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Chinese Checker Versions of the Pythagorean Theorem

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Abstract

In this paper, we give Chinese checker versions of the Pythagorean Theorem, and show that the converses of these Chinese checker versions of the Pythagorean Theorem are not true. Finally, we give a necessary and sufficient condition for a triangle in the Chinese checker plane to have a right angle.

Mathematics Subject Classification: 51K05, 51K99

Keywords: Chinese checker distance, Pythagorean theorem, Protractor geometry

1 Introduction

In the game of Chinese checkers (see [14]), checkers are allowed to move in the vertical (north and south), horizontal (east and west), and diagonal (northeast, northwest, southeast and southwest) directions. In [9], Krause asked how to develop a distance function that measures the length of ways mimicing the movements of the Chinese checkers, from a point to another in the Cartesian coordinate plane. Later, Chen [3] defined the distance, named it Chinese checker distance, and then proved that the Chinese checker distance is a metric. If $P = (x_1, y_1)$ and $Q = (x_2, y_2)$ are two points in $\mathbb{R}^2$, the *Chinese checker* (CC) *distance* between P and Q is

$$d_C(P, Q) = \max\{|x_1 - x_2|, |y_1 - y_2|\} + (\sqrt{2} - 1)\min\{|x_1 - x_2|, |y_1 - y_2|\}$$

while the well-known Euclidean distance between P and Q is

$$d_E(P, Q) = [(x_1 - x_2)^2 + (y_1 - y_2)^2]^{1/2}.$$

A *metric geometry* consists of a set $\mathcal{P}$, whose elements are called *points*, together with a collection $\mathcal{L}$ of non-empty subsets of $\mathcal{P}$, called *lines*, and a distance function d, such that

1) Every two distinct points in $\mathcal{P}$ lie on a unique line,
2) There exist three points in $\mathcal{P}$, which do not lie all on one line,
3) There exists a bijective function $f : l \to \mathbb{R}$ for all lines in $\mathcal{L}$ such that $|f(P) - f(Q)| = d(P,Q)$ for each pair of points P and Q on l.

A metric geometry defined above is denoted by $\{\mathcal{P}, \mathcal{L}, d\}$. However, if a metric geometry satisfies the plane separation axiom below, and it has an angle measure function m, then it is called *protractor geometry* and denoted by $\{\mathcal{P}, \mathcal{L}, d, m\}$.

4) For every l in $\mathcal{L}$, there are two subsets H_1 and H_2 of $\mathcal{P}$ (called *half planes* determined by l) such that
(i) $H_1 \cup H_2 = \mathcal{P} - l$ ($\mathcal{P}$ with l removed),
(ii) H_1 and H_2 are disjoint and each is convex,
(iii) If $A \in H_1$ and $B \in H_2$, then $[AB] \cap l \neq \varnothing$.

If L_E is the set of all lines in the Cartesian coordinate plane, and m_E is the standard angle measure function in the Euclidean plane, then $\{\mathbb{R}^2, L_E, d_C, m_E\}$, called *CC plane*, is a model of protractor geometry (This can be shown easily: the proof is similar to that of taxicab plane; refer to [10] or [5] to see that the taxicab plane is a model of protractor geometry). CC plane is also in the class of non-Euclidean geometries since it fails to satisfy the side-angle-side axiom (see Figure 1). However, CC plane is almost the same as Euclidean plane $\{\mathbb{R}^2, L_E, d_E, m_E\}$ since the points are the same, the lines are the same, and the angles are measured in the same way. Since CC geometry has a distance function different from that in Euclidean geometry, it is interesting to study the CC analogues of topics that include the distance concept in Euclidean geometry (see [1], [2], [4], [6], [7], [8], [11], [12] and [13] for some works on this subject).

