Amalgamation for Reducts of Polyadic Equality Algebras, both a Negative Result and a Positive Result

Tarek Sayed Ahmed

Department of Mathematics, Faculty of Science
Cairo University, Giza, Egypt
rutahmed@gmail.com

Abstract

Let $G \subseteq \omega^\omega$ be a semigroup. $G$ polyadic algebras with equality, or simply $G$ algebras, are reducts of polyadic algebras with equality obtained by restricting the similarity type and axiomatization of polyadic algebras to substitutions in $G$, and possibly weakening the axioms governing the diagonal elements. Such algebras were introduced in the context of 'finitizing' first order logic with equality. We show that when $G = \{[i,j],[i|j],suc,pred\}$ then the class of $G$ algebras fails to have the amalgamation property. On the other hand, when $G$ is a strongly rich semigroup then a natural superclass of the class of $G$ polyadic equality algebras, obtained by discarding one of the equations holding in $G$ algebras (namely, $x.d_{ij} \leq s_{[i|j]}x$), has the super amalgamation property.

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1 Main results

Throughout $\alpha$ is an infinite countable ordinal and $G \subseteq \alpha^\alpha$. $[i|j]$ denotes the replacement that sends $i$ to $j$ and leaves the rest of the elements fixed. We always assume that $\{[i|j] : i,j \in \alpha\} \subseteq G$. $[G]$ denotes the semigroup of $(\alpha^\alpha, \circ)$ generated by $G$. Let $G^*$ denotes the set of all finite words on $G$, i.e. $G^* = \bigcup_{n \in \omega}^nG$. For $u, w \in G^*$, $u\circ w$, or simply $uw$, denotes the concatenation of $u$ and $w$. Recall that $(G^*, \cap)$ is the free semigroup generated by $G$. We let $\hat{\cdot} : G^* \rightarrow [G]$ denote the unique anti-homomorphism extending the identity inclusion $Id : G \rightarrow [G]$. For each $u \in G$ assume that $s_u(x)$ is a unary term. Let $w \in G^*$. Suppose that $w = \langle w_0, \cdots, w_{n-1} \rangle$, with $w_i \in G$. Then $s_w(x)$
denotes the unary term \( s_{w_{u-1}}(\cdots(s_{w_0}(x))\cdots) \). \( \Sigma_G \) denotes the following set of equations in one variable:

\[
\Sigma_G := \{ s_u(x) = s_w(x) : u, w \in G^* \text{ and } u = w \}.
\]

**Definition 1**.

(i) By a \( G \) polyadic algebra of dimension \( \alpha \), or a \( GPA_\alpha \) for short, we understand an algebra of the form

\[
\mathfrak{A} = \langle A, +, 
\ldots, 0, 1, c_i, s_\tau \rangle_{\tau \in \alpha, \tau \in [G]}
\]

where \( c_i \) (\( i \in \alpha \)) and \( s_\tau \) (\( \tau \in [G] \)) are unary operations on \( A \) such that postulates \( (P_1 - P_{10}) \) below hold for \( \tau, \sigma \in [G] \) and all \( i, j, k \in \alpha \).

\( (P_1) \) \( \langle A, +, \ldots, 0, 1 \rangle \) is a boolean algebra, henceforth denoted by \( B[A] \).

\( (P_2) \) \( x \leq c_i x = c_i c_i x, c_i (x + y) = c_i x + c_i y, c_i (-c_i x) = -c_i x, c_i c_j x = c_j c_i x \).

In other words \( c_i \) is an additive closure operator, and \( c_i, c_j \) commute.

\( (P_3) s_\tau \) is a boolean endomorphism.

\( (P_4) \Sigma_G. \) In particular \( s_\tau s_\sigma x = s_{\tau \circ \sigma} x \) and \( s_{I\delta} x = x \).

\( (P_5) s_i c_i x = s_{\tau[i][j]} c_i x \).

\( \tau[i][j] \) is the transformation that agrees with \( \tau \) on \( \alpha \setminus \{i\} \) and \( \tau[i][j](i) = j \).

\( (P_6) s_i c_i x = c_j s_\tau x \) if \( \tau^{-1}(j) \in \{i\} \).

\( (P_7) c_i s_{\tau[i][j]} x = s_{\tau[i][j]} x \).

\( (P_8) s_{\tau[i][j]} c_i x = c_i x \).

\( (P_9) s_{\tau[i][j]} c_k x = c_k s_{\tau[i][j]} x \) whenever \( k \notin \{i, j\} \).

\( (P_{10}) c_i s_{\tau[i][j]} x = c_j s_{\tau[i][j]} x \).

(ii) By a \( G \) polyadic equality algebra of dimension \( \alpha \), a \( GPEA_\alpha \) for short, we understand an algebra of the form

\[
\mathfrak{B} = \langle B, +, 
\ldots, 0, 1, c_i, s_\tau, d_{ij} \rangle_{i,j \in \alpha, \tau \in [G]}
\]

such that \( d_{ij} \in A \) for all \( i, j \in \alpha \) and \( \langle A, +, \ldots, 0, 1, c_i, s_\tau \rangle_{i \in \alpha, \tau \in [G]} \) is a \( G \) algebra such that the following hold for all \( k, l \in \alpha \) and all \( \tau \in [G] \):

\( (P_{11}) d_{kk} = 1 \)

\( (P_{12}) s_\tau d_{kl} = d_{\tau(k), \tau(l)} \).

\( (P_{13}) x \cdot d_{kl} \leq s_{\tau[k][l]} x \)
GPEA_\alpha also stands for the class of all GPEA_\alpha’s. Motivated by the complexity of hitherto existing axiomatizations of the generic algebraic counterparts of first order logic with equality, namely cylindric algebras and polyadic algebras [1], GPEA_\alpha was introduced in the attempt to “finitize” first order logic with equality, cf. [4], [7], [8], [6], and [5]. It is shown in [7] that if G is any strongly rich semigroup then the classes GPA_\alpha have the super amalgamation property (in the sense of the coming definition). An example of a strongly rich semigroup is the one generated by \{[i|j], suc, pred : i, j \in \omega\}. Here suc and pred are the successor and predecessor functions on \omega, respectively. In this paper we prove two contrasting results concerning the amalgamation property for G polyadic equality algebras. But first the relevant definitions:

**Definition 2.** Let W be a class of partially ordered algebras. W has the super amalgamation property, or SUPAP for short, if

(i) for all \mathcal{A}_0, \mathcal{A}_1 and \mathcal{A}_2 \in W, and all monomorphisms \ i_1 and \ i_2 of \mathcal{A}_0 into \mathcal{A}_1, \mathcal{A}_2, respectively, there exists \mathcal{A} \in W, a monomorphism \ m_1 from \mathcal{A}_1 into \mathcal{A} and a monomorphism \ m_2 from \mathcal{A}_2 into \mathcal{A} such that \ m_1 \circ i_1 = m_2 \circ i_2, and

(ii) for all \ x \in A_j, for all \ y \in A_k, if \ m_j(x) \leq m_k(y) then there exists \ z \in A_0 such that \ x \leq i_j(z) and \ i_k(z) \leq y where \ \{j, k\} = \{1, 2\}.

In this case we say that \mathcal{A} is a super amalgam of \mathcal{A}_1 and \mathcal{A}_2 over \mathcal{A}_0, via \ m_1 and \ m_2, or even simply a super amalgam.

We will exclusively deal with Boolean algebras with operators hence the partial order dealt with will always be the Boolean order. Recall that W has the amalgamation property or AP for short, if W satisfies (i). W has the strong AP, or SAP for short if in addition to (i), we have \ m_1 \circ i_1(A_0) = m_1(A_1) \cap m_2(A_2).

We first show that

**Theorem 3.** Let T denote the semigroup generated by \{[i|j], suc, pred : i, j \in \omega\}. Then the class TPEA_\omega fails to have the amalgamation property.

This is in deep contrast to the result in [7] addressing the equality free case. Before formulating our positive result, we recall from [7] Definition 1.4 the notion of strongly rich semigroup. T as in Theorem 3 is an example. We use the notation in op.cit.

