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# Integers for the Number of Maximal Independent Sets in Graphs

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**Abstract**

Let $G$ be a simple undirected graph. Denote by $\mathrm{mi}(G)$ (respectively, $\mathrm{xi}(G)$) the number of maximal (respectively, maximum) independent sets in $G$. In this paper we determine the third and fouth largest value of $\mathrm{mi}(G)$ among all graphs of order $n$. Moreover, the extremal graphs achieving these values are also determined.

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**Keywords:** maximal (maximum) independent set, extremal graph

## 1 Introduction

An *independent set* is a subset $S$ of $V(G)$ such that no two vertices in $S$ are adjacent in $G$. A *maximal independent set* is an independent set that is not a proper subset of any other independent set. A *maximum independent set* is an independent set of maximum size among all independent sets of $G$. Note that a maximum independent set is maximal but the converse does not always hold. Denote by $\mathrm{mi}(G)$ (respectively, $\mathrm{xi}(G)$) the number of maximal (respectively, maximum) independent sets in $G$.

Due to the important roles in designing algorithms, the enumeration of the maximal independent sets in graphs have been studied widely. Erdős and Moser raised the problem of determining the maximum value of $\mathrm{mi}(G)$ among all graphs of order $n$ and the extremal graphs achieving the maximum value. This problem was solved by Moon and Moser [17]. Later researchers focused on the problem for special classes of graphs: for connected graphs independently by Füredi [4] and Griggs et al. [5]; for trees independently by Meir and Moon [16], Sagan [18], and Wilf [20]; for forest by Jou and Chang [11]; for (connected) graphs with at most one cycle by Jou and Chang [11]; for bipartite graphs by Liu [15]; for triangle-free graphs by Hujter and Tuza [8] and for connected triangle-free graphs by Chang and Jou [1]. Recently, Sagan and Vatter[19] and Ying et al. [21] solved the problem for graphs with at most $r$ cycles. For other related, including algorithmic, results on $\mathrm{mi}(G)$, see [3, 6, 10, 12, 13]. Unlike the parameter $\mathrm{mi}(G)$, there are few results for the parameter $\mathrm{xi}(G)$, see [2, 7, 14].

## 2 Preliminary

**Proposition 2.1** [8] *For any vertex $x$ in a graph $G$,*
(1) $\mathrm{mi}(G) \leq \mathrm{mi}((G-x)+\mathrm{mi}((G-N[x])$;
(2) *If $x$ is a leaf adjacent to $y$, then* $\mathrm{mi}((G) = \mathrm{mi}((G-N[x])+\mathrm{mi}((G-N[y])$.

**Proposition 2.2** [4] *For any $n \geq 6$,* $\mathrm{mi}((C_n) = \mathrm{mi}((C_{n-2})+\mathrm{mi}((C_{n-3})$.

**Proposition 2.3** [8] *For any two vertex disjoint graphs $G$ and $H$,* $\mathrm{mi}((G \cup H) = \mathrm{mi}((G)\mathrm{mi}((H)$.

Here for two vertex disjoint graphs $G$ and $H$, we denote by $G \cup H$ the union of $G$ and $H$. For an integer $n \geq 2$, define the graph $G_1(n)$ as follows.

$$G_1(n) = \begin{cases} sK_3, & \text{if } n = 3s; \\ K_4 \cup (s-1)K_3, \text{ or } 2K_2 \cup (s-1)K_3, & \text{if } n = 3s+1; \\ K_2 \cup sK_3, & \text{if } n = 3s+2. \end{cases}$$

Let $g_1(n) = \mathrm{mi}(G_1(n))$. From the preceding propositions, we have

$$g_1(n) = \begin{cases} 3^s, & \text{if } n = 3s; \\ 4 \cdot 3^{s-1}, & \text{if } n = 3s+1; \\ 2 \cdot 3^s, & \text{if } n = 3s+2. \end{cases}$$

For any graph of order $n$, we have the following result.

**Theorem 2.4** [17] *If $G$ is a graph of order $n \geq 2$, then* $\mathrm{mi}((G) \leq g_1(n)$. *Furthermore, the equality holds if and only if $G \cong G_1(n)$.*

Denote by $K_m * K_n$ the graph obtained from $K_m \cup K_n$ by connecting a single vertex of one component to a single vertex of the other. For example, see the graph $K_4 * K_4$ as illustrated in Figure 1.

