Chinese Checker Versions
of the Pythagorean Theorem

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Abstract
In this paper, we give Chinese checker versions of the Pythagorean
Theorem, and show that the converses of these Chinese checker versions
of the Pythagorean Theorem are not true. Finally, we give a necessary
and sufficient condition for a triangle in the Chinese checker plane to
have a right angle.

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gometry

1 Introduction
In the game of Chinese checkers (see [14]), checkers are allowed to move in the
vertical (north and south), horizontal (east and west), and diagonal (northeast,
northwest, southeast and southwest) directions. In [9], Krause asked how to
develop a distance function that measures the length of ways mimicking the
movements of the Chinese checkers, from a point to another in the Cartesian
coordinate plane. Later, Chen [3] defined the distance, named it Chinese
checker distance, and then proved that the Chinese checker distance is a metric.
If $P = (x_1, y_1)$ and $Q = (x_2, y_2)$ are two points in $\mathbb{R}^2$, the Chinese checker (CC)
distance between $P$ and $Q$ is
\[ d_C(P, Q) = \max \{|x_1 - x_2|, |y_1 - y_2|\} + (\sqrt{2} - 1) \min \{|x_1 - x_2|, |y_1 - y_2|\} \]

while the well-known Euclidean distance between $P$ and $Q$ is
\[ d_E(P, Q) = [(x_1 - x_2)^2 + (y_1 - y_2)^2]^{1/2}. \]
A metric geometry consists of a set \( \mathcal{P} \), whose elements are called points, together with a collection \( \mathcal{L} \) of non-empty subsets of \( \mathcal{P} \), called lines, and a distance function \( d \), such that

1) Every two distinct points in \( \mathcal{P} \) lie on a unique line,
2) There exist three points in \( \mathcal{P} \), which do not lie all on one line,
3) There exists a bijective function \( f : \mathcal{L} \to \mathbb{R} \) for all lines in \( \mathcal{L} \) such that \( |f(P) - f(Q)| = d(P, Q) \) for each pair of points \( P \) and \( Q \) on \( l \).

A metric geometry defined above is denoted by \( \{\mathcal{P}, \mathcal{L}, d\} \). However, if a metric geometry satisfies the plane separation axiom below, and it has an angle measure function \( m \), then it is called protractor geometry and denoted by \( \{\mathcal{P}, \mathcal{L}, d, m\} \).

4) For every \( l \) in \( \mathcal{L} \), there are two subsets \( H_1 \) and \( H_2 \) of \( \mathcal{P} \) (called half planes determined by \( l \)) such that
   (i) \( H_1 \cup H_2 = \mathcal{P} - l \) (\( \mathcal{P} \) with \( l \) removed),
   (ii) \( H_1 \) and \( H_2 \) are disjoint and each is convex,
   (iii) If \( A \in H_1 \) and \( B \in H_2 \), then \( [AB] \cap l \neq \emptyset \).

If \( L_E \) is the set of all lines in the Cartesian coordinate plane, and \( m_E \) is the standard angle measure function in the Euclidean plane, then \( \{\mathbb{R}^2, L_E, d_C, m_E\} \), called CC plane, is a model of protractor geometry (This can be shown easily: the proof is similar to that of taxicab plane; refer to [10] or [5] to see that the taxicab plane is a model of protractor geometry). CC plane is also in the class of non-Euclidean geometries since it fails to satisfy the side-angle-side axiom (see Figure 1). However, CC plane is almost the same as Euclidean plane \( \{\mathbb{R}^2, L_E, d_E, m_E\} \) since the points are the same, the lines are the same, and the angles are measured in the same way. Since CC geometry has a distance function different from that in Euclidean geometry, it is interesting to study the CC analogues of topics that include the distance concept in Euclidean geometry (see [1], [2], [4], [6], [7], [8], [11], [12] and [13] for some works on this subject).