**Definition.** (a) Let \ T \subseteq \langle \alpha, \circ \rangle be a semigroup. We say that T is rich if T satisfies the following conditions:

1. \ (\forall i, j \in \alpha)(\forall \tau \in T)\tau[i|j] \in T.
(2) There exists $\sigma, \pi \in T$ such that
\[(\pi \circ \sigma = Id, Rng(\sigma) \neq \alpha),\]
and
\[(\forall \tau \in T)(\sigma \circ \tau \circ \pi)[(\alpha \setminus Rng(\sigma))\{Id\}] \in T.\]

(b) Let $T \subseteq \langle \alpha, \circ \rangle$ be a rich semigroup. Let $\sigma$ and $\pi$ be as in (a) above. If $\sigma$ and $\pi$ satisfy (i), (ii) below:

(i) $(\forall n \in \omega) |supp(\sigma^n \circ \pi^n)| < \alpha$.

(ii) $(\forall n \in \omega)[supp(\sigma^n \circ \pi^n) \subseteq \alpha \setminus Rng(\sigma^n)]$;

then we say that $T$ is a strongly rich semigroup.

In contrast to Theorem 3, we now have:

**Theorem 4**. Let $G$ be any strongly rich semigroup. Then $V = Mod(1-12)$ has the super amalgamation property. In particular, $V$ has the strong amalgamation property.

## 2 Proof of Theorem 3

Throughout, unless otherwise specified, $G$ is a set of transformations on $\omega$. In view of $\Sigma_G$ we identify $G$ algebras with $[G]$ algebras, where $[G]$ is the semigroup generated by $G$. At certain occasions $G$ will be specified to be $\{[i,j], \text{pred, suc} : i, j \in \omega\}$. Amalgamation in varieties can be pinned down to congruences on free algebras. Congruences correspond to ideals. This prompts:

**Definition 5**. Let $\mathfrak{A} \in GPEA_{\alpha}$. A subset $I$ of $\mathfrak{A}$ is an ideal iff the following conditions are satisfied:

(i) $0 \in I$

(ii) If $x, y \in I$, then $x + y \in I$

(iii) If $x \in I$ and $y \leq x$ then $y \in I$

(iv) For all $i < \alpha$ and $\tau \in G$ if $x \in I$ then $c_i x$ and $s_\tau x \in I$.

It can be checked that ideals function properly, that is ideals correspond to congruences the usual way. For $X \subseteq \mathfrak{A}$, the ideal generated by $X$, $Ig_{\mathfrak{A}}X$ is the smallest ideal containing $X$, i.e the intersection of all ideals containing $X$. We
need to characterize principal ideals, i.e. ideals generated by a single element. Before that, a piece of notation. For \( \Gamma = \{ i_0 \ldots i_{n-1} \}, c(\Gamma) x = c_{i_0}c_{i_1} \ldots c_{i_{n-1}}x. \)

**Lemma 6.** Let \( \mathfrak{A} \in GPEA_\omega \) and \( x \in A. \) Then \( \mathcal{I}\mathfrak{A}\{x\} = \{ y \in A : y \leq c(\Gamma)(s_{\tau_1}x + \ldots + s_{\tau_n}x) : \text{for some finite } \Gamma \text{ and } \tau_1 \ldots \tau_n \in G \}. \)

**Proof.** Let \( H \) denote the set of elements on the right hand side. It is easy to check \( H \subseteq \mathcal{I}\mathfrak{A}\{x\}. \) Conversely, assume that \( y \in H, i < \omega \) and \( \tau \in G. \) It is clear that \( c_i y \in H. \) \( H \) is closed under substitutions by noting that \( s_\tau c(\Gamma)x = c(\Delta)s_\tau x \) if \( \Gamma = \tau^{-1} \Delta. \) Now let \( z, y \in H. \) Assume that \( z \leq c(\Gamma)(s_{\tau_1}x + \ldots + s_{\tau_n}x) \) and \( y \leq c(\Delta)(s_{\sigma_1}x + \ldots + s_{\sigma_n}x), \) then

\[
z + y \leq c(\Gamma \cup \Delta)(s_{\tau_1}x + \ldots + s_{\tau_n}x + s_{\sigma_1}x + \ldots + s_{\sigma_n}x).
\]

The Lemma is proved. \( \blacksquare \)

The following about ideals in \( G \) algebras will be frequently used.

- If \( \mathfrak{A} \subseteq \mathfrak{B} \) are \( GPEAs \) and \( I \) is an ideal of \( \mathfrak{A}, \) then \( \mathcal{I}\mathfrak{B}(I) = \{ b \in B : \exists a \in I(b \leq a) \}. \)

- If \( I \) and \( J \) are ideals of a \( GPEA \) then the ideal generated by \( I \cup J \) is \( I + J = \{ x : x \leq i + j : \text{for some } i \in I, j \in J \}. \)

For a class \( K \) and a set \( X, \mathfrak{A}_X K \) denotes the \( K \) algebra freely generated by \( X, \) or the \( K \) free algebra on \( |X| \) generators. As a wide spread custom, we identify \( X \) with \( |X|. \) We understand the notion of free algebras in the sense of \( [2] \) Definition 0.4.19. In particular, free \( K \) algebras may not be in \( K. \) However, they are always in \( HSP(K), \) the variety generated by \( K. \)

For an algebra \( \mathfrak{A} \) and \( X \subseteq A \) we write \( \mathfrak{A}^{(X)} \) for the subalgebra of \( \mathfrak{A} \) generated by \( X. \) For better readability, we write \( \mathfrak{A}^{(x)} \) for \( \mathfrak{A}^{(|\{x\}|)} \)

**Theorem 7.** Let \( K \subseteq GPEA_\omega \) be a class such that \( H K = S K = K. \) If \( K \) has the amalgamation property, then the following condition holds for \( \mathfrak{A} = \mathfrak{A}_X K \) for any cardinal \( \mu > 0. \) For any \( X_1, X_2 \subseteq \mu, \) if \( x \in \mathfrak{A}^{(X_1)} \) and \( z \in \mathfrak{A}^{(X_2)} \) and \( x \leq z \) then there is a \( y \in \mathfrak{A}^{(X_1 \cap X_2)} \), a finite \( \Gamma \subseteq \omega \) and a finite \( H \subseteq G \) such that

\[
x \leq y \leq c(\Gamma) \sum_{\tau \in H} s_\tau z.
\]

**Proof.** We write \( R \in Co\mathfrak{A} \) if \( R \) is a congruence relation \( \mathfrak{A}. \) For \( X \subseteq A, \) then by \( (\mathfrak{A}/R)^{(X)} \) we understand the subalgebra of \( \mathfrak{A}/R \) generated by \( \{ x/R : x \in X \}. \) We first prove the following congruence extension property. For any \( X_1, X_2 \subseteq \mu \) if \( R \in Co\mathfrak{A}^{(X_1)} \) and \( S \in Co\mathfrak{A}^{(X_2)} \) and

\[
R \cap 2^{A^{(X_1 \cap X_2)}} = S \cap 2^{A^{(X_1 \cap X_2)}},
\]
then there exists a congruence $T$ on $\mathfrak{A}$ such that

$$T \cap A^{(X_1)} = R \text{ and } T \cap A^{(X_2)} = S.$$ 

So let $X_1, X_2, R$ and $S$ be as specified. Then $\mathfrak{A}^{(X_1)}/R$ and $\mathfrak{A}^{(X_2)}/S$ are in $\mathbf{K}$, since $\mathfrak{A}$ is in $\mathbf{K}$ and $\mathbf{K}$ is closed under forming subalgebras and homomorphic images. We now have

$$(\mathfrak{A}^{(X_1)}/R)(X_1 \cap X_2) \cong \mathfrak{A}^{(X_1 \cap X_2)}/(R \cap A^{(X_1 \cap X_2)}) = \mathfrak{A}^{(X_1 \cap X_2)}/(S \cap A^{(X_1 \cap X_2)}) \cong (\mathfrak{A}^{(X_2)}/S)(X_1 \cap X_2).$$

