On Relative Defects of Wronskians

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Abstract

The aim of this paper is to compare the Valiron defect with the relative Nevanlinna defect of wronskians generated by a transcendental meromorphic function.

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1 Introduction, Definitions and Notations.

Let \( f \) be a meromorphic function defined in the open complex plane \( \mathbb{C} \). For \( a \in \mathbb{C} \cup \{\infty\} \) we denote by \( n(t, a; f) \) (\( \overline{n}(t, a; f) \)) the number of \( a \)-points (distinct \( a \)-points) of \( f \) in \( |z| \leq t \), where an \( \infty \)-point is a pole of \( f \). We put

\[
N(r; a; f) = \int_0^r \frac{n(t; a; f) - n(0; a; f)}{t} dt + n(0; a; f) \log r
\]

and

\[
\overline{N}(r; a; f) = \int_0^r \frac{\overline{n}(t; a; f) - \overline{n}(0; a; f)}{t} dt + \overline{n}(0; a; f) \log r.
\]
The function $N(r, a; f)(\bar{N}(r, a; f))$ is called the counting function of $a$-points (distinct $a$-points) of $f$. In many occasions $N(r, \infty; f)$ and $\bar{N}(r, \infty; f)$ are denoted by $N(r, f)$ and $\bar{N}(r, f)$ respectively.

We also put

$$m(r, f) = \frac{1}{2\pi} \int_0^{2\pi} \log^+ |f(re^{i\theta})| d\theta,$$

where

$$\log^+ x = \log x, \text{ if } x \geq 1$$

$$= 0, \text{ if } 0 \leq x < 1.$$

For $a \in \mathbb{C}$ we denote by $m(r, \frac{1}{f-a})$ by $m(r, a; f)$ and we mean by $m(r, \infty; f)$ the function $m(r, f)$, which is called the proximity function of $f$.

The function $T(r, f) = m(r, f) + N(r, f)$ is called the characteristic function of $f$. If $a \in \mathbb{C} \cup \{\infty\}$, the quantity

$$\delta(a; f) = 1 - \limsup_{r \to \infty} \frac{N(r, a; f)}{T(r, f)} = \liminf_{r \to \infty} \frac{m(r, a; f)}{T(r, f)}$$

is called the Nevanlinna deficiency of the value $a$.

From the second fundamental theorem it follows that the set of values of $a \in \mathbb{C} \cup \{\infty\}$ for which $\delta(a; f) > 0$ is countable and $\sum_{a \neq \infty} \delta(a; f) + \delta(\infty; f) \leq 2$ (cf. [1, p.43]). If in particular, $\sum_{a \neq \infty} \delta(a; f) + \delta(\infty; f) = 2$, we say that $f$ has the maximum deficiency sum.

Similarly, the Valiron deficiency $\Delta(a, f)$ of the value ‘$a$’ is defined as

$$\Delta(a; f) = 1 - \liminf_{r \to \infty} \frac{N(r, a; f)}{T(r, f)} = \limsup_{r \to \infty} \frac{m(r, a; f)}{T(r, f)}.$$

Milloux [3] introduced the concept of absolute defect of ‘$a$’ with respect to $f'$. Later Xiong [5] extended this definition. He introduced the term

$$\delta_R^{(k)}(a; f) = 1 - \limsup_{r \to \infty} \frac{N(r, a; f^{(k)})}{T(r, f)},$$

for $k = 1, 2, 3, \ldots$ and called it the relative Nevanlinna defect of ‘$a$’ with respect to $f^{(k)}$. Xiong [5] has shown various relations between the usual defects and the relative defects for meromorphic functions. Singh [4] introduced the term relative defect for distinct zeros and poles and established various relations between it, relative defects and the usual defects.

We may now recall the following definition.
Definition 1. The order $\rho_f$ and lower order $\lambda_f$ of a meromorphic function $f$ are defined as follows:

$$
\rho_f = \limsup_{r \to \infty} \frac{\log T(r, f)}{\log r} \quad \text{and} \quad \lambda_f = \liminf_{r \to \infty} \frac{\log T(r, f)}{\log r}.
$$

If $\rho_f < \infty$ then $f$ is of finite order.

The term $S(r, f)$ denotes any quantity satisfying $S(r, f) = o\{T(r, f)\}$ as $r \to \infty$ through all values of $r$ if $f$ is of finite order and except possibly for a set of $r$ of finite linear measure otherwise. We do not explain the standard definitions and notations of the value distribution and the Nevanlinna theory as those are available in [1].

