An Analytic Approximation to the Solution
of Fuzzy Heat Equation by Adomian
Decomposition Method

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Abstract

Adomian decomposition method has been applied to solve many functional equations so far. In this article, we have used this method to solve the fuzzy heat equation, which governs on numerous scientific and engineering experimentations. Some special cases of the equation are solved as examples to illustrate ability and reliability of the method.

Keywords: Adomian decomposition method, Fuzzy heat equation, Fuzzy number

1. Introduction

The concept of fuzzy derivative was first introduced by S.L. Chang, L.A. Zadeh in [1]. It was followed up by D. Dubois, H. Prade [2], who defined any used the extension principle. Other methods have been discussed by M.L. Puri, D.A. Ralescu [3] and R. Goetschel, W. Voxman [4]. The fuzzy differential equations and fuzzy initial value problem were regularly treated by O. Kaleva in [5], [6] and by S. Seikkala [7], … . The numerical methods for solving fuzzy differential equations are introduced by M. Ma, M. Friedman, A. Kandel [8] by the standard
Euler method. Here we are going to propose a method for approximation of the solution the fuzzy heat equation using Adomian decomposition method, since the analytical methods commonly used for solving the fuzzy heat equation are very restricted and can be used in special cases. This paper is organized as follows: In section 2 we bring some basic definitions of fuzzy numbers and fuzzy derivative which have been discussed by B. Bede, SG. Gal [9] and we will use in this paper. In section 3 we define a fuzzy heat equation and In section 4 we discuss Adomian decomposition method. In section 5, we have some examples of this method and the conclusion is drown in section 6.

2. Preliminaries

We begin this section with defining the notation we will use in the paper. We place a \( \sim \) sign over a letter to denote a fuzzy subset of the real numbers. We write \( \tilde{A}(x) \), a number in \([0, 1]\), for the membership function of \( A \) evaluates at \( x \). An \( \alpha \)-cut of \( \tilde{A} \), written by \( \tilde{A}[\alpha] \), is defined as \( \{ x | \tilde{A}(x) \geq \alpha \} \), for \( 0 < \alpha \leq 1 \) and \( \alpha \)-cuts of fuzzy numbers are always closed and bounded. We represent an arbitrary fuzzy number by an ordered pair of functions \((u(\alpha), \overline{u}(\alpha))\), which satisfies the following requirements:

1. \( u(\alpha) \) is a bounded left continuous non decreasing function over \([0,1]\).
2. \( \overline{u}(\alpha) \) is a bounded left continuous non increasing function over \([0,1]\).
3. \( u(\alpha) \leq \overline{u}(\alpha), 0 \leq \alpha \leq 1 \).

A crisp number \( \alpha \) is simply represented by \( u(\alpha) = \overline{u}(\alpha) = \alpha, 0 \leq \alpha \leq 1 \).

For arbitrary fuzzy numbers \( x = (\underline{x}(\alpha), \overline{x}(\alpha)) \), \( y = (\underline{y}(\alpha), \overline{y}(\alpha)) \) and real number \( k \).

1. \( x = y \) if and only if \( \underline{x}(\alpha) = \underline{y}(\alpha) \) and \( \overline{x}(\alpha) = \overline{y}(\alpha) \).
2. \( x + y = (\underline{x}(\alpha) + \underline{y}(\alpha), \overline{x}(\alpha) + \overline{y}(\alpha)) \)
3. \( kx = \begin{cases} (k \underline{x}, k \overline{x}) , & k \geq 0, \\ (k \overline{x}, k \underline{x}), & k < 0. \end{cases} \)

Let \( F \) be the set of all upper semicontinuous normal convex fuzzy numbers with bounded \( \alpha \)-level sets.

**Definition.** [9] Let \( F : (a,b) \rightarrow F^n \) and \( t \in (a,b) \). We say that \( F \) is differentiable at \( t \), if we have 2 forms as follows:

1. it exists an element \( F'(t) \in F^n \) such that, for all \( h > 0 \) sufficiently near to 0,
   
   \[
   \lim_{h \to 0^+} \frac{F(t+h) - F(t)}{h} = \lim_{h \to 0^+} \frac{F(t) - F(t-h)}{h} = F'(t) \quad (2.1)
   \]