Thus $\mathbf{K}$ has the amalgamation property there exist $\mathfrak{B} \in \mathbf{K}$ with a set of generators $Y$ and $Y_1, Y_2$ with $Y_1 \cup Y_2 = Y$, such that

$$\mathfrak{B}^{(Y_1)} \cong \mathfrak{A}^{(X_1)}/R$$

and

$$\mathfrak{B}^{(Y_2)} \cong \mathfrak{A}^{(X_2)}/S.$$ 

Therefore $\mathfrak{B}^{(Y_1)}$ is a homomorphic image of $\mathfrak{A}^{(X_1)}$ and $\mathfrak{B}^{(Y_2)}$ is a homomorphic image of $\mathfrak{A}^{(X_2)}$. Since $\mathfrak{A}$ is free it follows that $\mathfrak{B}$ is a homomorphic image of $\mathfrak{A}$. Hence there exists $T \in \text{Co}\mathfrak{A}$ such that

$$\mathfrak{B} \cong \mathfrak{A}/T.$$ 

It follows thus that

$$\mathfrak{B}^{(Y_1)} \cong \mathfrak{A}^{(X_1)}/T \cap A^{(X_1)}$$

and

$$\mathfrak{B}^{(Y_2)} \cong \mathfrak{A}^{(X_2)}/T \cap A^{(X_2)}$$

whence

$$T \cap A^{(X_1)} = R \text{ and } T \cap A^{(X_2)} = S.$$ 

Now let $x \in A^{(X_1)}$, $z \in A^{(X_2)}$ and assume that $x \leq z$. Then

$$x \in (\mathfrak{g}^{(X_1)}\{z\}) \cap \mathfrak{A}^{(X_1)}.$$ 

Let

$$M = \mathfrak{g}^{(X_1)}\{z\} \text{ and } N = \mathfrak{g}^{(X_2)}(M \cap A^{(X_1 \cap X_2)}).$$

Then

$$M \cap A^{(X_1 \cap X_2)} = N \cap A^{(X_1 \cap X_2)}.$$ 

By identifying ideals with congruences, and using the congruence extension property, there is an ideal $P$ of $\mathfrak{A}$ such that

$$P \cap A^{(X_1)} = N \text{ and } P \cap A^{(X_2)} = M.$$
It follows that
\[ \mathfrak{g}^\mathfrak{X}(N \cup M) \cap A^{(X_1)} \subseteq P \cap A^{(X_1)} = N. \]

Hence
\[ (\mathfrak{g}^\mathfrak{X}(z)) \cap A^{(X_1)} \subseteq N. \]

and we have
\[ x \in \mathfrak{g}^\mathfrak{X}(X_1)[Id^\mathfrak{X}(X_2) \{z\} \cap A^{(X_1 \cap X_2)}]. \]

This implies that there is an element \( y \) such that
\[ x \leq y \in A^{(X_1 \cap X_2)} \]

and \( y \in \mathfrak{g}^\mathfrak{X}(X_2) \{z\} \). Lemma 6 gives a finite \( \Gamma \subseteq \omega \) and a finite \( H \subseteq G \) such that
\[ x \leq y \leq c(\Gamma) \sum_{\tau \in H} s_{\tau z}. \]

The theorem is proved. \( \blacksquare \)

The \( y \) in the above will be referred to as an interpolant of \( x \) and \( z \), and we say that \( x \) and \( z \) can be interpolated in \( A^{(X_1 \cap X_2)} \).

Theorem 8. If \( \mathfrak{A} \in GPEA_\omega \) then \( \mathfrak{RdCA} \mathfrak{A} \in CA_\omega \).

Proof. We prove the cylindric axioms. The proof is taken from [2] Theorem 5.4.3, but we spell out more details. We check that \((C_0 - C_7)\) holds. \((C_0 - C_5)\) holds by definition. First we show that \( c_id_{kl} = 1 \) for all \( l, k \in \omega \).

\[ 1 = d_{ll} = s_{[k][l]}d_{kl} \leq s_{[k][l]}c_kd_{kl} = s_{Id}c_kd_{kl} = c_kd_{kl}. \]

Next we show that \( x.d_{kl} = s_{[k][l]}x.d_{kl} \) for every \( l, k < \omega \) and \( x \in A \). We have \( x.d_{kl} \leq s_{[k][l]}x \) and \( x.d_{kl} \leq d_{kl} \). Therefore \( \leq \) follows. On the other hand
\[ s_{[k][l]}x.d_{kl} - x \leq s_{[k][l]}x.s_{[k][l]} - x = 0 \]

so the conclusion follows. Now if \( k \neq l \) we have \( c_k s_{[k][l]} x = s_{[k][l]} x \) for any \( x \in A \). Hence
\[ c_k(x.d_{kl}) = c_k(s_{[k][l]}x.d_{kl}) = c_k(c_k s_{[k][l]} x.d_{kl}) = s_{[k][l]}x.c_kd_{kl} = s_{[k][l]}x. \]

Now
\[ c_k(d_{ik}.d_{jk}) = s_{[k][i]}d_{jk} = d_{ji} = d_{ij} \]
Also
\[ c_i(d_{ij}.x)c_i(d_{ij}.x - x) = s_{[i][j]}x.s_{[i][j]}x - x = 0 \]

So we proved the cylindric axioms. ■

In fact it is proved in [9] that if \( G \) is a rich semigroup, then \( \mathfrak{Rd_{CA}} \) is a representable \( \mathfrak{CA} \). We are now assured that our algebras are adequate to algebraize first order logic with equality. From now on we freely use (consequences of) \( \mathfrak{CA} \) axioms from the monograph [2]. In conformity with the notation adopted therein, we write \( s_i^j \) for the more cumbersome \( s_{[i][j]} \). \( s_i^j \) is term definable in \( \mathfrak{CA} \), by \( c_j(x.d_{ij}) \).

**Lemma 9.** Let \( \mathfrak{A} \) be any \( GPEA_\omega \). Then for any \( x, y, z \in A \) we have
\[ c_0(x.c_1y).c_0(x - c_1y) \leq c_0c_1(c_1z.s_0^1c_1z - d_{01}) + c_0(x - c_1z). \]

**Proof.** Let
\[ a = x.c_1y - c_0(x - c_1z), \]
\[ b = x - c_1y - c_0(x - c_1z) \]

Then we have
\[ c_1a.c_1b \leq c_1(x.c_1y).c_1(x - c_1y) \text{ by } [2] 1.2.7 \]
\[ = c_1x.c_1y.c_1x - c_1y \text{ by } [2] 1.2.11 \]

and so
\[ c_1a.c_1b = 0 \quad (1) \]

From the inclusion \( x - c_1z \leq c_0(x - c_1z) \) we get
\[ x - c_0(x - c_1z) \leq c_1z. \]

Thus \( a, b \leq c_1z \) and hence, by [2] 1.2.9,
\[ c_1a, c_1b \leq c_1z \quad (2) \]

We now compute:
\[ c_0a.c_0b \leq c_0c_1a.c_0c_1b \text{ by } [2] 1.2.7 \]
\[ = c_0c_1a.c_1s_0^1c_1b \text{ by } [2] 1.5.8 (i), [2] 1.5.9 (i) \]
\[ = c_1(c_0c_1a.s_0^1c_1b) \]
\[ = c_0c_1(c_0c_1a.s_0^1c_1b) \]
\[ = c_0c_1(c_1a.s_0^1c_1b.(-d_{01} + d_{01})) \]
\[ = c_0c_1[(c_1a.s_0^1c_1b. - d_{01}) + (c_1a.s_0^1c_1b.d_{01})] \]
\[ = c_0c_1[(c_1a.s_0^1c_1b. - d_{01}) + (c_1a.c_1b.d_{01})] \text{ by } [2] 1.5.5 \]
\[ = c_0c_1(c_1a.s_0^1c_1b. - d_{01}) \text{ by } (1) \]
\[ \leq c_0c_1(c_1z.s_0^1c_1z. - d_{01}) \text{ by } (2), [2] 1.2.7 \]
We have proved that
\[ c_0[x. c_1 y. - c_0(x. - c_1 z)] c_0[x. - c_1 y. - c_0(x. - c_1 z)] \leq c_0 c_1 (c_1 z.s^0_1 c_1 z. - d_{01}). \]

In view of [2] 1.2.11 and axiom \( c_3 \) this gives
\[ c_0(x. c_1 y). c_0(x. - c_1 y). - c_0(x. - c_1 z) \leq c_0 c_1 (c_1 z.s^0_1 c_1 z. - d_{01}). \]

The conclusion of the lemma now follows.

