r, f) + S (r, f). \quad (1)
\]

Now by Nevanlinna’s first fundamental theorem and by Lemma 5 it follows from (1) that
\[
m \left( r, \frac{1}{f} \right) \leq T \left( r, \frac{f - a P_0 [f]}{P_0 [f]} \right) - N \left( r, \frac{f - a P_0 [f]}{P_0 [f]} \right) + (\gamma P_0 - 1) m (r, f) + S (r, f)
\]
i.e.,
\[
m \left( r, \frac{1}{f} \right) \leq T \left( r, \frac{P_0 [f]}{f - a} \right) - N \left( r, \frac{f - a P_0 [f]}{P_0 [f]} \right) + (\gamma P_0 - 1) m (r, f) + S (r, f)
\]
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\[ i.e., m\left(r, \frac{1}{f}\right) \leq N\left(r, P_0[f]\right) + m\left(r, \frac{P_0[f]}{(f-a)P_0}\right) + (\gamma P_0 - 1)m(r, f-a) - N\left(r, \frac{f-a}{P_0[f]}\right) + (\gamma P_0 - 1)m(r, f) + S(r, f) \]

\[ i.e., m\left(r, \frac{1}{f}\right) \leq N\left(r, P_0[f]\right) - N\left(r, \frac{1}{f-a}\right) + 2(\gamma P_0 - 1)m(r, f) + S(r, f) \]

In view of \{p.34, [3]\} it follows from (2) that

\[ m\left(r, \frac{1}{f}\right) \leq N\left(r, P_0[f]\right) + N\left(r, \frac{1}{f-a}\right) - N\left(r, f-a\right) - N\left(r, \frac{1}{P_0[f]}\right) + 2(\gamma P_0 - 1)m(r, f) + S(r, f) \]

\[ i.e., \liminf_{r \to \infty} \frac{m\left(r, \frac{1}{f}\right)}{T(r, f)} \leq \liminf_{r \to \infty} \left\{ \frac{N\left(r, P_0[f]\right)}{T(r, f)} - \frac{N\left(r, f\right)}{T(r, f)} - \frac{N\left(r, \frac{1}{f-a}\right)}{T(r, f)} \right\} + \limsup_{r \to \infty} \frac{N\left(r, \frac{1}{f-a}\right)}{T(r, f)} + 2(\gamma P_0 - 1) \limsup_{r \to \infty} \frac{m(r, f)}{T(r, f)} \]

\[ i.e., \liminf_{r \to \infty} \frac{m\left(r, \frac{1}{f}\right)}{T(r, f)} \leq \liminf_{r \to \infty} \frac{N\left(r, P_0[f]\right)}{T(r, f)} - \liminf_{r \to \infty} \frac{N\left(r, f\right)}{T(r, f)} - \liminf_{r \to \infty} \frac{N\left(r, \frac{1}{f-a}\right)}{T(r, f)} + \limsup_{r \to \infty} \frac{N\left(r, \frac{1}{f-a}\right)}{T(r, f)} + 2(\gamma P_0 - 1) \limsup_{r \to \infty} \frac{m(r, f)}{T(r, f)} \]

\[ i.e., \delta(0; f) \leq \{1 - \Delta^P_R(\infty; f)\} - \{1 - \Delta(\infty; f)\} - \{1 - \Delta^P_R(0; f)\} + \{1 - \delta(a; f)\} + 2(\gamma P_0 - 1) \Delta(\infty; f) \]

\[ i.e., \delta(0; f) + \Delta^P_R(\infty; f) + \delta(a; f) \leq (2\gamma P_0 - 1) \Delta(\infty; f) + \Delta^P_R(0; f) \]

This proves the theorem.
Remark 3. The sign ‘\( \leq \)’ in Theorem 1 cannot be replaced by ‘\(<\)’ only. This is evident from the following example.