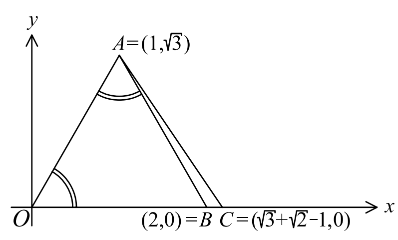

Figure 1. In figure $d_C(A,O) = d_C(A,B) = d_C(O,C) = \sqrt{3} + \sqrt{2} - 1$ and $\measuredangle(OAB) = \measuredangle(AOC) = \pi/3$, but $d_C(A,B) \neq d_C(A,C)$.

2 CC Versions of the Pythagorean Theorem

It is well-known that if ABC is a triangle with right angle A in the Euclidean plane, then $\mathbf{a}^2 = \mathbf{b}^2 + \mathbf{c}^2$ where $\mathbf{a} = d_E(B, C)$, $\mathbf{b} = d_E(A, C)$ and $\mathbf{c} = d_E(A, B)$; this is the **Pythagorean Theorem**. Also it is well-known that its converse is true in the Euclidean plane. A CC version of the Pythagorean Theorem for a right triangle ABC would be an equation that relates the three CC distances a, b, c between pairs of vertices, where $a = d_C(B, C)$, $b = d_C(A, C)$ and $c = d_C(A, B)$. Two CC versions of the Pythagorean Theorem that depend on two parameters in addition to the CC distances between the vertices of a right triangle, were given in [7]. Here, we give CC versions of the Pythagorean Theorem that depend on only one parameter in addition to the CC distances between the vertices of a right triangle. We also show that the converses of these CC versions of the Pythagorean Theorem are false in the CC plane.

The following equation, which relates the Euclidean distance to the CC distance between two points in the Cartesian coordinate plane, plays an important role in our arguments.

Lemma 2.1 *For any two points P and Q in the Cartesian plane that do not lie on a vertical line, if m is the slope of the line through P and Q, then*

$$d_E(P, Q) = \rho(m) d_C(P, Q) \tag{1}$$

where $\rho(m) = (1 + m^2)^{1/2} \big/ (\max\{1, |m|\} + (\sqrt{2} - 1) \min\{1, |m|\})$. If P and Q lie on a vertical line, then by definition, $d_E(P, Q) = d_C(P, Q)$.

Proof. Let $P = (x_1, y_1)$ and $Q = (x_2, y_2)$ with $x_1 \neq x_2$; then $m = (y_2 - y_1) \big/ (x_2 - x_1)$. Equation (1) is derived by a straightforward calculation with m and the coordinate definitions of $d_E(P, Q)$ and $d_C(P, Q)$ given in Section 1.

■

Another useful fact that can be verified by direct calculation is:

Lemma 2.2 *For any real number $m \neq 0$, let $m' = -1/m$. Then*

$$\rho(m) = \rho(m'). \tag{2}$$

In all that follows, unless otherwise stated, ABC is a triangle in the Cartesian coordinate plane with vertices labeled in counterclockwise order, with right angle at A. The Euclidean distances between pairs of vertices are denoted $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$, and the corresponding CC distances are a, b, c, as defined earlier. We first note that although the Euclidean distances $\mathbf{b}$ and $\mathbf{c}$ are, in general, different from the corresponding CC distances b and c, corresponding ratios of these distances are equal.

Lemma 2.3 $\mathbf{b}/\mathbf{c} = b/c$.

Proof. If the legs AB and AC of ABC are parallel to the coordinate axes, then $\mathbf{b} = b$ and $\mathbf{c} = c$, and the two ratios are equal. If one of the legs of ABC is not parallel to a coordinate axis, then neither is the other. If the slope of AB is m, then the slope of AC is $m' = -1/m$, since the legs are perpendicular. By equation (1), $\mathbf{c} = \rho(m)c$ and $\mathbf{b} = \rho(m')b$. But then equation (2) implies that $\mathbf{b}/\mathbf{c} = b/c$. ■

Our main results follow; these give relations between the three CC distances a, b, c that depend only on one parameter, namely, the slope of one of the legs or the slope of the hypotenuse of right triangle ABC. If a leg or the hypotenuse of ABC is parallel to a coordinate axis, then there is a relation between a, b, and c that does not depend on any other parameter.