We now turn to concrete versions of \( Gal \) algebras. The class of \( G \) polyadic equality set algebras, or \( GPESA \) for short, is defined in [7, 1.1]. An algebra is in \( GPESA \) if it is a subalgebra of an algebra of the form
\[ \langle \wp(V), \cup, \cap, \sim, \emptyset, V, C_i, S_\tau, D_{ij} \rangle_{i,j<\alpha, \tau \in G} \]

where \( V \subseteq ^\alpha U \), for a set \( U \), is a compressed space in the sense of [2] 3.1.5 and the extra non-boolean operations defined for \( i, j < \alpha \), \( \tau \in G \) and \( X \subseteq V \) by
\[ C_i X = \{ t \in V : (\exists s \in X) s_j = t_j \ \forall j \neq i \}, \]
\[ S_\tau X = \{ \sigma \in V : \sigma \circ \tau \in X \}, \]
and
\[ D_{ij} = \{ s \in V : s_i = s_j \}. \]

\( GPESA \) is not always closed under ultraproducts [4]. It is proved by Sain in op.cit that when \( G \) is rich then the variety generated by \( GPESA \) coincides with \( GPEA \).

**Lemma 10.** Let \( \gamma \geq 4 \). Let \( G = \{ [i,j], \text{pred, suc} : i, j \in \alpha \} \). Let \( K \) be a class of algebras such that \( GPESA \subseteq K \subseteq GPEA \). Let \( r, s \) and \( t \) be the elements of \( \mathfrak{A} = \wp_{\gamma} K \) defined by the formulas
\[ r = c_0(x. c_1 y). c_0(x. - c_1 y), \]
\[ s = c_0 c_1 (c_1 z.s^0_1 c_1 z. - d_{01}) + c_0(x. - c_1 z) \]
\[ t = c_0 c_1 (c_1 w.s^0_1 c_1 w. - d_{01}) + c_0(x. - c_1 w) \]
where \( x, y, z, \) and \( w \) are the first four free generators of \( \mathfrak{A} \). Then there exists no \( u \in \mathfrak{A} \) such that, for some finite \( \Gamma \subseteq \gamma \), and \( H \subseteq G \)
\[ r \leq u \leq c_0(\Gamma) \sum_{\tau \in H} S_\tau(s.t) \]

**Proof.** Set \( \mathfrak{A} = \wp_{\gamma} K \) Assume that \( \gamma \geq 4 \). We must show that the inclusion
\[ r \leq s.t \] (3)
cannot be interpolated by an element of $\mathfrak{A}^{(x)}$. Let

$$\mathfrak{B} = (\wp(\omega), \cup, \cap, \sim, \emptyset, \omega, C_\kappa, D_{\kappa\lambda}, S_\tau)_{\kappa, \lambda < \omega, \tau \in G}$$

that is $\mathfrak{B}$ is the $G$ full set algebra in the space $\omega^\omega$. Let $E$ be the set of all equivalence relations on $\omega$, and for each $R \in E$ set

$$X_R = \{ \varphi : \varphi \in \omega \} \text{ and, for all } \xi, \eta < \omega, \varphi_\xi = \varphi_\eta \text{ iff } \xi R \eta \}.$$

Let

$$C = \{ \bigcup_{R \in L} X_R : L \subseteq E \}.$$

$C$ is clearly closed under the formation of arbitrary unions, and since

$$\sim \bigcup_{R \in L} X_L = \bigcup_{R \in E\setminus L} X_R$$

for every $L \subseteq E$, we see that $C$ is closed under the formation of complements with respect to $\omega^\omega$. Thus $C$ is a Boolean subuniverse (indeed, a complete Boolean subuniverse) of $\mathfrak{B}$; moreover, it is obvious that

$$X_R \text{ is an atom of } (C, \cup, \cap, \sim, 0, \omega^\omega) \text{ for each } R \in E. \quad (4)$$

For all $\kappa, \lambda < \omega$ we have $D_{\kappa\lambda} = \bigcup\{X_R : (\kappa, \lambda) \in R \in E\}$ and hence $D_{\kappa\lambda} \in B$. Also,

$$C_\kappa X_R = \bigcup\{X_S : S \in E, 2(\omega \sim \{\kappa\}) \cap S = 2(\omega \sim \{\kappa\}) \cap R\}$$

for any $\kappa < \omega$ and $R \in E$. Thus, because $C_\kappa$ is completely additive (cf.[2] 1.2.6(i)) and the remark preceding it), we see that $C$ is closed under the operation $C_\kappa$ for every $\kappa < \omega$. It is easy to show that $C$ is closed under substitutions. For any $\tau \in \omega^\omega$,

$$S_\tau X_R = \bigcup\{X_S : S \in E, \forall i, j < \omega (iRj \leftrightarrow \tau(i)S\tau(j))\}.$$

The set on the right may of course be empty. Therefore, we have shown that

$$C \text{ is a subuniverse of } \mathfrak{B}. \quad (5)$$

To prove that (3) can’t be interpolated by an element of $\mathfrak{A}^{(x)}$, it suffices to show that there is a subset $Y$ of $\omega^\omega$ such that

$$X_{Id} \cap f(r) \neq 0 \text{ for every } f \in \text{Hom}(\mathfrak{A}, \mathfrak{B}) \text{ such that } f(x) = X_{Id} \text{ and } f(y) = Y. \quad (6)$$
and also that for every finite \( \Gamma \subseteq \omega \), for each \( H \subseteq G \) there are subsets \( Z, W \) of \( \omega \) such that

\[
X_{Id} \sim C_{(\Gamma)} \sum_{\tau \in H} S_{\tau} g(s.t) \neq 0 \quad \text{for every} \quad g \in \text{Hom}(\mathfrak{A}, \mathfrak{B})
\]

such that \( g(x) = X_{Id}, g(z) = Z \) and \( g(w) = W \).\(^{(7)}\)

Here \( \text{Hom}(\mathfrak{A}, \mathfrak{B}) \) stands for the set of all homomorphisms from \( \mathfrak{A} \) to \( \mathfrak{B} \). For suppose, on the contrary, that these conditions are not sufficient. Then there exists a finite \( \Gamma \subseteq \omega \), a finite \( H \subseteq G \), and an interpolant \( u \in \mathfrak{A}(x) \) and there also exist \( Y, Z, W \subseteq \omega \) such that \( (6) \) and \( (7) \) hold. Take any \( h \in \text{Hom}(\mathfrak{A}, \mathfrak{B}) \) such that \( h(x) = X_{Id}, h(y) = Y, h(z) = Z \), and \( h(w) = W \). This is possible by the freeness of \( \mathfrak{A} \). Then using the fact that \( X_{Id} \cap h(r) \) is non-empty by \( (6) \) we get

\[
X_{Id} \cap h(u) = h(x. u) \supseteq h(x. r) \neq 0.
\]

And using the fact that \( X_{Id} \sim C_{(\Gamma)} \sum_{\tau \in S_{\tau}} h(s.t) \) is non-empty by \( (7) \) we get

\[
X_{Id} \sim hu = h(x. - u) \supseteq h(x. - c_{(\Gamma)} \sum_{\tau \in H} s_{\tau}(s.t)) \neq 0.
\]

However, in view of \( (4) \), it is impossible for \( X_{Id} \) to intersect both \( h(u) \) and its complement since \( h(u) \in C \) and \( X_{Id} \) is an atom; to see that \( h(u) \) is indeed contained in \( C \) recall that \( u \in \mathfrak{A}(x) \), and then observe that because of \( (5) \) and the fact that \( X_{Id} \in C \) we must have

\[
h[\mathfrak{A}(x)] \subseteq C \tag{8}\]

Therefore, \( (6) \) and \( (7) \) are sufficient conditions for \( (3) \) not to be interpolated by an element of \( \mathfrak{A}(x) \).