Example 1. Let \( f = \exp z \), \( n_{01} = 1 \) and except for \( i = 0, j = 1 \); all other \( n_{ij} \) for each \( j \) and for \( i = 0, 1, 2, \ldots k \).

Then \( \Delta(\infty; f) = \Delta^{P_0}(0; f) = \Delta^{P_0}(\infty; f) = 1 \)
and \( \delta(0; f) = \delta(\infty; f) = 1. \)

So \( \delta(a; f) = 0. \) Also \( \gamma_{P_0} = 1. \)

Then \( (2\gamma_{P_0} - 1) \Delta(\infty; f) + \Delta^{P_0}(0; f) = 2. \)

Theorem 2. If \( f \) be a transcendental meromorphic function with \( \rho_f < \infty \) and \( \sum_{a \neq \infty} \Theta(a; f) = 2 \) then

\[ \delta(\infty; f) + \Delta^{P_0}(\infty; f) + \delta(0; f) \leq \gamma_{P_0} \Delta(\infty; f) + \Delta^{P_0}(0; f) + \Delta^{P_0}(\infty; f) \Gamma_{P_0} \]

Proof. Since \( f = P_0[f] \frac{f}{P_0[f]} \) we get that

\[ m(r, f) \leq m(r, P_0[f]) + m \left( r, \frac{f}{P_0[f]} \right). \tag{3} \]

Now by Nevanlinna’s first fundamental theorem and by Lemma 4 we obtain from (3) that

\[ m(r, f) \leq m(r, P_0[f]) + T \left( r, \frac{f}{P_0[f]} \right) - N \left( r, \frac{f}{P_0[f]} \right) \]
\[ + \gamma_{P_0} - 1 \right) m(r, f) - N \left( r, \frac{f}{P_0[f]} \right) + O(1). \tag{4} \]

Now in view of \( \{p.34, [3]\} \) it follows from (4) that

\[ m(r, f) \leq m(r, P_0[f]) + N(r, P_0[f]) + N \left( r, \frac{1}{f} \right) - N(r, f) \]
\[ - N \left( r, \frac{1}{P_0[f]} \right) + (\gamma_{P_0} - 1) m(r, f) + S(r, f) + O(1) \]
i.e., \( \lim \inf_{r \to \infty} \frac{m(r, f)}{T(r, f)} \leq \lim \inf_{r \to \infty} \left\{ \frac{N(r, P_0[f])}{T(r, f)} - \frac{N(r, f)}{T(r, f)} - \frac{N\left(r, \frac{1}{P_0[f]}\right)}{T(r, f)} \right\} \)

\[ + \lim \sup_{r \to \infty} \left\{ \frac{N\left(r, \frac{1}{f}\right)}{T(r, f)} + \frac{m(r, P_0[f])}{T(r, f)} + (\gamma_{P_0} - 1) \frac{m(r, f)}{T(r, f)} \right\} \]

i.e., \( \lim \inf_{r \to \infty} \frac{m(r, f)}{T(r, f)} \leq \lim \inf_{r \to \infty} \frac{N(r, P_0[f])}{T(r, f)} - \lim \inf_{r \to \infty} \frac{N(r, f)}{T(r, f)} - \lim \inf_{r \to \infty} \frac{N\left(r, \frac{1}{P_0[f]}\right)}{T(r, f)} \)

\[ + \lim \sup_{r \to \infty} \frac{N\left(r, \frac{1}{f}\right)}{T(r, f)} + \lim \sup_{r \to \infty} \frac{m(r, P_0[f])}{T(r, f)} + (\gamma_{P_0} - 1) \lim \sup_{r \to \infty} \frac{m(r, f)}{T(r, f)} \quad (5) \]