For $n \geq 6$, define the graph $G_2(n)$ as follows.

$$G_2(n) = \begin{cases} (K_3 * K_3) \cup (s-2)K_3, \text{ or } 3K_2 \cup (s-2)K_3, & \\ \text{or } K_4 \cup K_2 \cup (s-2)K_3, & \text{if } n = 3s; \\ (K_4 * K_3) \cup (s-2)K_3, & \text{if } n = 3s+1; \\ (K_3 * K_3) \cup (s-2)K_3 \cup K_2, \text{ or } 4K_2 \cup (s-2)K_3, & \\ \text{or } K_4 \cup 2K_2 \cup (s-2)K_3, \text{ or } 2K_4 \cup (s-2)K_3, & \text{if } n = 3s+2. \end{cases}$$

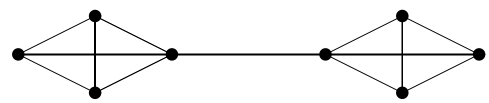

Figure 1: The graph $K_4 * K_4$

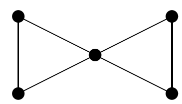

Figure 2: The graph $H_1$

Let $g_2(n) = \mathrm{mi}(G_2(n))$. From the preceding propositions, we have

$$g_2(n) = \begin{cases} \frac{11}{12} g_1(n), & \text{if } n = 3s+1; \\ \frac{8}{9} g_1(n), & \text{otherwise.} \end{cases}$$

Recently, Jin and Li [9] proved the following results for graphs of order $n$.

**Theorem 2.5** [9] *Let $G$ be a graph of order $n \geq 3$ and $G \ncong G_1(n)$. Then* $\mathrm{mi}((G) \leq g_2(n)$. *Furthermore, the equality holds if and only if $G \cong G_2(n)$.*

For $n \geq 6$, define the graph $G_3(n)$ as follows.

$$G_3(n) = \begin{cases} (K_4 * K_3) \cup (s-3)K_3 \cup K_2, \text{ or} & \\ (K_3 * 2K_3) \cup (s-3)K_3, & \text{if } n = 3s; \\ (K_3 * K_3) \cup (s-3)K_3 \cup 2K_2, \text{ or} & \\ (s-3)K_3 \cup 5K_2, \text{ or } 2K_4 \cup (s-3)K_3 \cup K_2, \text{ or} & \\ (K_3 * K_3) \cup (s-3)K_3 \cup K_4, & \text{if } n = 3s+1; \\ (K_3 * K_2) \cup (s-1)K_3, \text{ or } C_5 \cup (s-1)K_3, & \\ \text{or } H_1 \cup (s-1)K_3, \text{ or } K_5 \cup (s-1)K_3, & \text{if } n = 3s+2, \end{cases}$$

where $H_1$ is illustrated in Figure 2.

Let $g_3(n) = \mathrm{mi}(G_3(n))$. From the preceding propositions, we have

$$g_3(n) = \begin{cases} \frac{22}{27} g_1(n), & \text{if } n = 3s; \\ \frac{8}{9} g_1(n), & \text{if } n = 3s+1; \\ \frac{5}{6} g_1(n), & \text{if } n = 3s+2. \end{cases}$$

For $n \geq 6$, define the graph $G_4(n)$ as follows.

$$G_4(n) = \begin{cases} (K_3 * K_3) \cup (K_3 * K_3) \cup (s-4)K_3, \text{or} & \\ 3K_4 \cup (s-4)K_3, \text{or } 6K_2 \cup (s-4)K_3, \text{or} & \\ (K_3 * K_3) \cup K_4 \cup K_2 \cup (s-4)K_3, \text{ or} & \\ K_4 \cup 4K_2 \cup (s-4)K_3, \text{or} & \\ 2K_4 \cup 2K_2 \cup (s-4)K_3, & \text{if } n = 3s; \\ (K_4 * 2K_3) \cup (s-3)K_3, & \text{if } n = 3s+1; \\ (K_4 * K_3) \cup 2K_2 \cup (s-3)K_3, \text{ or} & \\ (K_3 * 2K_3) \cup K_2 \cup (s-3)K_3, & \text{if } n = 3s+2. \end{cases}$$

Let $g_4(n) = \mathrm{mi}(G_4(n))$. From the preceding propositions, we have

$$g_4(n) = \begin{cases} \frac{64}{81} g_1(n), & \text{if } n = 3s; \\ \frac{31}{36} g_1(n), & \text{if } n = 3s+1; \\ \frac{22}{27} g_1(n), & \text{if } n = 3s+2. \end{cases}$$