Let \( \sigma \in \omega \) be such that \( \sigma_0 = 0 \), and \( \sigma_\kappa = \kappa + 1 \) for every non-zero \( \kappa < \omega \). Let \( \tau = \sigma \upharpoonright (\omega \sim \{0\}) \cup \{(0, 1)\} \). Then \( \sigma, \tau \in X_{Id} \). Take

\[
Y = \{\sigma\}.
\]

Then

\[
\sigma \in X_{Id} \cap C_1 Y \quad \text{and} \quad \tau \in X_{Id} \sim C_1 Y
\]

and hence

\[
\sigma \in C_0(X_{Id} \cap C_1 Y) \cap C_0(X_{Id} \sim C_1 Y). \tag{9}\]

Therefore, we have \( \sigma \in f(r) \) for every \( f \in \text{Hom}(\mathfrak{A}, \mathfrak{B}) \) such that \( f(x) = X_{Id} \) and \( f(y) = Y \), and that \( (6) \) holds.
We now want to show that for any given finite $\Gamma \subseteq \omega$ and $H \subseteq G$, there exist sets $Z, W \subseteq \omega$ such that (7) holds; it is clear that no generality is lost if we assume that $0, 1 \in \Gamma$, so we make this assumption. Take
\[ Z = \{ \varphi : \varphi \in X_{Id}, \varphi_0 < \varphi_1 \} \cap C(\Gamma)\{Id\} \]
and
\[ W = \{ \varphi : \varphi \in X_{Id}, \varphi_0 > \varphi_1 \} \cap C(\Gamma)\{Id\}. \]
We first show that
\[ \text{Id} \in X_{Id} \sim C(\Gamma)g(s.t) \]
for any $g \in \text{Hom}(A, B)$ such that $g(x) = X_{Id}$, $g(z) = Z$, and $g(w) = W$; to do this we simply compute the value of $C(\Gamma)g(s.t)$. For the purpose of this computation we make use of the following property of ordinals: if $\Delta$ is any non-empty set of ordinals, then $\bigcap \Delta$ is the smallest ordinal in $\Delta$, and if, in addition, $\Delta$ is finite, then $\bigcup \Delta$ is the largest element ordinal in $\Delta$. Also, in this computation we shall assume that $\varphi$ always represents an arbitrary sequence in $\omega$. Then, setting
\[ \Delta \varphi = \Gamma \sim [\Gamma \sim \{0, 1\}] \]
for every $\varphi$, we successively compute:
\[
\begin{align*}
C_1 Z &= \{ \varphi : |\Delta \varphi| = 2, \varphi_0 = \bigcap \Delta \varphi \} \cap C(\Gamma)\{Id\}, \\
(X_{Id} \sim C_1 Z) \cap C(\Gamma)\{Id\} &= \{ \varphi : |\Delta \varphi| = 2, \varphi_0 = \bigcup \Delta \varphi, \varphi_1 = \bigcap \Delta \varphi \} \cap C(\Gamma)\{Id\}, \\
\text{and, finally}, \\
C_0(X_{Id} \sim C_1 Z) \cap C(\Gamma)\{Id\} &= \{ \varphi : |\Delta \varphi| = 2, \varphi_1 = \bigcap \Delta \varphi \} \cap C(\Gamma)\{Id\}. \tag{11}
\end{align*}
\]
Similarly, we obtain
\[
\begin{align*}
C_0(X_{Id} \sim C_1 W) \cap C(\Gamma)\{Id\} &= \{ \varphi : |\Delta \varphi| = 2, \varphi_1 = \bigcup \Delta \varphi \} \cap C(\Gamma)\{Id\}.
\end{align*}
\]
The last two formulas together give
\[
C_0(X_{Id} \sim C_1 Z) \cap C_0(X_{Id} \sim C_1 W) \cap C(\Gamma)\{Id\} = 0. \tag{12}
\]
Continuing the computation we successively obtain:

\[ C_1 Z \cap D_{01} = \{ \varphi : |\Delta \varphi| = 2, \varphi_0 = \varphi_1 = \bigcap \Delta \varphi \} \cap C(\Gamma)\{Id\}; \]

\[ S^0_1 C_1 Z = \{ \varphi : |\Delta \varphi| = 2, \varphi_1 = \bigcap \Delta \varphi \} \cap C(\Gamma)\{Id\}; \]

\[ C_1 Z \cap S^0_1 C_1 Z = \{ \varphi : |\Delta \varphi| = 2, \varphi_0 = \varphi_1 = \bigcap \Delta \varphi \} \cap C(\Gamma)\{Id\}; \]

hence we finally get

\[ C_0 C_1 (C_1 Z \cap S^0_1 C_1 Z \sim D_{01}) = C_0 C_1 0 = 0, \tag{13} \]

and similarly we get

\[ C_0 C_1 (C_1 W \cap S^0_1 C_1 W \sim D_{01}) = 0 \tag{14} \]

Now take \( g \) to be any homomorphism from \( \mathcal{A} \) into \( \mathcal{B} \) such that \( g(x) = X_{Id}, \ g(z) = Z \) and \( g(w) = W \). Let \( a = g(s.t) \). From (13) and (14) \( a = C_0(X_{Id} \sim C_1 Z) \cap C_0(X_{Id} \sim C_1 W) \). Then from (12), we have

\[ a \cap C(\Gamma)\{Id\} = \emptyset. \]

Then applying \( C(\Gamma) \) to both sides of this equation we get

\[ C(\Gamma)a \cap C(\Gamma)\{Id\} = \emptyset. \]

Thus (10) holds. It can be shown, in a similar fashion, that

\[ s_{suc}a \cap C(\Gamma)\{Id\} = \emptyset \]

and

\[ s_{pred}a \cap C(\Gamma)\{Id\} = \emptyset. \]

Therefore \( Id \notin C(\Gamma)s_{suc}a \) and \( Id \notin C(\Gamma)s_{pred}a \). Now let \( \Gamma \) be a finite subset of \( \omega \) and let \( i, j \in \omega \). We show that the \( X_{Id} \sim C(\Gamma)(S_{suc}a + S_{pred}a + S^j_i a) \) is not empty. We have taken the special case when \( H = \{ s_{suc}, s_{pred}, S^j_i \} \). The general case can be done in the same way. But let us proceed with this special case. We have

\[ C(\Gamma)(S_{suc}a + S_{pred}a + S^j_i a) \subseteq C(\Gamma)S_{suc}a + C(\Gamma)S_{pred}a + C(\Gamma)S^j_i a \]

Let \( \Delta = \Gamma \cup \{ i \} \). Then

\[ C(\Gamma)S_{suc}a + C(\Gamma)S_{pred}a + C(\Gamma)S^j_i a \subseteq C(\Delta)S_{suc}a + C(\Delta)S_{pred}a + C(\Delta)S^j_i a. \]
So given finite \( \Gamma \) and \( H \) as above, we take \( \Delta = \Gamma \cup \{ i \} \), we put
\[
Z = \{ \varphi : \varphi \in X_{Id}, \varphi_0 < \varphi_1 \} \cap C(\Delta)\{Id\}
\]
and
\[
W = \{ \varphi : \varphi \in X_{Id}, \varphi_0 > \varphi_1 \} \cap C(\Delta)\{Id\},
\]
and proceed as above. Then we eventually obtain that
\[
Id \in X_{Id} \setminus C(\Delta)(S_{suc}a + S_{pred}a + a).
\]
Hence
\[
Id \in X_{Id} \setminus C(\Gamma)(S_{suc}a + S_{pred}a + S_i a).
\]
We have shown that for any finite \( \Gamma \subseteq \omega \) and any \( H \subseteq G \), there are \( Z, W \subseteq \omega \omega \) such that (7) holds.

Combined with Theorem 7 we have the result in the abstract.

But we shall go further. Given two class \( K_1 \subseteq K_2 \) we say that \( K_1 \) has the amalgamation property with respect to \( K_2 \) if an amalgam can always be found in \( K_2 \). (\( K_1 \) itself may not have the amalgamation property.) The class of \( G \) set algebras is always properly contained in \( G \) algebras, for the latter is a finite schema axiomatizable variety while the former is not. When \( G \) consists only of finite transformations, then the class of \( G \) set algebras is not finite schema axiomatizable (a classical result of Sain), whereas if it contains at least one infinitary substitution, then it is not closed under ultrapowers, another result of Sain \cite{4}. We now have:

**Theorem 11.** Let \( K \) be a class of algebras such that \( GPEA_\omega \subseteq K \subseteq GPEA_\omega \). Then \( K \) does not have the amalgamation property with respect to \( GPEA_\omega \).