Now by Lemma 1 we obtain from (5) that
\[ \delta(\infty; f) \leq \{1 - \Delta_{\rho_0}^R(\infty; f)\} - \{1 - \Delta(\infty; f)\} - \{1 - \Delta_{\rho_0}^R(0; f)\} \]

\[ + \{1 - \delta(0; f)\} + \lim \sup_{r \to \infty} \frac{m(r, P_0[f])}{T(r, P_0[f])} \lim_{r \to \infty} \frac{T(r, P_0[f])}{T(r, f)} + (\gamma_{P_0} - 1) \Delta(\infty; f) \]

i.e., \( \delta(\infty; f) + \Delta_{\rho_0}^R(\infty; f) + \delta(0; f) \leq \gamma_{P_0} \Delta(\infty; f) + \Delta_{\rho_0}^P(0; f) + \Delta_{\rho_0}^P(\infty; f) \Gamma_{P_0}. \)

Thus the theorem is established.

Using the first part of Lemma 3, we may establish the next theorem under the same conditions in Theorem 2.

**Theorem 3.** Let \( f \) be a transcendental meromorphic function of finite order \( \rho_f \) and \( \sum_{a \neq \infty} \Theta(a; f) = 2. \) Then
\[ \delta(\infty; f) + \delta(0; f) + 1 \leq \gamma_{P_0} \Delta(\infty; f) + \Delta_{\rho_0}^P(0; f) + \Gamma_{P_0}. \]
Proof. Using the first part of Lemma 3 and the inequality (5) it follows that

\[
\delta(\infty; f) \leq \left\{ 1 - \Delta_P^R(\infty; f) \right\} - \left\{ 1 - \Delta(\infty; f) \right\} - \left\{ 1 - \Delta_P^R(0; f) \right\} \\
+ \left\{ 1 - \delta(0; f) \right\} + \Delta_P^R(\infty; f) - (1 - \Gamma_P^R) + (\gamma_P^R - 1) \Delta(\infty; f)
\]

i.e., \( \delta(\infty; f) + \delta(0; f) + 1 \leq \gamma_P^R \Delta(\infty; f) + \Delta_P^R(0; f) + \Gamma_P^R. \)

Thus the theorem is proved.

Theorem 4. If \( f \) be a transcendental meromorphic function with \( \rho_f < \infty \), \( \delta(\infty; f) = 1 \) and \( \Theta(\infty; f) = \sum_{a \neq \infty} \delta_p(a; f) = 1 \) then

\[
1 + \Delta_P^R(\infty; f) + \delta(0; f) \leq \Delta_P^R(0; f) + \gamma_P^R \Delta_P^R(\infty; f) + \gamma_P^R.
\]

The proof of the theorem is omitted because it can be carried out in the line of Theorem 2 and with the help of Lemma 2.

Remark 4. If we omit the condition \( \delta(\infty; f) = 1 \) of Theorem 4 and the other conditions remaining the same, using the first part of Lemma 4 we may establish the following theorem without proof.

Theorem 5. Let \( f \) be a transcendental meromorphic function of finite order \( \rho_f \) and \( \Theta(\infty; f) = \sum_{a \neq \infty} \delta_p(a; f) = 1 \). Then

\[
\delta(\infty; f) + \delta(0; f) + 1 \leq \gamma_P^R \Delta(\infty; f) + \Delta_P^R(0; f) + \gamma_P^R.
\]

Theorem 6. Let \( a, b \neq 0, \infty \) be any two distinct complex numbers. Then for any meromorphic function \( f \) of finite order \( \rho_f \),

\[
2 \delta(a; f) + \delta(b; f) + 2 \Delta_P^R(\infty; f) \leq (3 \gamma_P^R - 1) \Delta(\infty; f) + 2 \Delta_P^R(0; f).
\]