![Figure 1](image.png)

**Figure 1.** In figure \( d_C(A, O) = d_C(A, B) = d_C(O, C) = \sqrt{3} + \sqrt{2} - 1 \) and \( \angle(OAB) = \angle(AOC) = \pi/3 \), but \( d_C(A, B) \neq d_C(A, C) \).
2 CC Versions of the Pythagorean Theorem

It is well-known that if $ABC$ is a triangle with right angle $A$ in the Euclidean plane, then $a^2 = b^2 + c^2$ where $a = d_E(B,C)$, $b = d_E(A,C)$ and $c = d_E(A,B)$; this is the Pythagorean Theorem. Also it is well-known that its converse is true in the Euclidean plane. A CC version of the Pythagorean Theorem for a right triangle $ABC$ would be an equation that relates the three CC distances $a$, $b$, $c$ between pairs of vertices, where $a = d_C(B,C)$, $b = d_C(A,C)$ and $c = d_C(A,B)$. Two CC versions of the Pythagorean Theorem that depend on two parameters in addition to the CC distances between the vertices of a right triangle, were given in [7]. Here, we give CC versions of the Pythagorean Theorem that depend on only one parameter in addition to the CC distances between the vertices of a right triangle. We also show that the converses of these CC versions of the Pythagorean Theorem are false in the CC plane.

The following equation, which relates the Euclidean distance to the CC distance between two points in the Cartesian coordinate plane, plays an important role in our arguments.

\[ d_E(P,Q) = \rho(m) d_C(P,Q) \] (1)

where $\rho(m) = (1 + m^2)^{1/2}/\left(\max\{1, |m|\} + (\sqrt{2} - 1) \min\{1, |m|\}\right)$. If $P$ and $Q$ lie on a vertical line, then by definition, $d_E(P,Q) = d_C(P,Q)$.

**Proof.** Let $P = (x_1, y_1)$ and $Q = (x_2, y_2)$ with $x_1 \neq x_2$; then $m = (y_2 - y_1)/(x_2 - x_1)$. Equation (1) is derived by a straightforward calculation with $m$ and the coordinate definitions of $d_E(P,Q)$ and $d_C(P,Q)$ given in Section 1.

Another useful fact that can be verified by direct calculation is:

**Lemma 2.2** For any real number $m \neq 0$, let $m' = -1/m$. Then

\[ \rho(m) = \rho(m'). \] (2)

In all that follows, unless otherwise stated, $ABC$ is a triangle in the Cartesian coordinate plane with vertices labeled in counterclockwise order, with right angle at $A$. The Euclidean distances between pairs of vertices are denoted $a$, $b$, $c$, and the corresponding CC distances are $a$, $b$, $c$, as defined earlier. We first note that although the Euclidean distances $b$ and $c$ are, in general, different from the corresponding CC distances $b$ and $c$, corresponding ratios of these distances are equal.
Lemma 2.3 \( b/c = b/c \).

**Proof.** If the legs \( AB \) and \( AC \) of \( ABC \) are parallel to the coordinate axes, then \( b = b \) and \( c = c \), and the two ratios are equal. If one of the legs of \( ABC \) is not parallel to a coordinate axis, then neither is the other. If the slope of \( AB \) is \( m \), then the slope of \( AC \) is \( m' = -1/m \), since the legs are perpendicular. By equation (1), \( c = \rho(m)c \) and \( b = \rho(m')b \). But then equation (2) implies that \( b/c = b/c \). \( \blacksquare \)

Our main results follow; these give relations between the three CC distances \( a, b, c \) that depend only on one parameter, namely, the slope of one of the legs or the slope of the hypotenuse of right triangle \( ABC \). If a leg or the hypotenuse of \( ABC \) is parallel to a coordinate axis, then there is a relation between \( a, b, \) and \( c \) that does not depend on any other parameter.