The following definitions are well known.

Definition 2. A meromorphic function $a = a(z)$ is called small with respect to $f$ if $T(r, a) = S(r, f)$.

Definition 3. Let $a_1, a_2, \ldots, a_k$ be linearly independent meromorphic functions and small with respect to $f$. We denote by $L(f) = W(a_1, a_2, \ldots, a_k, f)$ the wronskian determinant of $a_1, a_2, \ldots, a_k, f$ i.e.,

$$
L(f) = \begin{vmatrix}
\begin{array}{cccc}
  a_1 & a_2 & \ldots & a_k \\
  a'_1 & a'_2 & \ldots & a'_k \\
  \vdots & \vdots & \ddots & \vdots \\
  a^{(k)}_1 & a^{(k)}_2 & \ldots & a^{(k)}_k \\
\end{array}
\end{vmatrix}
\begin{array}{c}
  f \\
  f' \\
  \vdots \\
  f^{(k)} \\
\end{array}
$$

In this paper we call the terms

$$
\delta_A^L(a; f) = 1 - \limsup_{r \to \infty} \frac{N(r, a; L(f))}{T(r, L(f))} = \liminf_{r \to \infty} \frac{m(r, a; L(f))}{T(r, L(f))},
$$

the usual Nevanlinna defect or the absolute Nevanlinna defect of the value ‘$a$’ with respect to $L(f)$,

$$
\Delta_A^L(a; f) = 1 - \liminf_{r \to \infty} \frac{N(r, a; L(f))}{T(r, L(f))} = \limsup_{r \to \infty} \frac{m(r, a; L(f))}{T(r, L(f))},
$$

the usual Valiron defect or the absolute Valiron defect of the value ‘$a$’ with respect to $L(f)$,

$$
\delta_R^L(a; f) = 1 - \limsup_{r \to \infty} \frac{N(r, a; L(f))}{T(r, f)}.
$$
the relative Nevanlinna defect of ‘a’ with respect to $L(f)$ and

$$\Delta^L_R(a; f) = 1 - \liminf_{r \to \infty} \frac{N(r, a; L(f))}{T(r, f)},$$

the relative Valiron defect of ‘a’ with respect to $L(f)$ and prove various relations among them.

Since the natural extension of a derivative is a differential polynomial, in this paper we prove our results for a special type of linear differential polynomials viz. the wronskians.

2 Lemmas.

In this section we present some lemmas which will be needed in the sequel.

**Lemma 1.** [2] Let $f$ be a transcendental meromorphic function having the maximum deficiency sum. Then

$$\lim_{r \to \infty} \frac{T(r, L(f))}{T(r, f)} = 1 + k - k\delta(\infty; f).$$

**Lemma 2.** Let $f$ be a transcendental meromorphic function having the maximum deficiency sum. Then for any $\alpha$,

$$\Delta^L_R(\alpha; f) = \{k\delta(\infty; f) - k\} + \limsup_{r \to \infty} \frac{m(r, \alpha; L(f))}{T(r, f)},$$

and

$$\delta^L_R(\alpha; f) = \{k\delta(\infty; f) - k\} + \liminf_{r \to \infty} \frac{m(r, \alpha; L(f))}{T(r, f)}.$$
Proof. In view of Lemma 1, we obtain that

\[
\Delta_R^P(\alpha; f) = 1 - \liminf_{r \to \infty} \frac{N(r, \alpha; L(f))}{T(r, f)}
\]

\[
= 1 - \liminf_{r \to \infty} \frac{N(r, \alpha; L(f))}{T(r, L(f))} \cdot \lim_{r \to \infty} \frac{T(r, L(f))}{T(r, f)}
\]

\[
= 1 - \liminf_{r \to \infty} \frac{N(r, \alpha; L(f))}{T(r, L(f))} \cdot \{1 + k - k \delta(\infty; f)\}
\]

\[
= \{1 + k - k \delta(\infty; f)\} \{1 - \liminf_{r \to \infty} \frac{N(r, \alpha; L(f))}{T(r, L(f))}\}
\]

\[
+ \{k \delta(\infty; f) - k\}
\]

\[
= \{1 + k - k \delta(\infty; f)\} \limsup_{r \to \infty} \frac{m(r, \alpha; L(f))}{T(r, L(f))} + \{k \delta(\infty; f) - k\}
\]

\[
= \{1 + k - k \delta(\infty; f)\} \{\limsup_{r \to \infty} \frac{m(r, \alpha; L(f))}{T(r, f)} \cdot \lim_{r \to \infty} \frac{T(r, f)}{T(r, L(f))}\}
\]

\[
+ \{k \delta(\infty; f) - k\}
\]

\[
= k \delta(\infty; f) - k + \limsup_{r \to \infty} \frac{m(r, \alpha; L(f))}{T(r, f)}.
\]