Proof. Considering the identity

\[
\frac{b - a}{f - a} = \frac{P_0[f]}{f - a} \left\{ \frac{f - a}{P_0[f]} - \frac{f - b}{P_0[f]} \right\},
\]

we obtain in view of Lemma 5 that

\[
m\left( r, \frac{b - a}{f - a} \right) \leq m\left( r, \frac{f - a}{P_0[f]} \right) + m\left( r, \frac{f - b}{P_0[f]} \right) + m\left( r, \frac{P_0[f]}{(f - a)^{\gamma_P^R}} \right) \\
+ (\gamma_P^R - 1) m\left( r, f - a \right)
\]
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\[ i.e., \, m\left( r, \frac{b-a}{f-a} \right) \leq T \left( r, \frac{f-a}{P_0[f]} \right) - N \left( r, \frac{f-a}{P_0[f]} \right) + T \left( r, \frac{f-b}{P_0[f]} \right) - N \left( r, \frac{f-b}{P_0[f]} \right) + (\gamma_{P_0} - 1) \, m \left( r, f \right) + S \left( r, f \right). \]  

(6)

Since \( m\left( r, \frac{1}{f-a} \right) \leq m\left( r, \frac{b-a}{f-a} \right) + O(1) \) and \( T \left( r, f \right) = T \left( r, \frac{1}{f} \right) + O(1) \), it follows from (6) that

\[ m\left( r, \frac{1}{f-a} \right) \leq T \left( r, \frac{P_0[f]}{f-a} \right) - N \left( r, \frac{f-a}{P_0[f]} \right) + T \left( r, \frac{P_0[f]}{f-b} \right) - N \left( r, \frac{f-b}{P_0[f]} \right) + (\gamma_{P_0} - 1) \, m \left( r, f \right) + S \left( r, f \right) + O(1) \]

i.e.,

\[ m\left( r, \frac{1}{f-a} \right) \leq N \left( r, \frac{P_0[f]}{f-a} \right) + m\left( r, \frac{P_0[f]}{(f-a)^{\gamma_{P_0}}} \right) + (\gamma_{P_0} - 1) \, m \left( r, f-a \right) - N \left( r, \frac{f-a}{P_0[f]} \right) + N \left( r, \frac{P_0[f]}{f-b} \right) + m\left( r, \frac{P_0[f]}{(f-b)^{\gamma_{P_0}}} \right) + (\gamma_{P_0} - 1) \, m \left( r, f-b \right) - N \left( r, \frac{f-b}{P_0[f]} \right) + (\gamma_{P_0} - 1) \, m \left( r, f \right) + S \left( r, f \right) + O(1) \]

i.e.,

\[ m\left( r, \frac{1}{f-a} \right) \leq N \left( r, \frac{P_0[f]}{f-a} \right) - N \left( r, \frac{f-a}{P_0[f]} \right) + N \left( r, \frac{P_0[f]}{f-b} \right) - N \left( r, \frac{f-b}{P_0[f]} \right) + 3 \left( \gamma_{P_0} - 1 \right) \, m \left( r, f \right) + S \left( r, f \right). \]  

(7)
In view of \{p.34, [3]\} we get from (7) that

\[
m \left( r, \frac{1}{f-a} \right) \leq N \left( r, P_0[f] \right) + N \left( r, \frac{1}{f-a} \right) - N \left( r, f - a \right) - N \left( r, \frac{1}{P_0[f]} \right) + N \left( r, \frac{1}{f-a} \right)
\]

\[
+ N \left( r, P_0[f] \right) + N \left( r, \frac{1}{f-b} \right) - N \left( r, f - b \right)
\]

\[
- N \left( r, \frac{1}{P_0[f]} \right) + 3 (\gamma_{P_0} - 1) m(r, f) + S(r, f)
\]

i.e.,

\[
m \left( r, \frac{1}{f-a} \right) \leq 2 N \left( r, P_0[f] \right) - 2 N \left( r, f \right) - 2 N \left( r, \frac{1}{P_0[f]} \right) + N \left( r, \frac{1}{f-a} \right) + N \left( r, \frac{1}{f-b} \right) + 3 (\gamma_{P_0} - 1) m(r, f)
\]

\[
+ S(r, f)
\]

