Neighborhoods of Certain Classes of Analytic Functions of Complex Order

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Abstract

By making use of the familiar concept of neighborhoods of analytic functions, the authors prove several inclusion relations associated with the \((n, \delta)\)-neighborhood of certain subclasses of analytic functions of complex order.

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1 Introduction and Motivation

Let \(A\) be the class of functions \(f\) normalized by

\[
f(z) = z + \sum_{n=2}^{\infty} a_n z^n,
\]

which are analytic in the open unit disk \(D = \{z \in C : |z| < 1\}\).

As usual, we denote by \(S\) the subclass of \(A\), consisting of functions which are also univalent in \(D\). We recall here the definitions of the wellknown classes of starlike function and convex functions respectively:

\[
S^* = \left\{ f \in A : \text{Re} \left( \frac{zf'(z)}{f(z)} \right) > 0, z \in D \right\},
\]

\[
K = \left\{ f \in A : \text{Re} \left( 1 + \frac{zf''(z)}{f'(z)} \right) > 0, z \in D \right\}.
\]
Let \( w \) be a fixed point in \( D \) and \( A(w) = \{ f \in H(D) : f(w) = f'(w) - 1 = 0 \} \).

In [18], Kanas and Ronning introduced the following classes

\[
S_w = \{ f \in A(w) : f \text{ is univalent in } D \}
\]

\[
SV_w^* = \left\{ f \in A(w) : \text{Re} \left( \frac{(z - w)f'(z)}{f(z)} \right) > 0, z \in D \right\}, \quad (2)
\]

\[
CV_w = \left\{ f \in A : 1 + \left( \text{Re} \left( \frac{(z - w)f''(z)}{f'(z)} \right) \right) > 0, z \in D \right\}. \quad (3)
\]

Later Acu and Owa [1] studied the classes extensively.

The class \( S_w^* \) is defined by geometric property that the image of any circular arc centered at \( w \) is starlike with respect to \( f(w) \) and the corresponding class \( S_w^c \) is defined by the property that the image of any circular arc centered at \( w \) is convex. We observed that the definitions are somewhat similar to the ones introduced by Goodman in [15] and [16] for uniformly starlike and convex functions, except that in this case the point \( w \) is fixed.

Let \( \Sigma_w \) denoted the subclass of \( A(w) \) consisting of the function of the form

\[
f(z) = \frac{1}{z - w} + \sum_{n=1}^{\infty} a_n(z - w)^n, \quad (a_n \geq 0, \ z \neq w). \quad (4)
\]

The function \( f \) in \( \Sigma_w \) is said to be starlike function of order \( \beta \) if and only if

\[
\text{Re} \left\{ -\frac{(z - w)f'(z)}{f(z)} \right\} > \beta \quad (z \in D) \quad (5)
\]

for some \( \beta (0 \leq \beta < 1) \). We denote by \( S_w^*(\beta) \) the class of all starlike functions of order \( \beta \). Similarly, a function \( f \) in \( \Sigma_w \) is said to be convex of order \( \beta \) if and only if

\[
\text{Re} \left\{ -1 - \frac{(z - w)f''(z)}{f'(z)} \right\} > \beta \quad (z \in D) \quad (6)
\]

for some \( \beta (0 \leq \beta < 1) \). We denote by \( C_w(\beta) \) the class of all convex functions of order \( \beta \).

We note that the class \( S_w^0(\beta) \) and various other subclasses of \( S_w^*(\beta) \) have been studied rather extensively by Nehari and Netanyahu [23], Acu and Owa [1], Clunie [8], Pommerenke[25,24], Miller[21], Royster [26], and others (cf.,
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For function $f$ to be in the class $\Sigma_w$, let us define the following:

$$I^0 f(z) = f(z),$$

$$I^1 f(z) = (z - w)f'(z) + \frac{2}{z - w},$$

$$I^2 f(z) = (z - w)\left(I^1 f(z)\right)' + \frac{2}{z - w},$$

and for $k=1,2,3,...$ we can write

$$I^k f(z) = (z - w)\left(I^{k-1} f(z)\right)'+ \frac{2}{z - w} = \frac{1}{z - w} + \sum_{n=1}^{\infty} n^k a_n(z - w)^n. \quad (7)$$

The differential operator $I^k$ was studied extensively by Ghanim and Darus [[11],[12],[13]], and in the case $w = 0$ was given by Frasin and Darus [10].