**Theorem 2.4** (i) If the legs of \( ABC \) are parallel to the coordinate axes, then

\[
a = \max\{b, c\} + (\sqrt{2} - 1) \min\{b, c\}.
\]

(ii) If the legs of \( ABC \) are not parallel to the coordinate axes, the hypotenuse \( BC \) is not vertical, and \( m \) is the slope of one leg, then

\[
\begin{align*}
(\max\{1, |m|\} + (\sqrt{2} - 1) \min\{1, |m|\})a &= \max\{|bm + c|, |cm - b|\} + \\
&= (\sqrt{2} - 1) \min\{|bm + c|, |cm - b|\}.
\end{align*}
\]

**Proof.** (i) This follows immediately from the definition of CC distance.

(ii) Let angle \( CBA \) be denoted \( \theta \); note that \( \theta \) is positive and acute, by the counterclockwise labeling of \( ABC \) (see Figure 2). Then \( \tan \theta = b/c = b/c \),

![Figure 2](image)

by Lemma 2.3. First suppose that \( m \) is the slope of \( AB \); then \( m' = -1/m \) is the slope of \( AC \). Let \( m_1 \) be the slope of \( BC \). It is well-known (or easily found, using the identity for the tangent of the difference of two angles) that \( \tan \theta = (m - m_1)/(1 + mm_1) \). Thus

\[
b/c = (m - m_1)/(1 + mm_1).
\]
Solving equation (5) for \( m_1 \) yields

\[
m_1 = \frac{cm - b}{bm + c}
\]  

(6)

where \( m \neq -c/b \). Applying equation (1) to the Pythagorean theorem \( a^2 = b^2 + c^2 \), and using Lemma 2.2 gives

\[
[(1 + m_1^2)^{1/2}/(\max\{1, |m_1|\}) + (\sqrt{2} - 1) \min\{1, |m_1|\}]^2 a^2 = \\
[(1 + m^2)^{1/2}/(\max\{1, |m|\}) + (\sqrt{2} - 1) \min\{1, |m|\}]^2 (b^2 + c^2)
\]  

(7)

which simplifies to

\[
(\max\{1, |m|\} + (\sqrt{2} - 1) \min\{1, |m|\})^2 a^2 = \\
[(\max\{1, |m_1|\} + (\sqrt{2} - 1) \min\{1, |m_1|\})^2/(1 + m_1^2)](1 + m^2)(b^2 + c^2)
\]  

(8)

Substituting for \( m_1 \) as given in equation (6), the right side of equation (8) can be simplified to \((\max\{|bm + c|, |cm - b|\} + (\sqrt{2} - 1) \min\{|bm + c|, |cm - b|\})^2\). Finally, taking the square root of both sides of the simplified equation produces equation (4). If the slope of \( AC \) is \( m \), then the slope of \( AB \) is \( m' = -1/m \), and our proof produces equation (4') which is equation (4) with \( m' \) replacing \( m \) throughout. But if equation (4') is multiplied by \(|m|\), equation (4) results. Thus equation (4) is true when \( m \) is the slope of either \( AB \) or \( AC \). □

**Corollary 2.5** If \( BC \), the hypotenuse of \( ABC \), is parallel to a coordinate axis, then

\[
a = (b^2 + c^2)/(\max\{b, c\} + (\sqrt{2} - 1) \min\{b, c\}).
\]  

(9)

**Proof.** This is a consequence of equation (7). If \( BC \) is parallel to the \( x \)-axis, then \( m_1 = 0 \), and if \( BC \) is parallel to \( y \)-axis, then we let \( m_1 \to \infty \) in the quotient \( [(1 + m_1^2)^{1/2}/(\max\{1, |m_1|\}) + (\sqrt{2} - 1) \min\{1, |m_1|\}] \). In either case, equation (7) becomes \( a^2 = [(1 + m^2)^{1/2}/(\max\{1, |m|\}) + (\sqrt{2} - 1) \min\{1, |m|\}]^2 (b^2 + c^2) \) where \( m \) is the slope of \( AB \). Let \( AD \) be the altitude from \( A \) (see Figure 3).