**Proof.** Seeking a contradiction assume that \( K \) has \( AP \) with respect to \( GPEA_\omega \). Let \( \mathfrak{A} = \mathfrak{F}_4GPEA_\omega \). Let \( \{ 0, 1, 2, 3 \} \) be the free generators of \( \mathfrak{A} \). Let \( r, s \) and \( t \) be as in the above Theorem 10. Let \( X_1 = \{ 0, 1 \} \) and \( X_2 = \{ 0, 2, 3 \} \). Then
\[
\mathfrak{A}^{(X_1 \cap X_2)} = Sg^\mathfrak{A}\{0\}.
\]
We have
\[
 r \in A^{(X_1)} \text{ and } s, t \in A^{(X_2)}.
\]
Let \( R \) be an ideal of \( \mathfrak{A} \) such that
\[
\mathfrak{A}/R \cong \mathfrak{F}_4K_\omega.
\]
By (16) we have

\[ r \in \mathfrak{I} \mathfrak{g}^\mathfrak{X} \{s,t\} \cap A^{(X_1)}. \]  

(18)

Let

\[ M = \mathfrak{I} \mathfrak{g}^{\mathfrak{X}(X_2)} \{s,t\} \cup (R \cap A^{(X_2)}); \]  

(19)

\[ N = \mathfrak{I} \mathfrak{g}^{\mathfrak{X}(X_1)} [(M \cap A^{(X_1 \cap X_2)}) \cup (R \cap A^{(X_1)})]. \]  

(20)

Then we have

\[ R \cap A^{(X_2)} \subseteq M \text{ and } R \cap A^{(X_1)} \subseteq N. \]  

(21)

From the first of these inclusions we get

\[ M \cap A^{(X_1 \cap X_2)} \supseteq (R \cap A^{(X_2)}) \cap A^{(X_1 \cap X_2)} = (R \cap A^{(X_1)}) \cap A^{(X_1 \cap X_2)}. \]  

By (20) we have

\[ N \cap A^{(X_1 \cap X_2)} = M \cap A^{(X_1 \cap X_2)}. \]  

From this we get

\[(\mathfrak{A}^{(X_2)}/M)^{(X_1 \cap X_2)} \cong \mathfrak{A}^{(X_1 \cap X_2)}/M \cap A^{(X_1 \cap X_2)} \]  

(22)

\[ = \mathfrak{A}^{(X_1 \cap X_2)}/N \cap A^{(X_1 \cap X_2)} \cong (\mathfrak{A}^{(X_1)}/N)^{(X_1 \cap X_2)} \]

From (17) and (21) we have \( \mathfrak{A}^{(X_2)}/M \) is in \( HSK \). By a similar argument \( \mathfrak{A}^{(X_1)}/N \) is in \( HSK \). By our assumption, there is a \( \mathfrak{B} \), a \( Y = \{y_0, y_1, y_2, y_3\} \) generating \( \mathfrak{B} \) and

\[ \mathfrak{B}^{(Y_1)} \cong \mathfrak{A}^{(X_1)}/N, \quad \mathfrak{B}^{(Y_2)} \cong \mathfrak{A}^{(X_2)}/M. \]

Here \( Y_1 = \{y_0, y_1\} \) and \( Y_2 = \{y_0, y_2, y_3\} \). Let \( P \) be the ideal of \( \mathfrak{A} \) such that \( \mathfrak{A}/P \cong \mathfrak{B} \). Then

\[ \mathfrak{A}^{(X_2)}/P \cap A^{(X_2)} \cong (\mathfrak{A}/P)^{(X_2)} \cong \mathfrak{B}^{(Y_2)} \cong \mathfrak{A}^{(X_2)}/M. \]

Thus

\[ P \cap A^{(X_1)} = N \]  

(23)

and

\[ P \cap A^{(X_2)} = M. \]  

(24)
In view of (16), (19), (23) we have \( s.t \in P \) and hence by (18) \( r \in P \). Consequently from (16) and (24) we get \( r \in N \). From (20) there exists elements
\[
u \in M \cap A^{(X_1 \cap X_2)}
\]
and \( b \in R \) such that
\[
r \leq u + b.
\]
Since \( u \in M \) by (19) there is a finite \( \Gamma \subseteq \omega \) and a finite \( H \subseteq G \) and a \( c \in R \) such that
\[
u \leq c_{(\Gamma)} \sum_{\tau \in H} s_{\tau}(s.t) + c.
\]
Let \( h \) be the homomorphism from \( A \) to \( \mathfrak{F}t_4K \) be such that \( h(i) = i \). (Strictly speaking \( h \) is not the identity function, it rather maps the first generator of \( A \) to the first generator of \( \mathfrak{F}t_4K \), and so on). Notice that \( kerh = R \). Then \( h(b) = h(c) = 0 \). It follows that
\[
h(r) \leq h(u) \leq c_{(\Gamma)} \sum_{\tau \in H} s_{\tau}(h(s).h(t)).
\]
This contradicts Lemma 10. By this the proof is complete.

\[\square\]

3 Proof of Theorem 4

We follow verbatim [7] until further notice. We only make sure that the presence of diagonals survives the proof in [7]. It turns out that this is fairly straightforward. Our second positive result formulated in Theorem 4 follows immediately from:

Proposition . Let \( X \) be a countable set. Let \( V = Mod(P_1 - P_{12}) \). Then \( \mathfrak{F}t_XV \) has the interpolation property.

Proof. Let \( A = \mathfrak{F}t_XV \) and let \( X_1, X_2 \subseteq X \). Let \( a \in \mathfrak{G}^X(X_1 \cap X_2) \) such that \( a \leq b \). We assume that no interpolant exists, and we arrive at a contradiction. An interpolant, in this case, is an element \( c \in \mathfrak{G}^X(X_1 \cap X_2) \) such that \( a \leq c \leq b \). Let \( n \leq \omega \). Then \( \alpha_n = \omega + n \) and \( M_n = \alpha_n \setminus \omega \). Note that when \( n \in \omega \), then \( M_n = \{\omega, \ldots, \omega + n - 1\} \). Let \( \tau \in T \). Then \( \tau_n = \tau \cup Id_{M_n} \). \( T_n \) denotes the subsemigroup of \( \langle \alpha_n, o \rangle \) generated by \( \{\tau_n : \tau \in T\} \cup \bigcup_{i,j \in \alpha_n} \{[i][j], [i], [j]\} \). For \( n \in \omega \), we let \( \rho_n : \alpha_n \to \omega \) be the bijection defined by \( \rho_n[i] = suc^\alpha n \) and \( \rho_n(\omega + i) = i \) for all \( i < n \). From now on, unless otherwise specified, \( n \) is a fixed finite ordinal, \( \alpha_n, M_n, T_n, \rho_n \) are as indicated above, and \( \mathfrak{A} \) is an arbitrary countable \( G \) algebra. (We do assume freeness yet.) Now let \( \mathfrak{A}_n \) be the algebra defined as follows:
\[
\mathfrak{A}_n = \langle A, +, \cdot, 0, 1, c^\alpha_n, d^\alpha_n, d^\alpha_n^{ij} \rangle_{i,j \in \alpha_n, v \in T_n}
\]
where for each $i, j \in \alpha_n$ and $v \in T_n$, 

$$c_i^{\alpha_n} := c_{\rho_v(i)}^\alpha, s_v^{\alpha_n} := s_v^\alpha \quad \text{and} \quad d_{ij}^{\alpha_n} = d_{\rho_v(i), \rho_v(j)}^\alpha.$$ 

Let $\mathfrak{A}_n$ be the following reduct of $\mathfrak{A}$ obtained by restricting the type of $\mathfrak{A}_n$ to the first $\omega$ dimensions:

$$\mathfrak{A}_n = \langle A_n, +, \cdot, 0, 1, c_i^{\alpha_n}, s_v^{\alpha_n}, d_{ij}^{\alpha_n} \rangle_{i, j < \omega}.$$ 

For $x \in A$, let $e_n(x) = s_{\text{suc}}(x)$. Then $e_n : \mathfrak{A} \to \mathfrak{A}_n$.