Further, Let $T_w^*$ denoted the subclass of $\Sigma_w$ consisting functions of the form

$$f(z) = \frac{1}{z - w} - \sum_{n=1}^{\infty} a_n(z - w)^n, \quad (a_n \geq 0). \quad (8)$$

Now, we define the $(n, \delta)$- neighborhood of the function $f \in T_w^*$ by

$$N_{n,\delta} (f; g) \quad (9)$$

$$= \left\{ g \in T_w^* : g(z) = \frac{1}{z - w} - \sum_{n=1}^{\infty} b_n(z - w)^n \quad and \quad \sum_{n=1}^{\infty} n |a_n - b_n| \leq \delta \right\}. \quad (10)$$

In particular, for the identity function

$$e(z) = \frac{1}{z - w} \quad (10)$$

we immediately have

$$N_{n,\delta} (e; g) \quad (11)$$
\[ g \in T_w : g(z) = \frac{1}{z-w} - \sum_{n=1}^{\infty} b_n (z-w)^n \text{ and } \sum_{n=1}^{\infty} n |b_n| \leq \delta. \]

There are many similar work that have been done in neighborhood like Goodman [17], Ruscheweyh [27] and also see ([2], [3], [4], and [28]).

The above concept of \((n, \delta)-neighborhoods\) was extended and applied recently to families of meromorphically multivalent functions by Liu and Srivastava ([19] and [20]). The main object of the present paper is to investigate the \((n, \delta)-neighborhoods\) of several subclasses of the class \(\Sigma_w\) of normalized analytic functions in \(D\) with missing coefficients.

Next, for the functions \(f_j (j = 1, 2)\) given by

\[ f_j(z) = \frac{1}{z-w} - \sum_{n=1}^{\infty} a_{n,j} (z-w)^n \]  

(12)

let \(f_1 * f_2\) denote the Hadamard product (or convolution) of \(f_1\) and \(f_2\), defined by

\[ (f_1 * f_2)(z) = \frac{1}{z-w} - \sum_{n=1}^{\infty} a_{n,1} a_{n,2} (z-w)^n = (f_2 * f_1)(z). \]  

(13)

Let \(T_w^* (\beta, k, \alpha)\) denote the subclass of \(T_w^*\) consisting of functions \(f\) which satisfy the following inequality:

\[ \left| \frac{1}{\beta} \left( \frac{I^{k+1}f(z)}{I^k f(z)} - 1 \right) \right| < \alpha \]  

(14)

\(((z-w)) \in D; \beta \in C \setminus \{0\}; k \geq 0; 0 < \alpha \leq 1).\]

In addition, let \(T_w^* (\beta, k, \alpha; \mu)\) denote the subclass of \(T_w^*\) consisting of functions \(f\) which satisfy the following inequality:

\[ \left| \frac{1}{\beta} \left( (1-\mu) \frac{I^k f(z)}{z-w} + \mu \left( I^{k+1} f(z) \right) - 1 \right) \right| < \alpha \]  

(15)

\(((z-w)) \in D; \beta \in C \setminus \{0\}; k \geq 0; 0 < \alpha \leq 1; 0 \leq \mu \leq 1).\]

2 Inclusion Relations Involving \(N_{n,\delta}(e)\)

In our investigation of the inclusion relations involving \(N_{n,\delta}(e)\), we shall require Theorem 2.1 and Theorem 2.2 below:
Theorem 2.1 Let the function \( f \in T_w^* \) be defined by (8). Then \( f \) is in the class \( T_w^*(\beta, k, \alpha) \) if and only if

\[
\sum_{n=1}^{\infty} n^k (n - 1 + \alpha |\beta|) a_n \leq \alpha |\beta|.
\]  

(16)

Proof: We first suppose that \( f \in T_w^*(\beta, k, \alpha) \). Then by appealing to the condition (14), we readily obtain

\[
\Re \left( \frac{I^{k+1} f(z)}{I^k f(z)} - 1 \right) > -\beta \alpha
\]

(17)
or, equivalently

\[
\Re \left( \frac{- \left( \sum_{n=1}^{\infty} n^{k+1} a_n (z - w)^n - \sum_{n=1}^{\infty} n^k a_n (z - w)^n \right)}{1 - \sum_{n=1}^{\infty} n^k a_n (z - w)^n} \right) > -\beta \alpha.
\]  

(18)

We now choose values of \((z - w)\) on the real axis and let \((z - w) \rightarrow 1\) through real values. Then the inequality (18) immediately yields the desired condition (16).