![Figure 3](image-url)

By similar triangles and Lemma 2.3, \(|m| = |AD/BD| = |AC/AB| = b/c\) when \( BC \) is horizontal, and \(|m| = c/b\) when \( BC \) is vertical. Thus \( a^2 = [(1 +
\[(b/c)^2 + \frac{1}{2} / \left( \max\{1, b/c\} + (\sqrt{2} - 1) \min\{1, b/c\}\right) = (b^2 + c^2), \text{ if } BC \text{ is horizontal, and } a^2 = [(1 + (c/b)^2)^{1/2} / \left( \max\{1, c/b\} + (\sqrt{2} - 1) \min\{1, c/b\}\right)](b^2 + c^2), \text{ if } BC \text{ is vertical.}\] Each of these equations simplifies to \[a^2 = (b^2 + c^2) / \left( \max\{b, c\} + (\sqrt{2} - 1) \min\{b, c\}\right), \text{ which is equivalent to equation (9).}\]

The next corollary also gives a CC version of the Pythagorean Theorem, with the slope of the hypotenuse as a parameter, instead of the slope of one of the legs.

**Corollary 2.6** If no side of \(ABC\) is parallel to a coordinate axis, and \(m_1\) is the slope of \(BC\), the hypotenuse of \(ABC\), then
\[
a / \left( \max\{1, |m_1|\} + (\sqrt{2} - 1) \min\{1, |m_1|\}\right) = (b^2 + c^2) / \left( \max\{|bm_1 - c|, |cm_1 + b|\} + (\sqrt{2} - 1) \min\{|bm_1 - c|, |cm_1 + b|\}\right)
\]

**Proof.** If \(m\) is the slope of \(AB\), then we can solve equation (5) for \(m\):
\[
m = (b + cm_1) / (c - bm_1)
\]
where \(m \neq c/b\). Substituting this value for \(m\) in equation (4) and simplifying yields equation (10).

**Remark 2.1** We note that when \(AB\) is parallel to the \(x\)-axis, our derivation of equation (4) in the proof of Theorem 2.4 is still valid, and since \(m = 0\), equation (4) reduces to equation (3). Similarly, for the case when \(BC\) is parallel to the \(x\)-axis, equation (10) reduces to equation (9). In addition, equations (3) and (9) for the cases when \(AB\) or \(BC\) is vertical agree with the limits obtained when \(m \to \infty\) in equation (4) or \(m_1 \to \infty\) in equation (10), respectively. To see this, first recall that equations (4) and (9) are derived from equation (8). Note that as \(m \to \infty\), 
\[
[(1 + m^2)^{1/2} / \left( \max\{1, |m|\} + (\sqrt{2} - 1) \min\{1, |m|\}\right)]]^2 \to 1
\]
and \(m_1 \to c/b\) (see equation (6)). Thus as \(m \to \infty\), equation (8) becomes
\[
a^2(1 + c^2/b^2) / \left( \max\{1, c/b\} + (\sqrt{2} - 1) \min\{1, c/b\}\right)^2 = b^2 + c^2,
\]
which simplifies to equation (3). Similarly, as \(m_1 \to \infty\), 
\[
[(1 + m_1^2)^{1/2} / \left( \max\{1, |m_1|\} + (\sqrt{2} - 1) \min\{1, |m_1|\}\right)]^2 \to 1
\]
and \(m \to -c/b\) (see equation (11)). In this case, as \(m_1 \to \infty\), equation (8) becomes equation (9).