Theorem 2.4 *(i) If the legs of ABC are parallel to the coordinate axes, then*

$$a = \max\{b, c\} + (\sqrt{2} - 1) \min\{b, c\}. \tag{3}$$

(ii) If the legs of ABC are not parallel to the coordinate axes, the hypotenuse BC is not vertical, and m is the slope of one leg, then

$$(\max\{1, |m|\} + (\sqrt{2} - 1) \min\{1, |m|\})a = \max\{|bm + c|, |cm - b|\} + (\sqrt{2} - 1) \min\{|bm + c|, |cm - b|\}. \tag{4}$$

Proof. (i) This follows immediately from the definition of CC distance.
(ii) Let angle CBA be denoted θ; note that θ is positive and acute, by the counterclockwise labeling of ABC (see Figure 2). Then $\tan\theta = \mathbf{b}/\mathbf{c} = b/c$,

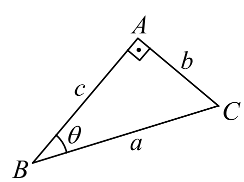

Figure 2

by Lemma 2.3. First suppose that m is the slope of AB; then $m' = -1/m$ is the slope of AC. Let m_1 be the slope of BC. It is well-known (or easily found, using the identity for the tangent of the difference of two angles) that $\tan\theta = (m - m_1)/(1 + mm_1)$. Thus

$$b/c = (m - m_1)/(1 + mm_1). \tag{5}$$

Solving equation (5) for m_1 yields

$$m_1 = (cm - b) \diagup (bm + c) \tag{6}$$

where $m \neq -c/b$. Applying equation (1) to the Pythagorean theorem $\mathbf{a}^2 = \mathbf{b}^2 + \mathbf{c}^2$, and using Lemma 2.2 gives

$$\begin{aligned}&[(1+m_1^2)^{1/2} \diagup (\max\{1, |m_1|\} + (\sqrt{2}-1)\min\{1, |m_1|\})]^2 a^2 = \\ &\qquad [(1+m^2)^{1/2} \diagup (\max\{1, |m|\} + (\sqrt{2}-1)\min\{1, |m|\})]^2 (b^2+c^2)\end{aligned} \tag{7}$$

which simplifies to

$$\begin{aligned}&(\max\{1, |m|\} + (\sqrt{2}-1)\min\{1, |m|\})^2 a^2 = \\ &\qquad [(\max\{1, |m_1|\} + (\sqrt{2}-1)\min\{1, |m_1|\})^2 \diagup (1+m_1^2)](1+m^2)(b^2+c^2)\end{aligned} \tag{8}$$

Substituting for m_1 as given in equation (6), the right side of equation (8) can be simplified to $(\max\{|bm+c|, |cm-b|\} + (\sqrt{2}-1)\min\{|bm+c|, |cm-b|\})^2$. Finally, taking the square root of both sides of the simplified equation produces equation (4). If the slope of AC is m, then the slope of AB is $m' = -1/m$, and our proof produces equation (4′) which is equation (4) with m' replacing m throughout. But if equation (4′) is multiplied by $|m|$, equation (4) results. Thus equation (4) is true when m is the slope of either AB or AC. ■

Corollary 2.5 *If BC, the hypotenuse of ABC, is parallel to a coordinate axis, then*

$$a = (b^2+c^2) \diagup (\max\{b, c\} + (\sqrt{2}-1)\min\{b, c\}). \tag{9}$$

Proof. This is a consequence of equation (7). If BC is parallel to the x-axis, then $m_1 = 0$, and if BC is parallel to y-axis, then we let $m_1 \to \infty$ in the quotient $[(1+m_1^2)^{1/2} \diagup (\max\{1, |m_1|\} + (\sqrt{2}-1)\min\{1, |m_1|\})]$. In either case, equation (7) becomes $a^2 = [(1+m^2)^{1/2} \diagup (\max\{1, |m|\} + (\sqrt{2}-1)\min\{1, |m|\})]^2 (b^2+c^2)$ where m is the slope of AB. Let AD be the altitude from A (see Figure 3).