This proves the first part of the lemma.
Similarly we can prove the second part of the lemma.

Lemma 3. [1] Let \( k \) be any positive integer and \( \Psi = \sum_{i=0}^{k} a_i f^{(i)} \), where \( a_i \) are meromorphic functions, such that \( T(r, a_i) = S(r, f) \) for \( i = 0, 1, 2, ..., k \). Then

\[
m(r, \frac{\Psi}{f}) = S(r, f).
\]

3 Theorems.
In this section we present the main results of the paper.

Theorem 1. Let \( f \) be a transcendental meromorphic function and ‘\( a \)’ be any non-zero finite complex number. Then

\[
\delta(0; f) + \Delta^L_R(\infty; f) + \delta(a; f) \leq \Delta(\infty; f) + \Delta^L_R(0; f).
\]

Proof. Let us consider the following identity

\[
\frac{a}{f} = 1 - \frac{f - a}{L(f)} \cdot \frac{L(f)}{f}
\]
Since \( m(r, \frac{1}{f}) \leq m(r, \frac{f-a}{L(f)}) + O(1) \), in view of Lemma 3 we get from the above identity that

\[
m(r, \frac{1}{f}) \leq m(r, \frac{f-a}{L(f)}) + m(r, \frac{L(f)}{f})
\]

i.e.,

\[
m(r, \frac{1}{f}) \leq m(r, \frac{f-a}{L(f)}) + S(r, f).
\] (1)

Now by Nevanlinna’s first fundamental theorem and by Lemma 3 it follows from (1) that

\[
m(r, \frac{1}{f}) \leq T(r, \frac{f-a}{L(f)}) - N(r, \frac{f-a}{L(f)}) + S(r, f)
\]

i.e.,

\[
m(r, \frac{1}{f}) \leq T(r, \frac{L(f)}{f-a}) - N(r, \frac{f-a}{L(f)}) + S(r, f)
\]

i.e.,

\[
m(r, \frac{1}{f}) \leq N(r, \frac{L(f)}{f-a}) + m(r, \frac{L(f)}{f-a}) - N(r, \frac{f-a}{L(f)})
\]

\[
+ S(r, f)
\]

i.e.,

\[
m(r, \frac{1}{f}) \leq N(r, \frac{L(f)}{f-a}) - N(r, \frac{f-a}{L(f)}) + S(r, f).
\] (2)

In view of \{p.34, [1]\} it follows from (2) that

\[
m(r, \frac{1}{f}) \leq N(r, L(f)) + N(r, \frac{1}{f-a}) - N(r, f-a)
\]

\[-N(r, \frac{1}{L(f)}) + S(r, f)
\]

i.e.,

\[
\liminf_{r \to \infty} \frac{m(r, \frac{1}{f})}{T(r, f)} \leq \liminf_{r \to \infty} \frac{N(r, L(f))}{T(r, f)} - \frac{N(r, f-a)}{T(r, f)}
\]

\[-\frac{N(r, \frac{1}{L(f)})}{T(r, f)} + \limsup_{r \to \infty} \frac{N(r, \frac{1}{f-a})}{T(r, f)}
\]

i.e.,

\[
\liminf_{r \to \infty} \frac{m(r, \frac{1}{f})}{T(r, f)} \leq \liminf_{r \to \infty} \frac{N(r, L(f))}{T(r, f)} - \liminf_{r \to \infty} \frac{N(r, f-a)}{T(r, f)}
\]

\[-\liminf_{r \to \infty} \frac{N(r, \frac{1}{L(f)})}{T(r, f)} + \limsup_{r \to \infty} \frac{N(r, \frac{1}{f-a})}{T(r, f)}
\]
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i.e., \[\delta(0; f) \leq \{1 - \Delta^L_R(\infty; f)\} - \{1 - \Delta(\infty; f)\} - \{1 - \Delta^L_R(0; f)\} + \{1 - \delta(a; f)\}\]

i.e., \[\delta(0; f) + \Delta^L_R(\infty; f) + \delta(a; f) \leq \Delta(\infty; f) + \Delta^L_R(0; f).\]

This proves the theorem.