# 3 Main Result

**Theorem 3.1** *Let $G$ be a graph of order $n$ and $G \ncong G_i(n), i = 1, 2, 3$. Then* $\mathrm{mi}((G) \leq g_4(n)$*. Furthermore, the equality holds if and only if $G \cong G_4(n)$.*

**Proof.** It is easy to see that the equality holds for any graph $G \cong G_4(n)$. We prove the theorem by induction on $n$. Since $g_1(3) = 3$, $g_1(4) = 4$ and $g_1(5) = 6$, it is easy to see that for any graph $G \ncong G_i(n), i = 1, 2, 3$, of order $n$ $(3 \leq n \leq 5)$, we have $\mathrm{mi}(G) < g_4(n)$. Suppose that the theorem holds for all graphs of order less than $n$. Now we consider a graph $G$ of order $n \geq 6$. First, we have the following remarks.

**Remark 1.** Suppose that $\delta(G) = 1$. Choose $x \in V(G)$ with $N(x) = \{y\}$. If $d(y) \geq 2$, from Proposition 2.1, by the induction hypothesis we have

$$\begin{aligned} \mathrm{mi}(G) &= \mathrm{mi}(G - x - y) + \mathrm{mi}(G - N[y]) \\ &\leq g_1(n-2) + g_1(n-3) \\ &\leq \begin{cases} \frac{7}{9} g_1(n), & \text{if } n \equiv 0 \ (mod\ 3); \\ \frac{5}{6} g_1(n), & \text{otherwise.} \end{cases} \\ &< g_4(n). \end{aligned}$$

So assume that $d(y) = 1$. Since $G \ncong G_i(n), i = 1, 2, 3$, we have that $G - x - y \ncong G_i(n-2), i = 1, 2$. Suppose that $G - x - y \cong G_3(n-2)$. If $n \equiv 0, \text{or } 2 \ (mod\ 3)$, then $G \cong G_4(n)$. If $n \equiv 1 \ (mod\ 3)$, then from Proposition 2.3 we have $mi(G) = \frac{5}{6} g_1(n) < g_4(n)$. Suppose that $G - x - y \ncong G_3(n-2)$, then by the induction hypothesis we have

$$\begin{aligned} g_4(n) &= \begin{cases} \frac{62}{81} g_1(n), & \text{if } n = 3s; \\ \frac{22}{27} g_1(n), & \text{if } n = 3s+1; \\ \frac{64}{81} g_1(n), & \text{if } n = 3s+2. \end{cases} \\ &< g_4(n), \end{aligned}$$

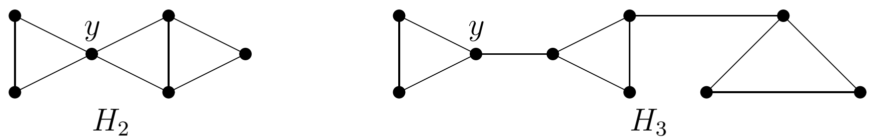


Figure 3: The graphs $H_2$ and $H_3$

i.e., the desired result.

**Remark 2.** Suppose that $G \cong C_n$ and $n \geq 6$. From Proposition 2.2, by the induction hypothesis we have

$$\begin{aligned}
\mathrm{mi}(G) &= \mathrm{mi}(C_{n-2}) + \mathrm{mi}(C_{n-3}) \\
&\leq \begin{cases} \frac{11}{12}g_1(n-2) + \frac{8}{9}g_1(n-3), & \text{if } n \equiv 0 \ (mod\ 3); \\ \frac{8}{9}g_1(n-2) + \frac{11}{12}g_1(n-3), & \text{if } n \equiv 1 \ (mod\ 3); \\ \frac{8}{9}g_1(n-2) + \frac{8}{9}g_1(n-3), & \text{if } n \equiv 2 \ (mod\ 3); \end{cases} \\
&\leq \begin{cases} \frac{19}{27}g_1(n), & \text{if } n \equiv 0 \ (mod\ 3); \\ \frac{3}{4}g_1(n), & \text{if } n \equiv 1 \ (mod\ 3); \\ \frac{20}{27}g_1(n), & \text{if } n \equiv 2 \ (mod\ 3); \end{cases} \\
&< g_4(n).
\end{aligned}$$

So assume that $\delta(G) \geq 2$ and $\Delta(G) \geq 3$. Next, we distinguish the following cases to complete the proof.