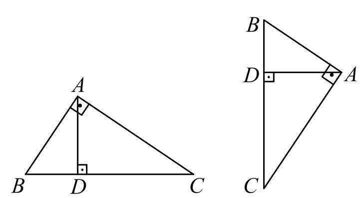

Figure 3

By similar triangles and Lemma 2.3, $|m| = |AD/BD| = |AC/AB| = b/c$ when BC is horizontal, and $|m| = c/b$ when BC is vertical. Thus $a^2 = [(1+$

$(b/c)^2)^{1/2} \diagup (\max\{1, b/c\} + (\sqrt{2}-1)\min\{1, b/c\})]^2(b^2+c^2)$, if BC is horizontal, and $a^2 = [(1+(c/b)^2)^{1/2} \diagup (\max\{1, c/b\} + (\sqrt{2}-1)\min\{1, c/b\})]^2(b^2+c^2)$, if BC is vertical. Each of these equations simplifies to $a^2 = (b^2+c^2)^2 \diagup (\max\{b, c\} + (\sqrt{2}-1)\min\{b, c\})^2$, which is equivalent to equation (9). ■

The next corollary also gives a CC version of the Pythagorean Theorem, with the slope of the hypotenuse as a parameter, instead of the slope of one of the legs.

Corollary 2.6 *If no side of ABC is parallel to a coordinate axis, and m_1 is the slope of BC, the hypotenuse of ABC, then*

$$a \diagup (\max\{1, |m_1|\} + (\sqrt{2}-1)\min\{1, |m_1|\}) = (b^2+c^2) \diagup \\ (\max\{|bm_1 - c|, |cm_1 + b|\} + (\sqrt{2}-1)\min\{|bm_1 - c|, |cm_1 + b|\}) \tag{10}$$

Proof. If m is the slope of AB, then we can solve equation (5) for m:

$$m = (b + cm_1) \diagup (c - bm_1) \tag{11}$$

where $m \neq c/b$. Substituting this value for m in equation (4) and simplifying yields equation (10). ■

Remark 2.1 We note that when AB is parallel to the x-axis, our derivation of equation (4) in the proof of Theorem 2.4 is still valid, and since $m = 0$, equation (4) reduces to equation (3). Similarly, for the case when BC is parallel to the x-axis, equation (10) reduces to equation (9). In addition, equations (3) and (9) for the cases when AB or BC is vertical agree with the limits obtained when $m \to \infty$ in equation (4) or $m_1 \to \infty$ in equation (10), respectively. To see this, first recall that equations (4) and (9) are derived from equation (8). Note that as $m \to \infty$, $[(1+m^2)^{1/2} \diagup (\max\{1, |m|\} + (\sqrt{2}-1)\min\{1, |m|\})]^2 \to 1$ and $m_1 \to c/b$ (see equation (6)). Thus as $m \to \infty$, equation (8) becomes $a^2(1+c^2/b^2) \diagup (\max\{1, c/b\} + (\sqrt{2}-1)\min\{1, c/b\})^2 = b^2+c^2$, which simplifies to equation (3). Similarly, as $m_1 \to \infty$, $[(1+m_1^2)^{1/2} \diagup (\max\{1, |m_1|\} + (\sqrt{2}-1)\min\{1, |m_1|\})]^2 \to 1$ and $m \to -c/b$ (see equation (11)). In this case, as $m_1 \to \infty$, equation (8) becomes equation (9).

Remark 2.2 If ABC is labeled in clockwise order, with right angle at A, then the roles of b and c are interchanged, and so equation (4) becomes

$$(\max\{1, |m|\} + (\sqrt{2}-1)\min\{1, |m|\})a = \max\{|bm - c|, |cm + b|\} + \\ (\sqrt{2}-1)\min\{|bm - c|, |cm + b|\}$$

and equation (10) becomes

$$a \diagup (\max\{1, |m_1|\} + (\sqrt{2}-1)\min\{1, |m_1|\}) = (b^2+c^2) \diagup \\ (\max\{|bm_1 + c|, |cm_1 - b|\} + (\sqrt{2}-1)\min\{|bm_1 + c|, |cm_1 - b|\}).$$