**Remark 1.** The sign \(\leq\) in Theorem 1 can not be replaced by \(\langle \) only. This is evident from the following example.

**Example 1.** Let \(f = \exp z\). Then \(\Delta(\infty; f) = \Delta^L_R(0; f) = \Delta^L_R(\infty; f) = 1\) and \(\delta(0; f) = \delta(\infty; f) = 1\). So \(\delta(a; f) = 0\). Then \(\delta(0; f) + \Delta^L_R(\infty; f) + \delta(a; f) = 2 = \Delta(\infty; f) + \Delta^L_R(0; f)\).

**Theorem 2.** If \(f\) be a transcendental meromorphic function having the maximum deficiency sum with \(\delta(\infty; f) = 1\) then

\[\Delta^L_R(\infty; f) + \delta(0; f) \leq \Delta^L_R(0; f) + \Delta^L_R(\infty; f).\]

**Proof.** Since \(f = L(f) \frac{L(f)}{f}\) we get that,

\[m(r, f) \leq m(r, L(f)) + m(r, \frac{f}{L(f)}). \quad (3)\]

Now by Nevanlinna’s first fundamental theorem and by Lemma 3 we obtain from (3) that

\[m(r, f) \leq m(r, L(f)) + T(r, \frac{f}{L(f)}) - N(r, \frac{1}{L(f)})\]

i.e., \(m(r, f) \leq m(r, L(f)) + T(r, \frac{L(f)}{f}) - N(r, \frac{f}{L(f)}) + O(1)\)

i.e., \(m(r, f) \leq m(r, L(f)) + N(r, \frac{L(f)}{f}) + m(r, \frac{L(f)}{f}) - N(r, \frac{f}{L(f)}) + O(1). \quad (4)\)

Now in view of \(p.34, [1]\) it follows from (4) that

\[m(r, f) \leq m(r, L(f)) + N(r, L(f)) + N(r, \frac{1}{L(f)}) - N(r, f) - N(r, \frac{1}{L(f)}) + S(r, f) + O(1)\]
Thus the theorem is proved.

Since $\delta(\infty; f) = 1$, then $\Delta(\infty; f) = 1$. So by Lemma 1 we obtain from (5) that

$$
\delta(\infty; f) \leq \{1 - \Delta_R^L(\infty; f)\} - \{1 - \Delta(\infty; f)\} - \{1 - \Delta_R^L(0; f)\} + \{1 - \delta(0; f)\} + \limsup_{r \to \infty} \frac{m(r, L(f))}{T(r, L(f))} \limsup_{r \to \infty} \frac{T(r, L(f))}{T(r, f)}
$$

i.e., $\delta(\infty; f) + \Delta_R^L(\infty; f) + \delta(0; f) \leq \Delta(\infty; f) + \Delta_R^L(0; f) + \Delta_R^L(\infty; f) - \delta(0; f) + k \delta(\infty; f)$

i.e., $1 + \Delta_R^L(\infty; f) + \delta(0; f) \leq 1 + \Delta_R^L(0; f) + \Delta_R^L(\infty; f)$

Thus the theorem is established.

**Remark 2.** If we omit the condition $\delta(\infty; f) = 1$ of Theorem 2 and the other conditions remaining the same, using the first part of Lemma 2 we may establish the next theorem.

**Theorem 3.** Let $f$ be a transcendental meromorphic function having the maximum deficiency sum. Then

$$
\delta(\infty; f) + \delta(0; f) + k \delta(\infty; f) \leq \Delta(\infty; f) + \Delta_R^L(0; f) + k.
$$

**Proof.** Using the first part of Lemma 2 and inequality (5) it follows that

$$
\delta(\infty; f) \leq \{1 - \Delta_R^L(\infty; f)\} - \{1 - \Delta(\infty; f)\} - \{1 - \Delta_R^L(0; f)\} + \{1 - \delta(0; f)\} + \Delta_R^L(\infty; f) - k \delta(\infty; f) + k
$$

i.e.,

$$
\delta(\infty; f) + \delta(0; f) + k \delta(\infty; f) \leq \Delta(\infty; f) + \Delta_R^L(0; f) + k.
$$