**Case 1.** $n = 3s$.

Choose $x \in V(G)$ with $d(x) = \Delta(G)$. If $d(x) \geq 5$, from Proposition 2.1, we have

$$\begin{aligned}
\mathrm{mi}(G) &\leq \mathrm{mi}(G - x) + \mathrm{mi}(G - N[x]) \\
&\leq g_1(n-1) + g_1(n-6) \\
&= \frac{7}{9}g_1(n) \\
&< \frac{64}{81}g_1(n).
\end{aligned}$$

If $d(x) = 4$ and $G - x \cong G_1(n-1)$, from $G \ncong G_3(n)$, we have that $G \cong H_2 \cup (s-2)K_3$ where the graph $H_2$ is illustrated in Figure 3. From Proposition 2.1 we have $\mathrm{mi}(G) = \frac{7}{9}g_1(n) < \frac{64}{81}g_1(n)$.

So let $d(x) = 3$. Since $G \ncong G_2(n)$, we have $G - x \ncong G_1(n-1)$. So if $G - x \cong G_2(n-1)$, we have $G \cong H_3 \cup (s-3)K_3$ where the graph $H_3$ is illustrated in Figure 3, or $G \cong G_4(n)$. When $G \cong G_4(n)$, we are done. When $G \cong H_3 \cup (s-3)K_3$, from Proposition 2.1 we have $\mathrm{mi}(G) = \frac{21}{27}g_1(n) < \frac{64}{81}g_1(n)$. If $G - x \cong G_3(n-1)$, then we have

$$\begin{aligned}
\mathrm{mi}(G) &\leq \mathrm{mi}(G - x) + \mathrm{mi}(G - N[x]) \\
&\leq \frac{5}{6}g_1(n-1) + g_1(n-4) \\
&= \frac{7}{9}g_1(n) < \frac{64}{81}g_1(n).
\end{aligned}$$

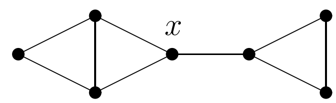


Figure 4: The graph $H_4$

If $G - y \ncong G_3(n-1), i = 1, 2, 3$, by the induction hypothesis, we have

$$\begin{array}{rcl} \mathrm{mi}(G) & \leq & \mathrm{mi}(G-x) + \mathrm{mi}(G - N[x]) \\ & \leq & \frac{64}{81} g_1(n-1) + g_1(n-4) \\ & < & \frac{64}{81} g_1(n). \end{array}$$

**Case 2.** $n = 3s + 1$.

Choose $x \in V(G)$ such that $d(x) = \Delta(G)$.

Suppose that $\Delta(G) = 3$. If $G - x \cong G_1(n-1)$, then since $G \ncong G_1(n)$, we have $G \cong H_4 \cup (s-2)K_3$ where $H_4$ is illustrated in Figure 4. From Proposition 2.1 we have $\mathrm{mi}(G) = \frac{3}{4} g_1(n) < \frac{31}{36} g_1(n)$.

If $G - x \cong G_2(n-1)$, then one of the component of $G$ must be isomorphic to one of the graphs illustrated in Figure 5 while other component of $G$ is $K_3$. From Proposition 2.1 one can check that $\mathrm{mi}(G) < \frac{31}{36} g_1(n)$.

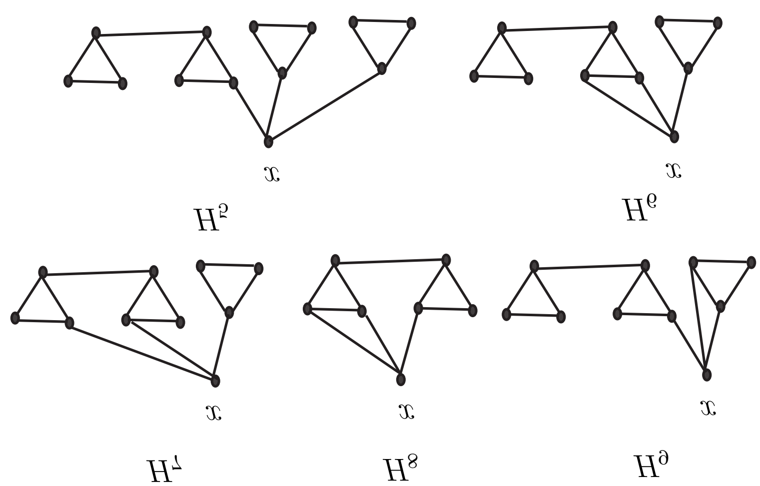

Figure 5: The graphs $H_5, H_6, H_7, H_8, H_9$

Since $\Delta(G_3(n-1)) = 4$ for $n \geq 6$, we have that $G - x \ncong G_3(n-1)$. So, assume that $G - x \ncong G_i(n-1), i = 1, 2, 3$, then by the induction hypothesis,