i.e.,

\[
\liminf_{r \to \infty} \frac{m \left( r, \frac{1}{f-a} \right)}{T(r,f)} \leq 2 \liminf_{r \to \infty} \left\{ \frac{N \left( r, P_0[f] \right)}{T(r,f)} - \frac{N \left( r, f \right)}{T(r,f)} - \frac{N \left( r, \frac{1}{P_0[f]} \right)}{T(r,f)} \right\}
\]

\[
+ \limsup_{r \to \infty} \left\{ \frac{N \left( r, \frac{1}{f-a} \right)}{T(r,f)} + \frac{N \left( r, \frac{1}{f-b} \right)}{T(r,f)} + 3 (\gamma_{P_0} - 1) \frac{m(r, f)}{T(r,f)} \right\}
\]

i.e.,

\[
\liminf_{r \to \infty} \frac{m \left( r, \frac{1}{f-a} \right)}{T(r,f)}
\]
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\[
\leq 2 \left\{ \liminf_{r \to \infty} \frac{N(r, P_0 [f])}{T(r, f)} - \liminf_{r \to \infty} \frac{N(r, f)}{T(r, f)} - \liminf_{r \to \infty} \frac{N(r, \frac{1}{P_0 [f]})}{T(r, f)} \right\}
\]

\[
+ \limsup_{r \to \infty} \frac{N(r, \frac{1}{f-a})}{T(r, f)} + \limsup_{r \to \infty} \frac{N(r, \frac{1}{f-b})}{T(r, f)} + 3 (\gamma P_0 - 1) \limsup_{r \to \infty} \frac{m(r, f)}{T(r, f)}
\]

i.e., \( \delta(a; f) \leq 2 \{ 1 - \Delta P_0^R (\infty; f) \} - 2 \{ 1 - \Delta (\infty; f) \} - 2 \{ 1 - \Delta P_0^R (0; f) \} + \{ 1 - \delta(a; f) \} + \{ 1 - \delta(b; f) \} + 3 (\gamma P_0 - 1) \Delta (\infty; f) \)

i.e., \( 2 \delta(a; f) + \delta(b; f) + 2 \Delta P_0^R (\infty; f) \leq (3 \gamma P_0 - 1) \Delta (\infty; f) + 2 \Delta P_0^R (0; f) \).

This proves the theorem.

**Theorem 7.** Let \( a \) be a finite complex number and \( b, c \) be two distinct non-zero complex numbers. Then for any meromorphic function \( f \) with finite order \( \rho_f \) and \( \sum_{a \neq \infty} \Theta(a; f) = 2 \),

\[
\delta(a; f) + \Gamma P_0 \{ \delta P_0^R (b; f) + \delta P_0^R (c; f) \} \leq (\gamma P_0 - 1) \Delta (\infty; f) + 2 \Gamma P_0.
\]

**Proof.** Since \( \frac{1}{f-a} = \frac{P_0 [f]}{f-a} \frac{1}{P_0 [f]} \), by Lemma 5 we obtain that

\[
m(r, \frac{1}{f-a}) \leq m(r, \frac{1}{P_0 [f]}) + m(r, \frac{P_0 [f]}{(f-a) P_0}) + (\gamma P_0 - 1) m(r, f)
\]

i.e., \( m(r, \frac{1}{f-a}) \leq m(r, \frac{1}{P_0 [f]}) + (\gamma P_0 - 1) m(r, f) + S(r, f) \). (8)

Applying Nevanlinna’s first fundamental theorem we get from (8) that

\[
m(r, \frac{1}{f-a}) \leq T(r, \frac{1}{P_0 [f]}) - N(r, \frac{1}{P_0 [f]}) + (\gamma P_0 - 1) m(r, f) + S(r, f).
\]