Thus the theorem is proved.
Theorem 4. Let $a, b \neq 0, \infty$ be any two distinct complex numbers. Then for any transcendental meromorphic function $f$

$$2\delta(a; f) + \delta(b; f) + 2\Delta_R^L(\infty; f) \leq 2\Delta(\infty; f) + 2\Delta_R^L(0; f).$$

Proof. Considering the identity

$$\frac{b - a}{f - a} = \frac{L(f)}{f - a} \left\{ \frac{f - a}{L(f)} - \frac{f - b}{L(f)} \right\},$$

we obtain in view of Lemma 3 that

$$m(r, \frac{b - a}{f - a}) \leq m(r, \frac{f - a}{L(f)}) + m(r, \frac{f - b}{L(f)}) + m(r, \frac{L(f)}{f - a}),$$

i.e.,

$$m(r, \frac{b - a}{f - a}) \leq T(r, \frac{f - a}{L(f)}) - N(r, \frac{f - a}{L(f)}) + T(r, \frac{f - b}{L(f)})$$

$$-N(r, \frac{f - b}{L(f)}) + S(r, f). \quad (6)$$

Since $m(r, \frac{1}{f - a}) \leq m(r, \frac{b - a}{f - a}) + O(1)$ and $T(r, f) = T(r, \frac{1}{f}) + O(1)$, it follows from (6) that

$$m(r, \frac{1}{f - a}) \leq T(r, \frac{L(f)}{f - a}) - N(r, \frac{f - a}{L(f)}) + T(r, \frac{L(f)}{f - b})$$

$$-N(r, \frac{f - b}{L(f)}) + S(r, f) + O(1)$$

i.e.,

$$m(r, \frac{1}{f - a}) \leq N(r, \frac{L(f)}{f - a}) + m(r, \frac{L(f)}{f - a}) - N(r, \frac{f - a}{L(f)})$$

$$+N(r, \frac{L(f)}{f - b}) + m(r, \frac{L(f)}{f - b}) - N(r, \frac{f - b}{L(f)})$$

$$+S(r, f) + O(1)$$

i.e.,

$$m(r, \frac{1}{f - a}) \leq N(r, \frac{L(f)}{f - a}) - N(r, \frac{f - a}{L(f)}) + N(r, \frac{L(f)}{f - b})$$

$$-N(r, \frac{f - b}{L(f)}) + S(r, f) + O(1). \quad (7)$$
In view of \{p.34, [1]\} we get from [7] that
\[
m(r, \frac{1}{f - a}) \leq N(r, L(f)) + N(r, \frac{1}{f - a}) - N(r, f - a) \\
- N(r, \frac{1}{L(f)}) + N(r, L(f)) + N(r, \frac{1}{f - b}) \\
- N(r, f - b) - N(r, \frac{1}{L(f)}) + S(r, f)
\]
i.e., \(m(r, \frac{1}{f - a}) \leq 2N(r, L(f)) - 2N(r, f) - 2N(r, \frac{1}{L(f)}) + N(r, \frac{1}{f - a}) + N(r, \frac{1}{f - b}) + S(r, f) + O(1)\)
i.e., \[\liminf_{r \to \infty} \frac{m(r, \frac{1}{f - a})}{T(r, f)} \leq 2\liminf_{r \to \infty} \left\{ \frac{N(r, L(f))}{T(r, f)} - \frac{N(r, f)}{T(r, f)} - \frac{N(r, \frac{1}{L(f)})}{T(r, f)} \right\} \]
+ \limsup_{r \to \infty} \left\{ \frac{N(r, \frac{1}{f - a})}{T(r, f)} + \frac{N(r, \frac{1}{f - b})}{T(r, f)} \right\}
i.e., \[\delta(a; f) \leq 2\left\{ 1 - \Delta_L^f(\infty; f) \right\} - 2\left\{ 1 - \Delta(\infty; f) \right\} \]
-2\left\{ 1 - \Delta_H^L(0; f) \right\} + \left\{ 1 - \delta(a; f) \right\} \\
+ \left\{ 1 - \delta(b; f) \right\}
i.e., \(2\delta(a; f) + \delta(b; f) + 2\Delta_H^L(\infty; f) \leq 2\Delta(\infty; f) + 2\Delta_H^L(0; f)\).