**Claim 1.** $e_n$ is an isomorphism from $\mathfrak{A}$ into $\mathfrak{A}_n$.

**Proof.** [7] Claim 2.6. We further check diagonals: $e_n$ preserves diagonal elements. Let $i, j < \omega$. Then by $(P_{12})$

$$e_n(d_{ij}^{\alpha_n}) = s_{\text{suc}}(d_{ij}) = d_{\text{suc}}(i, \text{suc}(j)) = d_{\rho_v(i), \rho_v(j)}^{\alpha_n} = d_{ij}^{\alpha_n}.$$ 

**Claim 2.** $e_n(\mathfrak{G}^\alpha Y) = \mathfrak{A}_n(\mathfrak{G}^{\alpha_n} e_n(Y))$ for all $Y \subseteq A$.

**Proof.** [7] Claim 2.7

For the sake of brevity, let $\alpha = \alpha_\omega = \omega + \omega$. Recall that $T_\omega$ is the semigroup generated by the set

$$\{\tau_\omega : \tau \in T\} \cup_{i, j \in \alpha} \{[i][j], [i, j]\}.$$ 

For $\sigma \in T_\omega$, and $n \in \omega$, let $[\sigma]_n = \sigma \upharpoonright \omega + n$. For each $n \in \omega$ let

$$\mathfrak{A}_n^+ = \langle A, +, \cdot, 0, 1, c_i^{\alpha_n^+}, s_\sigma^{\alpha_n^+}, d_{ij}^{\alpha_n^+} \rangle_{i, j < \omega, \sigma \in T_\omega}$$

be an expansion of $\mathfrak{A}_n$ such that $B|\mathfrak{A}_n^+ := B|\mathfrak{A}$ ($= B|\mathfrak{A}_n$); and for each $\sigma \in T_\omega$ and $i, j \in \alpha$, $s_\sigma^{\alpha_n^+} := s_{\sigma|n}^{\alpha_n}$ iff $[\sigma]_n \in T_n$, $c_i^{\alpha_n^+} := c_i^{\alpha_n}$ iff $i < \omega + n$, and $d_{ij}^{\alpha_n^+} := d_{ij}^{\alpha_n}$ iff $i, j < \omega + n$.

Fix $F$ to be any non-principal ultrafilter on $\omega$. Now forming the ultraproduct of the $\mathfrak{A}_n^+$'s relative to $F$, let

$$\mathfrak{A}^+ = \prod_{n \in \omega} \mathfrak{A}_n^+/F.$$ 

Next we neatly embed $\mathfrak{A}$ into $\mathfrak{A}^+$. Towards this end, for $x \in \mathfrak{A}$ we let

$$e(x) = \langle e_n(x) : n \in \omega \rangle/F.$$ 

Then of course $e : \mathfrak{A} \to \mathfrak{A}^+$. Now we let $\mathfrak{A}_\omega \mathfrak{A}^+$ be the following reduct of $\mathfrak{A}^+$, obtained by restricting its similarity type to the first $\omega$ dimensions:

$$\mathfrak{A}_\omega \mathfrak{A}^+ = \langle A^+, +, \cdot, 0, 1, c_i^{\alpha^+}, s_\sigma^{\alpha^+}, d_{ij}^{\alpha^+} \rangle_{i, j < \omega, \tau \in T}.$$
Recall that for $\tau \in \mathbb{T}$, $\tau_\omega = \tau \cup \text{Id}_{\alpha \cdot \omega} \in \mathbb{T}_\omega$. In particular, $\mathfrak{N}_{\alpha \cdot \omega}^{\mathbb{A}^+}$ is meaningful. Now that $\mathfrak{N}_{\alpha \cdot \omega} \mathbb{A}^+$ has the same similarity type as $\mathbb{A}$, we can compare the two algebras. And indeed we have:

**Claim 3.** $e$ is an isomorphism from $\mathbb{A}$ into $\mathfrak{N}_{\alpha \cdot \omega} \mathbb{A}^+$.

**Proof.** [7] Claim 2.9. We also check diagonals:

$$
eq e(d_{ij}^\mathfrak{N}) = \langle e_n(d_{ij})^\mathbb{A} : n \in \omega \rangle / F$$

$$= \langle d_{p_n(i),p_n(j)}^\mathfrak{N} : n \in \omega \rangle / F$$

$$= \langle d_{ij}^\mathfrak{N} : n \in \omega \rangle / F$$

$$= \langle d_{ij}^{\mathbb{A}^+} : n \in \omega \rangle / F$$

$$= d_{ij}^{\mathbb{A}^+}.$$

We embed $\mathbb{A}$ into $\mathfrak{N}_{\alpha \cdot \omega} \mathbb{A}^+ = \{ a \in \mathbb{A}^+ : c_i a = a \text{ for all } i, \omega \leq i < \omega + \omega \}$ via $e$,

**Claim 4.** $e(\mathfrak{N}_{\alpha \cdot \omega} \mathbb{A}^+) = \mathfrak{N}_{\alpha \cdot \omega} \mathbb{A}^+$ for all $Y \subseteq A$.

**Proof.** [7] Claim 2.10

From now on we assume that $\mathbb{A}$ is a subalgebra of $\mathfrak{N}_{\alpha \cdot \omega} \mathbb{A}^+$, i.e. we identify $\mathbb{A}$ with $e(\mathbb{A})$, regarding $e : \mathbb{A} \to \mathfrak{N}_{\alpha \cdot \omega} \mathbb{A}^+$ as the inclusion map. In particular, we have $\mathbb{A} \subseteq \mathfrak{N}_{\alpha \cdot \omega} \mathbb{A}^+$ and for all $Y \subseteq A$ we have (*)

$$\mathfrak{N}_{\alpha \cdot \omega} \mathbb{A}^+ \mathbb{A}^+ = \mathfrak{N}_{e(\mathbb{A})} \mathfrak{N}_{e(\mathbb{A})} \mathbb{A}^+ \mathbb{A}^+ = \mathfrak{N}_{\alpha \cdot \omega} \mathbb{A}^+ \mathbb{A}^+ Y.$$ 

In particular, $\mathbb{A} = \mathfrak{N}_{\alpha \cdot \omega} \mathbb{A}^+$. Also we fix sets $X,X_1$ and $X_2$ such that $X$ generates $\mathbb{A}$, and $X_1, X_2 \subseteq X$.

**Claim 5.** Let $a \in \mathfrak{N}_{\alpha \cdot \omega} \mathbb{A}^+ X_1$, $b \in \mathfrak{N}_{\alpha \cdot \omega} \mathbb{A}^+ X_2$ such that $a \leq c$. Assume that there is no $b \in \mathfrak{N}_{\alpha \cdot \omega} \mathbb{A}^+(X_1 \cap X_2)$ such that $\mathbb{A} \models a \leq b \leq c$. Then there is no $b \in \mathfrak{N}_{\alpha \cdot \omega} \mathbb{A}^+(X_1 \cap X_2)$ such that $\mathbb{A}^+ \models a \leq b \leq c$.

**Proof.** [7] Claim 2.15

We have $X$ is a generating set of $\mathbb{A}$, and $X_1, X_2 \subseteq X$. Further fix $a \in \mathfrak{N}_{\alpha \cdot \omega} \mathbb{A}^+(X_1)$ and $c \in \mathfrak{N}_{\alpha \cdot \omega} \mathbb{A}^+(X_2)$, for which no interpolant exists in $\mathfrak{N}_{\alpha \cdot \omega} \mathbb{A}^+(X_1 \cap X_2)$ (equivalently) no interpolant exists in $\mathfrak{N}_{\alpha \cdot \omega} \mathbb{A}^+(X_1 \cap X_2)$.) Recall that $\alpha = \omega + \omega$. For $Y \subseteq \mathbb{A}$, let $\mathbb{A}^+(Y)$ stand for $\mathfrak{N}_{\alpha \cdot \omega} \mathbb{A}^+ Y$. Recall that we assumed that $A \subseteq A^+$, that $\mathbb{A}$ is generated by $X$ (not freely yet) and that $X_1, X_2 \subseteq X$. Now arrange $\alpha \times \mathbb{A}^+(X_1)$ and $\alpha \times \mathbb{A}^+(X_2)$ into $\omega$-termed sequences:

$$\langle (k_i, x_i) : i \in \omega \rangle$$

and

$$\langle (l_i, y_i) : i \in \omega \rangle$$

respectively.
This is possible since both $\mathfrak{A}^+(X_1)$ and $\mathfrak{A}^+(X_2)$, being subalgebras of $A^+$, are countable and $|\alpha| = \omega$.