2. it exists an element \( F'(t) \in F^n \) such that, for all \( h < 0 \) sufficiently near to 0,
   
   \[
   \lim_{h \to 0^-} \frac{F(t+h) - F(t)}{h} = \lim_{h \to 0^-} \frac{F(t) - F(t-h)}{h} = F'(t) \quad (2.2)
   \]
Theorem. Let \( F : \mathcal{T} \to \mathcal{F} \) be a function and denote \( [F(t)]^\alpha = [f_\alpha(t), g_\alpha(t)] \), for each \( \alpha \in [0,1] \). Then

(i) If \( F \) is differentiable in the first form (1), then \( f_\alpha \) and \( g_\alpha \) are differentiable functions and

\[
[F'(t)]^\alpha = [f'_\alpha(t), g'_\alpha(t)] \tag{2.3}
\]

(ii) If \( F \) is differentiable in the second form (2), then \( f_\alpha \) and \( g_\alpha \) are differentiable functions and

\[
[F'(t)]^\alpha = [g'_\alpha(t), f'_\alpha(t)] \tag{2.4}
\]

Proof: [10].

Consider the FPDE

\[
\varphi(D_x, D_t) \bar{U}(x,t) = \bar{F}(x,t, \bar{K}) \tag{2.5}
\]

Subject to certain boundary conditions where the operator \( \varphi(D_x, D_t) \) will be a polynomial, with constant coefficients, in \( D_x \) and \( D_t \), where \( D_x(D_t) \) stands for the partial differential with respect to \( x \) (t).

Let \( I_1 = [0,M_1] \) and \( I_2 = [0,M_2] \) for some \( M_1(M_2) > 0 \), be two intervals.

\( \bar{F}(x,t, \bar{K}) \) will be a fuzzy continuous function for \((x,t) \in I_1 \times I_2 \) and \( \bar{K} = (\bar{k}_1, \ldots, \bar{k}_n) \) that \( \bar{k}_i \) is a triangular fuzzy number in the interval \( J_i \),

\[ 1 \leq i \leq n, \quad \bar{K} \in J = \prod_{i=1}^n J_i. \]

\( \bar{U} \) maps \( I_1 \times I_2 \) into fuzzy number. That is, \( \bar{U}(x,t) = \bar{Z} \) where \( \bar{Z} \) is a fuzzy number.

Eq. (2.5) is subject to certain boundary conditions,

\[
\bar{U}(0,t) = \bar{c}_1, \quad \bar{U}(x,0) = \bar{c}_2, \quad \bar{U}(M_1,t) = \bar{c}_3, \ldots, \quad \bar{U}(0,t) = \bar{g}_1(t; \bar{c}_2), \quad \bar{U}(x,0) = \bar{f}_1(x; \bar{c}_2)
\]

\[
\ldots, \quad \bar{U}_i(x,0) = \bar{f}_i(x; \bar{c}_2), \quad \bar{U}_i(0,t) = \bar{g}_i(t; \bar{c}_2), \quad \bar{c}_2, \ldots, \ldots.
\]

We let the \( \bar{g}_i \) and \( \bar{f}_i \) are fuzzy number functions. Also \( \bar{C} = (\bar{c}_1, \ldots, \bar{c}_n) \) that \( \bar{c}_i \) is a triangular fuzzy number in the interval \( L_i \), \( 1 \leq i \leq n, \quad \bar{C} \in L = \prod_{i=1}^n L_i. \)

\( \bar{K} = \prod_{i=1}^n \bar{K}_i, \quad \bar{C} = \prod_{i=1}^n \bar{C}_i, \quad \bar{U}(x,t)[\alpha] = [\bar{U}(x,t; \alpha)], \quad \bar{U}(x,t; \alpha) \).

We assume that the \( \bar{U}(x,t; \alpha) \) and \( \bar{U}(x,t; \alpha) \) have continuous partial so that \( \varphi(D_x, D_t) \bar{U}(x,t; \alpha) \) and \( \varphi(D_x, D_t) \bar{U}(x,t; \alpha) \) are continuous for each \((x,t) \in I_1 \times I_2 \) and \( \alpha \in [0,1] \). Also we have the following alternative.