Conversely, by applying the hypothesis (16) and letting \(|z - w| = 1\), we find that

\[
\frac{|I^{k+1} f(z)|}{|I^k f(z)| + 1} = \frac{- \sum_{n=1}^{\infty} n^{k+1} a_n (z - w)^n + \sum_{n=1}^{\infty} n^k a_n (z - w)^n}{1 - \sum_{n=1}^{\infty} n^k a_n (z - w)^n}
\]

\[
\leq \frac{\sum_{n=1}^{\infty} n^{k+1} a_n |z - w|^n - \sum_{n=1}^{\infty} n^k a_n |z - w|^n}{1 - \sum_{n=1}^{\infty} n^k a_n |z - w|^n}
\]

\[
\leq \frac{\sum_{n=1}^{\infty} n^{k+1} a_n - \sum_{n=1}^{\infty} n^k a_n}{1 - \sum_{n=1}^{\infty} n^k a_n}
\]

\[
\leq |\beta| \alpha.
\]

This implies

\[
\sum_{n=1}^{\infty} n^{k+1} a_n - \sum_{n=1}^{\infty} n^k a_n + \alpha \beta \sum_{n=1}^{\infty} n^k a_n \leq \alpha |\beta|.
\]

Thus

\[
\sum_{n=1}^{\infty} n^k (n - 1 + \alpha \beta) a_n \leq \alpha |\beta|
\]

Then, we have \( f \in T_w^*(\beta, k, \alpha) \).

Similarly, we can prove the following result:
Theorem 2.2 Let the function $f \in T_w^*$ be defined by (8). Then $f$ is in the class $T_w^*(\beta, k, \alpha; \mu)$ if and only if
\[
\sum_{n=1}^{\infty} n^k (n\mu - \mu + 1) a_n \leq \alpha |\beta| \quad (19)
\]

Proof: We first suppose that $f \in T_w^*(\beta, k, \alpha; \mu)$. Then by appealing to the condition (15), we readily obtain
\[
\Re \left( (1 + \mu) \frac{I^k f(z)}{z - w} + \mu \left( I^{k+1} f(z) - 1 \right) \right) < \alpha |\beta|
\]
or, equivalently
\[
\Re \left\{ \frac{1}{z - w} + \sum_{n=1}^{\infty} n^k a_n (z - w)^n - \mu \sum_{n=1}^{\infty} n^k a_n (z - w)^n \right. \\
+ \mu \sum_{n=1}^{\infty} n^{k+1} a_n (z - w)^n - (z - w) \left. \right\} < \alpha |\beta| |z - w| \quad (20)
\]

We now choose values of $(z - w)$ on the real axis and let $(z - w) \to 1$ through real values. Then the inequality (20) immediately yields the desired condition (19).

Conversely, by applying the hypothesis (19) and letting $|z - w| = 1$, we find that
\[
\left| \left( (1 - \mu) \frac{I^k f(z)}{z - w} + \mu \left( I^{k+1} f(z) - 1 \right) \right) \right| \\
= \left| \left( \frac{1}{z - w} + \sum_{n=1}^{\infty} n^k a_n (z - w)^n - \mu \sum_{n=1}^{\infty} n^k a_n (z - w)^n \right. \\
+ \mu \sum_{n=1}^{\infty} n^{k+1} a_n (z - w)^n - (z - w) \left. \right| \right| \\
\leq \left( \frac{1}{|z - w|} + \sum_{n=1}^{\infty} n^k a_n |z - w|^n - \mu \sum_{n=1}^{\infty} n^k a_n |z - w|^n \\
+ \mu \sum_{n=1}^{\infty} n^{k+1} a_n |z - w|^n - |z - w| \right)
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\[ \sum_{n=1}^{\infty} n^k a_n - \mu \sum_{n=1}^{\infty} n^k a_n + \mu \sum_{n=1}^{\infty} n^{k+1} a_n < \alpha |\beta|. \]

Thus, we have

\[ \sum_{n=1}^{\infty} n^k (n\mu - \mu + 1) a_n \leq \alpha |\beta| \]

Hence, we have \( f \in T_w^* (\beta, k, \alpha; \mu) \).