We now give an example that shows the converse of Theorem 2.4, and therefore the converse of Corollary 2.6, are false. That is, there are triangles ABC for which equation (4) holds, but have no right angle. The example refers to Figure 4, in which three different CC circles are shown. Recall that a CC circle with center A and radius r is the set of all points whose CC distance to A is r. This locus of points is a regular octagon with center A, each side having slope $\pm(\sqrt{2} \pm 1)$, and each diagonal through the center having length $2r$. Just as for a Euclidean circle, the center A and one point at a CC distance r from A completely determine the CC circle.

Example Let ABC be a triangle labeled in counterclockwise order such that BC is parallel to x-axis and A is inside of the CC circle with diameter BC. Let $d_C(B,C) = a$, $d_C(A,C) = b$, $d_C(A,B) = c$. Obviously, $\angle A$ is an obtuse angle. Let C_1 and C_2 denote the CC circles with radius a and centers B and C, respectively. Let m denote the slope of the line AB. Chose the point C' on C_1, such that $\angle BAC'$ is right angle (see Figure 4). Since C' lies on both C_1 and the CC circle C_3 with radius b and center A, we have $d_C(B,C') = a$ and $d_C(A,C') = b$. Applying Theorem 2.4 to right triangle ABC', one gets that

$$(\max\{1,|m|\} + (\sqrt{2}-1)\min\{1,|m|\})a = \max\{|bm+c|, |cm-b|\} + (\sqrt{2}-1)\min\{|bm+c|, |cm-b|\}$$

for triangle ABC which has no right angle. Thus, the converse of Theorem 2.4 is not true in the CC plane.

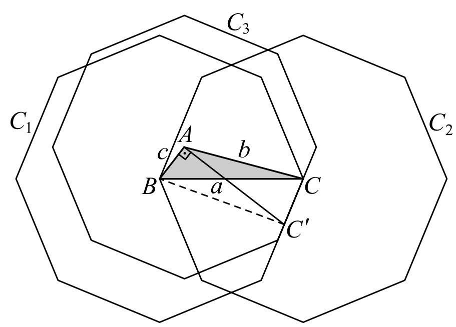

Figure 4

The following theorem gives a necessary and sufficient condition for a triangle in the CC plane to have a right angle. The sufficient condition is essentially a restatement of the converse of the Pythagorean Theorem.

Theorem 2.7 *Let ABC be a triangle in the CC plane with no side parallel to y-axis. Let $d_C(B,C) = a$, $d_C(A,C) = b$ and $d_C(A,B) = c$, and let m_1, m,* *and m' denote the slopes of the lines BC, AB and AC, respectively. Then $\angle A$ is a right angle if and only if*

$$\rho(m_1)a^2 = \rho(m)(b^2 + c^2) = \rho(m')(b^2 + c^2) \tag{12}$$

where $\rho(x) = (1 + x^2)^{1/2} \big/ (\max\{1, |x|\} + (\sqrt{2} - 1)\min\{1, |x|\})$.

Proof. If equation (12) holds, then $\rho(m) = \rho(m')$ and

$$\rho(m_1)a^2 = \rho(m)b^2 + \rho(m)c^2 = \rho(m')b^2 + \rho(m')c^2.$$

Therefore

$$\rho(m_1)a^2 = \rho(m)b^2 + \rho(m')c^2. \tag{13}$$

Applying equation (1) to equation (13) gives (by Lemma 2.1) $\mathbf{a}^2 = \mathbf{b}^2 + \mathbf{c}^2$, where $\mathbf{a} = d_E(B, C)$, $\mathbf{b} = d_E(A, C)$ and $\mathbf{c} = d_E(A, B)$. Since the converse of the Pythagorean Theorem is true, $\angle A$ is a right angle.

Now suppose, conversely, that $\angle A$ is a right angle. Then $m' = -1/m$, and so $\rho(m) = \rho(m')$ by Lemma 2.2. Equation (12) is just equation (7), derived in the proof of Theorem 2.4. ■

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