Case I. If we consider \( \varphi(D_x, D_t) \bar{U}(x,t) \), then by using the derivative in form (1):

\[
\left[ \varphi(D_x, D_t) \bar{U}(x,t) \right][\alpha] = \left[ \varphi(D_x, D_t) \bar{U}(x,t; \alpha), \varphi(D_x, D_t) \bar{U}(x,t; \alpha) \right]
\]

we should solve the system of fuzzy partial differential equations

\[
\varphi(D_x, D_t) \bar{U}(x,t; \alpha) = \bar{F}(x,t; \alpha), \tag{2.6}
\]

\[
\varphi(D_x, D_t) \bar{U}(x,t; \alpha) = \bar{F}(x,t; \alpha), \tag{2.7}
\]
For all \((x,t) \in I_1 \times I_2\) and all \(\alpha \in [0,1]\), where
\[
\begin{align*}
F(x,t;\alpha) &= \min \{F(x,t;k) \mid k \in \tilde{K}[\alpha]\}, \\
\tilde{F}(x,t;\alpha) &= \max \{F(x,t;k) \mid k \in \tilde{K}[\alpha]\}.
\end{align*}
\]
(2.8)
(2.9)

We append to equations (2.6) and (2.7) any boundary conditions, for example, if they were \(\bar{U}(0,t) = \bar{C}\) and \(\bar{U}(x,0) = \bar{f}(x,0)\), then we add
\[
\bar{U}(0,t;\alpha) = \bar{C}(\alpha), \quad \bar{U}(x,0;\alpha) = \bar{f}(x;\alpha)
\]
(2.10)

to Eq. (2.6) and
\[
\bar{U}(0,t;\alpha) = \bar{C}(\alpha), \quad \bar{U}(x,0;\alpha) = \bar{f}(x;\alpha)
\]
(2.11)
to Eq. (2.7) where \(\bar{C}[\alpha] = [C(\alpha), \bar{C}(\alpha)]\) and \(\bar{f} = [f(\alpha), \bar{f}(\alpha)]\).

**Case II.** If we consider \(\varphi(D_x,D_t)\bar{U}(x,t)\), then by using the derivative in form (2)
\[
\begin{align*}
[\varphi(D_x,D_t)\bar{U}(x,t)][\alpha] &= [\phi(D_x,D_t)\bar{U}(x,t;\alpha), \phi(D_x,D_t)\bar{U}(x,t;\alpha)]
\end{align*}
\]
we should solve the system of fuzzy partial differential equations:
\[
\begin{align*}
\varphi(D_x,D_t)\bar{U}(x,t;\alpha) &= \bar{F}(x,t;\alpha), \\
\varphi(D_x,D_t)\bar{U}(x,t;\alpha) &= \tilde{F}(x,t;\alpha),
\end{align*}
\]
(2.12)
(2.13)
for all \((x,t) \in I_1 \times I_2\) and for each \(\alpha \in [0,1]\), where \(\bar{F}(x,t;\alpha)\) and the boundary conditions are the same as case I.

### 3 Fuzzy parabolic equation

Consider the fuzzy parabolic equation with the indicated initial conditions:
\[
\frac{\partial \bar{U}}{\partial t} = \frac{\partial^2 \bar{U}}{\partial x^2} + \bar{k}(\alpha), \quad 0 < x < 1, \quad t > 0,
\]
(3.1)

where
\[
\bar{U}(x,0) = \bar{f}(x), \quad 0 < x < 1.
\]
Before solving this equation by Adomain method we have four different cases:

(a) If both \(\frac{\partial^2 \bar{U}}{\partial x^2}\) and \(\frac{\partial \bar{U}}{\partial t}\) are differentiable in form (1), then by Eqs. (2.6) and (2.7), we have:
\[
\begin{align*}
\frac{\partial \bar{U}}{\partial t} &= \frac{\partial^2 \bar{U}}{\partial x^2} + k(\alpha), \\
\frac{\partial \bar{U}}{\partial t} &= \frac{\partial^2 \bar{U}}{\partial x^2} + \tilde{k}(\alpha),
\end{align*}
\]
(3.2)

where
\[
\bar{U}(x,0;\alpha) = \bar{f}(x;\alpha), \quad 0 < x < 1, \quad \alpha \in [0,1], \\
\bar{U}(x,0;\alpha) = \tilde{f}(x;\alpha), \quad 0 < x < 1, \quad \alpha \in [0,1].
\]
(b) If both \( \frac{\partial^2 \tilde{U}}{\partial x^2} \) and \( \frac{\partial \tilde{U}}{\partial t} \) are differentiable in form (2), then by Eqs. (2.12) and (2.13) we have:
\[
\begin{align*}
\frac{\partial \tilde{U}}{\partial t} &= \frac{\partial^2 \tilde{U}}{\partial x^2} + k(\alpha), \\
\frac{\partial \tilde{U}}{\partial t} &= \frac{\partial^2 \tilde{U}}{\partial x^2} + \bar{k}(\alpha),
\end{align*}
\]
\(0 < x < 1, \quad t > 0, \quad \alpha \in [0,1].\)
The boundary conditions are the same as case (a).