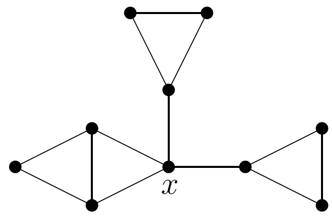


Figure 6: The graph $H_{10}$

we have

$$\begin{aligned} \mathrm{mi}(G) &\le \mathrm{mi}(G-x)+\mathrm{mi}(G-N[x]) \\ &\le \tfrac{64}{81}g_1(n-1)+g_1(n-4) \\ &= \tfrac{64}{81}3^{s-1}+3^{s-1} \\ &= \tfrac{91}{108}g_1(n) < \tfrac{31}{36}g_1(n). \end{aligned}$$

Now we consider the case $\Delta(G)\ge 4$. If $G-x\ncong G_1(n-1)$, from Theorems 2.4 and 2.5 we have

$$\begin{aligned} \mathrm{mi}(G) &\le \mathrm{mi}(G-x)+\mathrm{mi}(G-N[x]) \\ &\le \tfrac{8}{9}g_1(n-1)+g_1(n-5) \\ &= \tfrac{5}{6}g_1(n) < \tfrac{31}{36}g_1(n). \end{aligned}$$

If $d(x)\ge 5$, from Theorem 2.4 we have

$$\begin{aligned} \mathrm{mi}(G) &\le \mathrm{mi}(G-x)+\mathrm{mi}(G-N[x]) \\ &\le g_1(n-1)+g_1(n-6) \\ &= \tfrac{31}{36}g_1(n), \end{aligned}$$

where the equality holds if and only if $G-x\cong G_1(n-1)$ and $G-N[x]\cong G_1(n-6)$, i.e., $G\cong G_4(n)$.

So, we assume that $d(x)=4$ and $G-x\cong G_1(n-1)$. Then, since $G\ncong G_2(n)$, we have $G\cong H_{10}\cup(s-3)K_3$ where $H_{10}$ is illustrated in Figure 6. From Proposition 2.1 we have $\mathrm{mi}(G)=\frac{3}{4}g_1(n)<\frac{31}{36}g_1(n)$.

**Case 3.** $n=3s+2$.

Choose $x\in V(G)$ with $d(x)=\Delta(G)$. If $d(x)\ge 5$, from Theorem 2.4 we have

$$\begin{aligned} \mathrm{mi}(G) &\le \mathrm{mi}(G-x)+\mathrm{mi}(G-N[x]) \\ &\le g_1(n-1)+g_1(n-6) \\ &= \tfrac{7}{9}g_1(n) < \tfrac{22}{27}g_1(n). \end{aligned}$$

If $d(x)=3$, since $\delta(G)\ge 2$ and $\Delta(G)=3$, we have $G-x\ncong G_i(n-1)$, for $i=1,2,3$. Hence, by the induction hypothesis, we have

$$\begin{aligned} \mathrm{mi}(G) &\le \mathrm{mi}(G-x)+\mathrm{mi}(G-N[x]) \\ &\le \tfrac{31}{36}g_1(n-1)+g_1(n-4) \\ &= \tfrac{43}{54}g_1(n) < \tfrac{22}{27}g_1(n). \end{aligned}$$

So, let $d(x) = 4$. If $G - x \ncong G_1(n-1)$, from Theorems 2.4 and 2.5 we have

$$\begin{aligned} \mathrm{mi}(G) &\leq \mathrm{mi}(G-x) + \mathrm{mi}(G-N[x]) \\ &\leq \tfrac{11}{12} g_1(n-1) + g_1(n-5) \\ &= \tfrac{7}{9} g_1(n) < \tfrac{22}{27} g_1(n). \end{aligned}$$

Suppose that $G - x \cong G_1(n-1)$. Since $G \ncong G_i(n), i = 1, 2, 3$, we have $G - N[x] \ncong G_i(n-5), i = 1, 2$. From Theorems 2.4 and 2.5 we have

$$\begin{aligned} \mathrm{mi}(G) &\leq \mathrm{mi}(G-x) + \mathrm{mi}(G-N[x]) \\ &< g_1(n-1) + \tfrac{8}{9} g_1(n-5) \\ &= \tfrac{22}{27} g_1(n). \end{aligned}$$

This completes the proof. ∎

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