Recall that $\Delta x = \{ i \in \alpha : x \neq c_i x \}$. we can define by recursion (or step by step) $\omega$-termed sequences of witnesses:

$$\langle u_i : i \in \omega \rangle$$ and
$$\langle v_i : i \in \omega \rangle$$

such that for all $i \in \omega$ we have:

$$u_i \in \alpha \setminus (\Delta a \cup \Delta c) \cup \bigcup_{j \leq i}(\Delta x_j \cup \Delta y_j) \cup \{ u_j : j < i \} \cup \{ v_j : j < i \}$$

and

$$v_i \in \alpha \setminus (\Delta a \cup \Delta c) \cup \bigcup_{j \leq i}(\Delta x_j \cup \Delta y_j) \cup \{ u_j : j \leq i \} \cup \{ v_j : j < i \}.$$

**Notation.** For a boolean algebra $\mathfrak{B}$ and $Y \subseteq B$, we write $fl^\mathfrak{B} Y$ to denote the boolean filter generated by $Y$ in $\mathfrak{B}$. We let

$$Y_1 = \{a\} \cup \{-c_i x_i + s_{u_i}^k x_i : i \in \omega\},$$

$$Y_2 = \{-c\} \cup \{-c_i y_i + s_{v_i}^i y_i : i \in \omega\},$$

$$H_1 = fl^{B\mathfrak{A}^+(X_1)} Y_1, \ H_2 = fl^{B\mathfrak{A}^+(X_2)} Y_2,$$

and

$$H = fl^{B\mathfrak{A}^+(X_1 \cap X_2)} [(H_1 \cap A^+(X_1 \cap X_2)) \cup (H_2 \cap A^+(X_1 \cap X_2))].$$

**Claim 6.** $H$ is a proper filter of $\mathfrak{A}^+(X_1 \cap X_2)$.

**Proof.** [7] Claim 2.18

Now let $H^*$ be a (proper boolean) ultrafilter of $\mathfrak{A}^+(X_1 \cap X_2)$ containing $H$. From $BA$ theory\footnote{BA abbreviates boolean algebra.} we get ultrafilters $F_1$ and $F_2$ of $\mathfrak{A}^+(X_1)$ and $\mathfrak{A}^+(X_2)$, respectively such that

$$H^* \subseteq F_1, \ H^* \subseteq F_2$$

and

$$F_1 \cap \mathfrak{A}^+(X_1 \cap X_2) = F_2 \cap \mathfrak{A}^+(X_1 \cap X_2).$$

Then by their construction, it is easy to see that $F \in \{ F_1, F_2 \}$ satisfies the following ($+$):

$$(\forall k < \alpha)(c_k x \in F \implies (\exists \lambda \notin \Delta x)s^k x \in F)$$
Also for all \( x \in \mathfrak{A}^+(X_1 \cap X_2) \) we have \( x \in F_1 \) iff \( x \in F_2 \). In particular, for \( i, j < \alpha \), we have (++)
\[
d_{ij} \in F_1 \text{ iff } d_{ij} \in F_2.
\]

**Claim 7.**

(i) Let \( V \) be the generalized weak space \( \bigcup_{\tau \in \mathcal{T}} \omega^{\alpha_\tau} \). For \( v \in V \), let \( v^* = v \cup \text{Id} \upharpoonright (\alpha \setminus \omega) \). Then \( v^* \in \mathcal{T}_\omega \).

(ii) For \( k \in \{1, 2\} \), let \( f_k \) be the function with domain \( \mathfrak{A}_k = \mathfrak{S} \mathfrak{g}^{\mathfrak{A}}(X_k) \) such that
\[
f_k(a) = \{ v \in V : s_{v^*(\mathfrak{A}^+(X_k))} a \in F_k \}.
\]
Then for all \( i, j < \omega \), \( f_1(d_{ij}) = f_2(d_{ij}) \). \( \mathcal{B}_k = \langle \mathcal{B}(V), c_i, s_\tau, f_k(d_{ij}) \rangle_{i, j \in \omega, \tau \in \mathcal{T}} \) is a G algebra and \( f_k \) is a homomorphism from \( \mathfrak{A}_k \) into \( \mathcal{B}_k \). Also \( \text{Id} \in f_1(a) \cap f_2(-c) \).

(iii) Let \( g = f_1 \cup f_2 \upharpoonright X \). Then \( g \) is a function. Moreover \( g \) extends to a homomorphism from \( \mathfrak{A} \) into \( \langle \mathcal{B}(V), c_i, s_\tau, f_1(d_{ij}) \rangle_{i, j \in \omega, \tau \in \mathcal{G}} \). In particular, \( g(a - c) \neq 0 \).

(i) Follows from the definitions.

(ii) That \( \text{Id} \in f_1(a) \cap f_2(-c) \) follows from \( a \in F_1 \) and \( -c \in F_2 \). Fix \( k \in \{0, 1\} \). We show that \( \mathcal{B}_k \) is a G algebra. Let \( d^k_{ij} = f_k(d_{ij}) \). Let \( i, j < \omega \).
Then
\[
d^k_{ii} = f_k(d_{ii}) = \{ \tau \in V : s_{v^*(\mathfrak{A}^+(X_k))} d_{ii} \in F_k \}
= \{ \tau \in V : d_{\tau(i), \tau(i)} \in F_k \} = V.
\]
Also it is easy to check that
\[
s_\tau d^k_{ij} = d^k_{\tau(i), \tau(j)}.\]
Indeed we have
\[
v \in s_\tau d^k_{ij} \text{ iff } (v \circ \tau) \in d^k_{ij}
\]
iff
\[
(v \circ \tau) \in f_k(d_{ij}) \text{ iff } s_{v^*(\mathfrak{A}^+(X_k))} d_{ii} \in F_k
\]
iff
\[
d_{v^*(\mathfrak{A}^+(X_k))} d_{ij} \in F_k.
\]
On the other hand
\[
v \in d^k_{\tau(i), \tau(j)} \text{ iff } v \in f_k(d_{\tau(i), \tau(j)})
\]
iff
\[ s^*_i(d_{ri,\tau j}) \in F_k. \]

Iff
\[ d_{\circ \tau (i), \circ \tau (j)} \in F_k. \]

Thus \( B_k \) is a \( G \) algebra. It is easy to check that \( f_k \) is a boolean homomorphism for \( k \in \{0,1\} \). Also \( f_k \) preserves the diagonal elements, by definition. \( f_k \) preserves cylindrifications is like [7].

(iii) To prove that \( g \) is a function, we have to show that \( f_1 \) and \( f_2 \) agree on \( X_1 \cap X_2 \). But this follows immediately from how \( f_1 \) and \( f_2 \) are defined, by noting that \( F_1 \) and \( F_2 \) agree on \( \mathfrak{A}^+(X_1 \cap X_2) \). Since \( \mathfrak{A} \) is freely generated by \( X = X_1 \cup X_2 \), it follows that \( g \) extends to a homomorphism on \( \mathfrak{A} \). Also by
\[
g(a - c) = g(a) \cap g(-c) = f_1(a) \cap f_2(-c),
\]
and \( Id \) belonging to \( f_1(a) \) and \( f_2(-c) \), we get that
\[
Id \in g(a - c) \text{ i.e. } g(a - c) \neq 0.
\]
This contradicts that \( a \leq c \).

The above proof survives the case when \( G \) is a strongly rich semigroup, cf [7] p.345.

References


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