(c) If \( \frac{\partial^2 \tilde{U}}{\partial x^2} \) is differentiable in form (1) and \( \frac{\partial \tilde{U}}{\partial t} \) is differential in form (2) then by Eqs. (2.6) and (2.7) and (2.12) and (2.13) we have:
\[
\begin{align*}
\frac{\partial \tilde{U}}{\partial t} &= \frac{\partial^2 \tilde{U}}{\partial x^2} + k(\alpha), \\
\frac{\partial \tilde{U}}{\partial t} &= \frac{\partial^2 \tilde{U}}{\partial x^2} + \bar{k}(\alpha),
\end{align*}
\]
\(0 < x < 1, \quad t > 0, \quad \alpha \in [0,1].\)
The boundary conditions are the same as case (a).

(d) If \( \frac{\partial^2 \tilde{U}}{\partial x^2} \) is differentiable in the first form (2) and \( \frac{\partial \tilde{U}}{\partial t} \) is differential in the first form (1) then by Eqs. (2.6) and (2.7) and (2.12) and (2.13) we have:
\[
\begin{align*}
\frac{\partial \tilde{U}}{\partial t} &= \frac{\partial^2 \tilde{U}}{\partial x^2} + k(\alpha), \\
\frac{\partial \tilde{U}}{\partial t} &= \frac{\partial^2 \tilde{U}}{\partial x^2} + \bar{k}(\alpha),
\end{align*}
\]
\(0 < x < 1, \quad t > 0, \quad \alpha \in [0,1].\)
The boundary conditions are the same as the first case.

4. Solution of the heat equation by Adomain method

Consider Eq. (3.1) with specified initial conditions. For solving this equation by Adomian decomposition method the equation should be in canonical form which can be derived by rewriting as follows:
\[
L \tilde{U} = \frac{\partial^2 \tilde{U}}{\partial x^2} + \bar{k},
\]
where \( L = \frac{\partial}{\partial t} \), with the inverse operator \( L^{-1} = \int_0^t (.) \, dt \).

Applying the inverse operator, we got:
\[ \tilde{U}(x,t) = \tilde{U}(x,0) + \int_0^t \left( \frac{\partial^2 \tilde{U}}{\partial x^2} + \tilde{k} \right) dt. \] (4.2)

Substituting initial conditions and (4.1) into (4.2) and from the definition of generalized differentiability in [9] we get the canonical form of the equation:

\[ \tilde{U}(x,t) = f(x) + \int_0^t \frac{\partial \tilde{U}}{\partial x} dt + \int_0^t \tilde{k} dt. \] (4.3)

Adomian decomposition method considers the solution \( \tilde{U} \) as the sum of a series as:

\[ \tilde{U} = \sum_{n=0}^{\infty} \tilde{U}_n \] (4.4)

And the integrand on the right side Eq. (3.5) as the sum of a series as:

\[ \frac{\partial \tilde{U}}{\partial x} = \sum_{n=0}^{\infty} A_n(\tilde{U}_0, \tilde{U}_1, ..., \tilde{U}_n) \] (4.5)

where \( A_n(\tilde{U}_0, \tilde{U}_1, ..., \tilde{U}_n) \) are called fuzzy Adomian polynomials and should be computed. Substituting from (4.4) and (4.5) into (4.3) we get:

\[ \sum_{n=0}^{\infty} \tilde{U}_n = f(x) + \int_0^t \tilde{k} dt + \int_0^t \sum_{n=0}^{\infty} A_n(\tilde{U}_0, \tilde{U}_1, ..., \tilde{U}_n) dt \] (4.6)

From (4.6) the following Adomian procedure can be defined.

\[ \tilde{U}_0 = f(x) + \int_0^t \tilde{k} dt, \] (4.7)

\[ \tilde{U}_{n+1} = \int_0^t A_n(\tilde{U}_0, \tilde{U}_1, ..., \tilde{U}_n) dt, \quad n = 0, 1, 2... \] (4.8)

In some cases as Example 1 after some steps the remaining terms would vanish and we derive the exact solution. Otherwise by computing some terms say \( m \),

\[ \tilde{U} \approx \sum_{n=0}^{m} \tilde{U}_n, \quad \text{where} \quad \tilde{U} = \lim_{m \to \infty} \sum_{n=0}^{m} \tilde{U}_n, \] an approximation to the solution would be achieved. Now for finding \( \tilde{U}_0, \tilde{U}_1, ..., \tilde{U}_n, \tilde{U}_{n+1}, \ldots \) we have four different cases.