Our first inclusion relation involving \( N_{n,\delta} (e) \) is given by Theorem 2.3 below.

**Theorem 2.3** If

\[ \delta := \frac{\alpha |\beta|}{(n-1 + \alpha |\beta|)} \quad (|\beta| < 1) \quad (21) \]

then

\[ T_w^* (\beta, k, \alpha) \subset N_{n,\delta} (e). \]

**Proof:** For a function \( f \in T_w^* (\beta, k, \alpha) \) of the form (8), Theorem 2.1 immediately yields

\[ n^k (n - 1 + \alpha |\beta|) a_n \leq \sum_{n=1}^{\infty} n^k (n - 1 + \alpha |\beta|) a_n \]

so that

\[ a_n \leq \frac{\alpha |\beta|}{(n-1 + \alpha |\beta|)}. \quad (22) \]

On the other hand, we also find from (16) and (22) that

\[ na_n \leq \alpha |\beta| + (\alpha |\beta| - 1) a_n \]

\[ \leq \alpha |\beta| + (\alpha |\beta| - 1) \left( \frac{\alpha |\beta|}{n^k ((n - 1 + \alpha |\beta|))} \right) \]

\[ \leq \frac{\alpha |\beta|}{(n-1 + \alpha |\beta|)}, \quad (|\beta| < 1). \]

that is

\[ a_n \leq \frac{\alpha |\beta|}{(n-1 + \alpha |\beta|)} =: \delta \]

which, in view of the (10), proves Theorem 2.3.

Similarly, by applying Theorem 2.2 instead of Theorem 2.1, we now prove Theorem 2.4 below.
Theorem 2.4 If

\[ \delta := \frac{\alpha |\beta|}{n\mu - \mu + \alpha |\beta|} \]  

(23)

then

\[ \psi_w(\gamma, k, \alpha; \mu) \subset N_{n, \delta}(e). \]

Proof: Suppose that a function \( f \in \psi_w(\beta, k; \alpha; \mu) \) is of the form (8). Then we find from the assertion (19) of Theorem 2.2 that

\[ n^k \left(n\mu - \mu + \alpha |\beta|\right) a_n \leq \sum_{n=1}^{\infty} n^k \left(n\mu - \mu + \alpha |\beta|\right) a_n \]

which yields the following coefficient inequality:

\[ a_n \leq \frac{\alpha |\beta|}{n\mu - \mu + \alpha |\beta|} \]  

(24)

Making use of (19) in conjunction with (24), we also have

\[ na_n \leq \alpha |\beta| + (\mu - 1) a_n \]

\[ \leq \alpha |\beta| + (\mu - 1) \left( \frac{\alpha |\beta|}{(n\mu - \mu + \alpha |\beta|)} \right) \]

\[ \leq \frac{\alpha |\beta|}{(n\mu - \mu + \alpha |\beta|)} \]

That is,

\[ na_n \leq \frac{\alpha |\beta|}{n\mu - \mu + \alpha |\beta|} \]

which, in light of the definition (10), completes the proof of Theorem 2.2.

Remark 1: By suitably specializing the various parameters involved in Theorem 2.3 and Theorem 2.2, we can derive the corresponding inclusion relations for many relatively more familiar function classes.
3 Neighborhoods For The Classes $S^\lambda_w(\beta, k, \alpha)$ and $\psi^\lambda_w(\beta, k, \alpha; \mu)$

In this section we determine the neighborhood for each of the classes $S^\lambda_w(\beta, k, \alpha)$ and $\psi^\lambda_w(\beta, k, \alpha; \mu)$ which we define as follows. A function $f \in S_w$ is said to be in the class $S^\lambda_w(\beta, k, \alpha)$ if there exists a function $g \in S^\lambda_w(\beta, k, \alpha)$ such that
\[
\left| \frac{f(z)}{g(z)} - 1 \right| < 1 - \lambda, \quad (z \in D; 0 \leq \lambda < 1).
\] (25)

Analogously, a function $f \in S_w$ is said to be in the class $\psi^\lambda_w(\beta, k, \alpha; \mu)$ if there exists a function $g \in \psi_w(\beta, k, \alpha; \mu)$ such that the inequality (25) holds true.