This proves the theorem.

\textbf{Theorem 5.} Let ‘a’ be a finite complex number and \(b, c\) be two distinct non zero complex numbers. Then for any transcendental meromorphic function \(f\) having the maximum deficiency sum
\[
\delta(a; f) + \left\{ 1 + k - k\delta(\infty; f) \right\} \{ \delta_b^L(b; f) + \delta_c^L(c; f) \} \leq 2\left\{ 1 + k - k\delta(\infty; f) \right\}.
\]
\textbf{Proof.} Since \( \frac{1}{f-a} = \frac{L_f}{f-a} \frac{L_f}{L_f} \), by Lemma 3 we obtain that
\[
m(r, \frac{1}{f-a}) \leq m(r, \frac{1}{L_f}) + m(r, \frac{L_f}{f-a})
\]
i.e., \( m(r, \frac{1}{f-a}) \leq m(r, \frac{1}{L_f}) + S(r, f) \).

Applying Nevanlinna’s first fundamental theorem we get from (8) that
\[
m(r, \frac{1}{f-a}) \leq T(r, \frac{1}{L_f}) - N(r, \frac{1}{L_f}) + S(r, f).
\]

Now by Nevanlinna’s second fundamental theorem it follows from (9) that
\[
m(r, \frac{1}{f-a}) \leq \bar{N}(r, \frac{1}{L_f}) + \bar{N}(r, \frac{1}{L_f - b}) + \bar{N}(r, \frac{1}{L_f - c}) - N(r, \frac{1}{L_f}) + S(r, f).
\]

Since \( \bar{N}(r, \frac{1}{L_f}) - N(r, \frac{1}{L_f}) \leq 0 \), we obtain from (10) in view of Lemma 1 that
\[
m(r, \frac{1}{f-a}) \leq \bar{N}(r, \frac{1}{L_f - b}) + \bar{N}(r, \frac{1}{L_f - c}) - N(r, \frac{1}{L_f}) + S(r, f)
\]
i.e., \( m(r, \frac{1}{f-a}) \leq N(r, \frac{1}{L_f - b}) + N(r, \frac{1}{L_f - c}) + S(r, f) \)

i.e., \( m(r, \frac{1}{f-a}) \leq T(r, \frac{1}{L_f - b}) + T(r, \frac{1}{L_f - c}) - m(r, \frac{1}{L_f - b}) - m(r, \frac{1}{L_f - c}) + S(r, f) \)
i.e., \( m(r, \frac{1}{f-a}) \leq 2T(r, L_f) - m(r, \frac{1}{L_f - b}) - m(r, \frac{1}{L_f - c}) + S(r, f) \)

i.e., \( \liminf_{r \to \infty} \frac{m(r, \frac{1}{f-a})}{T(r, f)} \leq 2 \liminf_{r \to \infty} \frac{T(r, L_f)}{T(r, f)} - \liminf_{r \to \infty} \frac{m(r, \frac{1}{L_f - b})}{T(r, f)} - \liminf_{r \to \infty} \frac{m(r, \frac{1}{L_f - c})}{T(r, f)} \).
\[ i.e., \liminf_{r\to\infty} \frac{m\left(r, \frac{1}{f-a}\right)}{T(r, f)} \leq 2\liminf_{r\to\infty} \frac{T(r, L(f))}{T(r, f)} - \liminf_{r\to\infty} \frac{m\left(r, \frac{1}{L(f)-b}\right)}{T(r, L(f))} \lim_{r\to\infty} \frac{T(r, L(f))}{T(r, f)} \]

\[ -\liminf_{r\to\infty} \frac{m\left(r, \frac{1}{L(f)-c}\right)}{T(r, L(f))} \lim_{r\to\infty} \frac{T(r, L(f))}{T(r, f)} \]

\[ i.e., \delta(a; f) \leq 2\{1 + k - k\delta(\infty; f)\} - \delta_A^L(b; f)\{1 + k - k\delta(\infty; f)\} \]

\[ -\delta_A^L(c; f)\{1 + k - k\delta(\infty; f)\} \]

i.e., \[ \delta(a; f) + \{1 + k - k\delta(\infty; f)\}\{\delta_A^L(b; f) + \delta_A^L(c; f)\} \leq 2\{1 + k - k\delta(\infty; f)\} \]

Thus the theorem is established.

**References**


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