**Remark 2.2** If \(ABC\) is labeled in clockwise order, with right angle at \(A\), then the roles of \(b\) and \(c\) are interchanged, and so equation (4) becomes
\[
(\max\{1, |m|\} + (\sqrt{2} - 1) \min\{1, |m|\})a = \max\{|bm - c|, |cm + b|\} + (\sqrt{2} - 1) \min\{|bm - c|, |cm + b|\}
\]
and equation (10) becomes
\[
a / \left( \max\{1, |m_1|\} + (\sqrt{2} - 1) \min\{1, |m_1|\}\right) = (b^2 + c^2) / \left( \max\{|bm_1 + c|, |cm_1 - b|\} + (\sqrt{2} - 1) \min\{|bm_1 + c|, |cm_1 - b|\}\right).
\]
We now give an example that shows the converse of Theorem 2.4, and therefore the converse of Corollary 2.6, are false. That is, there are triangles $ABC$ for which equation (4) holds, but have no right angle. The example refers to Figure 4, in which three different CC circles are shown. Recall that a CC circle with center $A$ and radius $r$ is the set of all points whose CC distance to $A$ is $r$. This locus of points is a regular octagon with center $A$, each side having slope $\pm(\sqrt{2} \pm 1)$, and each diagonal through the center having length $2r$. Just as for a Euclidean circle, the center $A$ and one point at a CC distance $r$ from $A$ completely determine the CC circle.

**Example** Let $ABC$ be a triangle labeled in counterclockwise order such that $BC$ is parallel to $x$-axis and $A$ is inside of the CC circle with diameter $BC$. Let $d_C(B,C) = a$, $d_C(A,C) = b$, $d_C(A,B) = c$. Obviously, $\angle A$ is an obtuse angle. Let $C_1$ and $C_2$ denote the CC circles with radius $a$ and centers $B$ and $C$, respectively. Let $m$ denote the slope of the line $AB$. Chose the point $C'$ on $C_1$, such that $\angle BAC'$ is right angle (see Figure 4). Since $C'$ lies on both $C_1$ and the CC circle $C_3$ with radius $b$ and center $A$, we have $d_C(B,C') = a$ and $d_C(A,C') = b$. Applying Theorem 2.4 to right triangle $ABC'$, one gets that

\[
(\max\{1, |m|\} + (\sqrt{2} - 1) \min\{1, |m|\})a = \max\{|bm + c| , |cm - b|\} + (\sqrt{2} - 1) \min\{|bm + c| , |cm - b|\}
\]

for triangle $ABC$ which has no right angle. Thus, the converse of Theorem 2.4 is not true in the CC plane.

![Figure 4](image)

The following theorem gives a necessary and sufficient condition for a triangle in the CC plane to have a right angle. The sufficient condition is essentially a restatement of the converse of the Pythagorean Theorem.

**Theorem 2.7** Let $ABC$ be a triangle in the CC plane with no side parallel to $y$-axis. Let $d_C(B,C) = a$, $d_C(A,C) = b$ and $d_C(A,B) = c$, and let $m_1$, $m$,
and \( m' \) denote the slopes of the lines \( BC \), \( AB \) and \( AC \), respectively. Then \( \angle A \) is a right angle if and only if
\[
\rho(m_1)a^2 = \rho(m)(b^2 + c^2) = \rho(m')(b^2 + c^2) \tag{12}
\]
where \( \rho(x) = (1 + x^2)^{1/2}/(\max\{1, |x|\} + (\sqrt{2} - 1) \min\{1, |x|\}) \).

**Proof.** If equation (12) holds, then \( \rho(m) = \rho(m') \) and
\[
\rho(m_1)a^2 = \rho(m)b^2 + \rho(m)c^2 = \rho(m')b^2 + \rho(m')c^2.
\]
Therefore
\[
\rho(m_1)a^2 = \rho(m)b^2 + \rho(m')c^2. \tag{13}
\]
Applying equation (1) to equation (13) gives (by Lemma 2.1) \( a^2 = b^2 + c^2 \), where \( a = d_E(B,C) \), \( b = d_E(A,C) \) and \( c = d_E(A,B) \). Since the converse of the Pythagorean Theorem is true, \( \angle A \) is a right angle.

Now suppose, conversely, that \( \angle A \) is a right angle. Then \( m' = -1/m \), and so \( \rho(m) = \rho(m') \) by Lemma 2.2. Equation (12) is just equation (7), derived in the proof of Theorem 2.4. \( \blacksquare \)

**References**


Chinese checker versions of the Pythagorean theorem


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