**Theorem 3.1** If $g \in S_w(\beta, k, \alpha)$ and
\[
\lambda = 1 - \frac{\delta n^k (n - 1 - \alpha |\beta|)}{n [n^k (n - 1 - \alpha |\beta|) - \alpha |\beta|]}, \quad n = 1, 2, 3, \ldots.
\] (26)
then
\[
N_{n,\delta}(g) \subset S^\lambda_w(\beta, k, \alpha)
\]

**Proof:** Suppose that $f \in N_{n,\delta}(g)$. We then find from the definition (9) that
\[
n |a_n - b_n| \leq \sum_{n=1}^{\infty} n |a_n - b_n| \leq \delta
\] (27)
which readily implies the coefficient inequality:
\[
|a_n - b_n| \leq \frac{\delta}{n}, \quad n = 1, 2, 3, \ldots.
\] (28)

Next, since $g \in S_w(\beta, k, \alpha)$, we have [cf. Equation (22)]
\[
b_n \leq \frac{\alpha |\beta|}{n^k (n - 1 - \alpha |\beta|)}, \quad n = 1, 2, 3, \ldots.
\] (29)

Letting $|z - w| \to 1$, so we have
\[
\left| \frac{f(z)}{g(z)} - 1 \right| < \frac{\sum_{n=1}^{\infty} |a_n - b_n|}{1 - \sum_{n=1}^{\infty} b_n}
\]
\[ \leq \frac{\delta}{n} \cdot \frac{n^k (n - 1 - \alpha |\beta|)}{n^k (n - 1 - \alpha |\beta|) - \alpha |\beta|} = 1 - \lambda \]

provided that \( \lambda \) is given precisely by (26). Thus, by definition, \( f \in S^\lambda_w (\beta, k, \alpha) \) for \( \lambda \) given by (26).

This evidently completes our proof of Theorem 3.1.

Our proof of Theorem 3.2 below is much akin to that of Theorem 3.1.

**Theorem 3.2** If \( g \in \psi_w (\gamma, k, \alpha; \mu) \) and

\[ \lambda = 1 - \frac{\delta n^k (n\mu - \mu + 1)}{n [n^k (n\mu - \mu + 1) - \alpha |\beta|]} \], \[ n = 1, 2, 3, \ldots \] \( (30) \)

then

\[ N_{n,\delta} (g) \subset \psi^\lambda_w (\beta, k, \alpha; \mu). \]

**Proof:** Suppose that \( f \in N_{n,\delta} (g) \). We then find from the definition (7) that

\[ n |a_n - b_n| \leq \delta \] \( (31) \)

which readily implies the coefficient inequality:

\[ |a_n - b_n| \leq \frac{\delta}{n}, \quad n = 1, 2, 3, \ldots \] \( (32) \)

Next, since \( g \in \psi_w (\beta, k, \alpha; \mu) \), we have [cf. Equation (24)]

\[ b_n \leq \frac{\alpha |\beta|}{n^k (n\mu - \mu + 1)} \] \( (33) \)

Letting \( |z - w| \to 1 \), so we have

\[ \left| \frac{f (z)}{g (z)} - 1 \right| < \frac{\sum_{n=1}^{\infty} |a_n - b_n|}{1 - \sum_{n=1}^{\infty} b_n} \]

\[ \leq \frac{\delta}{n} \cdot \frac{n^k (n\mu - \mu + 1)}{n^k (n\mu - \mu + 1) - \alpha |\beta|} = 1 - \lambda \]

provided that \( \lambda \) is given precisely by (30). Thus, by definition, \( f \in \psi_w (\beta, k, \alpha; \mu) \) for \( \lambda \) given by (30).
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This evidently completes our proof of Theorem 3.2.

**Remark 2:** As we indicated previously in Section 2 and Remark 1, Theorem 3.1 and Theorem 3.2 can readily be specialized to deduce the corresponding neighborhood results for many simpler function classes.

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**References**


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