(9)
Now by Nevanlinna’s second fundamental theorem it follows from (9) that
\[
m(r, \frac{1}{f - a}) \leq \bar{N}(r, \frac{1}{P_0[f]}) + \bar{N}(r, \frac{1}{P_0[f] - b}) + \bar{N}(r, \frac{1}{P_0[f] - c}) - N(r, \frac{1}{P_0[f]}) + (\gamma P_0 - 1)m(r, f) + S(r, f).
\] (10)

Since \(\bar{N}(r, \frac{1}{P_0[f]}) - N(r, \frac{1}{P_0[f]}) \leq 0\), we obtain from (10) in view of Lemma 1 that
\[
m(r, \frac{1}{f - a}) \leq \bar{N}(r, \frac{1}{P_0[f] - b}) + \bar{N}(r, \frac{1}{P_0[f] - c}) + (\gamma P_0 - 1)m(r, f) + S(r, f)
\]
i.e., \(m(r, \frac{1}{f - a}) \leq \bar{N}(r, \frac{1}{P_0[f] - b}) + \bar{N}(r, \frac{1}{P_0[f] - c}) + (\gamma P_0 - 1)m(r, f) + S(r, f)
\]
i.e., \(m(r, \frac{1}{f - a}) \leq T(r, \frac{1}{P_0[f] - b}) + T(r, \frac{1}{P_0[f] - c}) - m(r, \frac{1}{P_0[f] - b}) - m(r, \frac{1}{P_0[f] - c}) + (\gamma P_0 - 1)m(r, f) + S(r, f)
\]
i.e., \(m(r, \frac{1}{f - a}) \leq 2T(r, P_0[f]) - m(r, \frac{1}{P_0[f] - b}) - m(r, \frac{1}{P_0[f] - c}) + (\gamma P_0 - 1)m(r, f) + S(r, f)
\]
i.e., \(\liminf_{r \to \infty} \frac{m(r, \frac{1}{f - a})}{T(r, f)} \leq 2\liminf_{r \to \infty} \frac{T(r, P_0[f])}{T(r, f)} - \liminf_{r \to \infty} \frac{m(r, \frac{1}{P[f] - b})}{T(r, f)} - \liminf_{r \to \infty} \frac{m(r, \frac{1}{P[f] - c})}{T(r, f)} + (\gamma P_0 - 1)\limsup_{r \to \infty} \frac{m(r, f)}{T(r, f)}
\)
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i.e., \( \liminf_{r \to \infty} \frac{m(r, \frac{1}{f-a})}{T(r, f)} \leq 2 \liminf_{r \to \infty} \frac{T(r, P_0 [f])}{T(r, f)} - \liminf_{r \to \infty} \frac{m(r, \frac{1}{f-b})}{T(r, P_0 [f])} \liminf_{r \to \infty} \frac{T(r, P_0 [f])}{T(r, f)} \)

\[ \text{i.e., } \delta(a; f) \leq 2 \Gamma_{P_0} - \delta^P_A (b; f) \cdot \Gamma_{P_0} - \delta^P_A (c; f) \cdot \Gamma_{P_0} + (\gamma_{P_0} - 1) \Delta(\infty; f) \]

\[ \text{i.e., } \delta(a; f) + \Gamma_{P_0} \left\{ \delta^P_A (b; f) + \delta^P_A (c; f) \right\} \leq (\gamma_{P_0} - 1) \Delta(\infty; f) + 2 \Gamma_{P_0}. \]

Thus the theorem is established.

In the line of Theorem 7 we may state the following theorem without proof.

**Theorem 8.** Let \( a' \) be a finite complex number and \( b, c \) be two distinct non-zero complex numbers. Then for any meromorphic function \( f \) with finite order \( \rho_f \) and \( \Theta(\infty; f) = \sum_{a \neq \infty} \delta_p(a; f) = 1 \),

\[ \delta(a; f) + \gamma_{P_0} \left\{ \delta^P_A (b; f) + \delta^P_A (c; f) \right\} \leq (\gamma_{P_0} - 1) \Delta(\infty; f) + 2 \gamma_{P_0}. \]

**References**


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