(a') If we consider case (a) and Adomian procedure, we have:

\[ \tilde{U}_{n+1}(x,t;\alpha) = \int_0^t \sum_{n=0}^{\infty} A_n(\tilde{U}_0(x,t;\alpha), ..., \tilde{U}_n(x,t;\alpha)) dt. \] (4.9)

\[ \tilde{U}_0(x,t;\alpha) = f(x,\alpha) + \int_0^t \tilde{k}(\alpha) dt, \]

\[ \tilde{U}_{n+1}(x,t;\alpha) = \int_0^t \sum_{n=0}^{\infty} A_n(\tilde{U}_0(x,t;\alpha), ..., \tilde{U}_n(x,t;\alpha)) dt. \] (4.10)

By solving (4.9) and (4.10) we can calculate the terms of a series as:

\[ \tilde{U}(x,t;\alpha) = \sum_{n=0}^{\infty} \tilde{U}_n(x,t;\alpha), \]

\[ \tilde{U}(x,t;\alpha) = \sum_{n=0}^{\infty} \tilde{U}_n(x,t;\alpha). \] (4.11)

so:
Analytic approximation to the solution

\[ \tilde{U}(x,t)[\alpha] = (\lim_{n \to \infty} \sum_{k=0}^{n} \tilde{U}_n(x,t;\alpha), \lim_{n \to \infty} \sum_{k=0}^{n} \tilde{U}_n(x,t;\alpha)) \]  

(4.12)

(b’) If we consider case (b) and Adomian procedure, we have:

\[ \tilde{U}_n(x,t;\alpha) = \tilde{f}(x,\alpha) + \sum_{k=0}^{n} \int_{0}^{t} k(\alpha) \, dt, \]

(4.13)

\[ \tilde{U}_{n+1}(x,t;\alpha) = \int_{0}^{t} \sum_{k=0}^{n} A_n(\tilde{U}_n(x,t;\alpha),...,\tilde{U}_n(x,t;\alpha)) \, dt. \]

(4.14)

From (4.13) and (4.14) first few terms of (4.11) are computed, so:

\[ \tilde{U}(x,t)[\alpha] = (\lim_{n \to \infty} \sum_{k=0}^{n} \tilde{U}_n(x,t;\alpha), \lim_{n \to \infty} \sum_{k=0}^{n} \tilde{U}_n(x,t;\alpha)) \]  

(4.15)

(c’) If we consider case (c) and Adomian procedure, we have:

\[ \tilde{U}_n(x,t;\alpha) = \tilde{f}(x,\alpha) + \sum_{k=0}^{n} \int_{0}^{t} k(\alpha) \, dt, \]

(4.16)

\[ \tilde{U}_{n+1}(x,t;\alpha) = \int_{0}^{t} \sum_{k=0}^{n} A_n(\tilde{U}_n(x,t;\alpha),...,\tilde{U}_n(x,t;\alpha)) \, dt. \]

(4.17)

From Eqs. (4.16) and (4.17) we can calculate the terms of (4.11).

\[ \tilde{U}(x,t)[\alpha] = (\lim_{n \to \infty} \sum_{k=0}^{n} \tilde{U}_n(x,t;\alpha), \lim_{n \to \infty} \sum_{k=0}^{n} \tilde{U}_n(x,t;\alpha)) \]  

(4.18)

(d’) If we consider case (d) and Adomian procedure, we have:

\[ \tilde{U}_n(x,t;\alpha) = \tilde{f}(x,\alpha) + \sum_{k=0}^{n} \int_{0}^{t} k(\alpha) \, dt, \]

(4.19)

\[ \tilde{U}_{n+1}(x,t;\alpha) = \int_{0}^{t} \sum_{k=0}^{n} A_n(\tilde{U}_n(x,t;\alpha),...,\tilde{U}_n(x,t;\alpha)) \, dt. \]

(4.20)

From (4.19) and (4.20) we can calculate the terms of (4.11), \( \tilde{U}(x,t)[\alpha] \) is the same as (4.12).
5. Examples

Example 5.1 Consider the following fuzzy equation with the indicated initial conditions:

\[ \frac{\partial \tilde{U}}{\partial t} (x,t) = \frac{\partial^2 \tilde{U}}{\partial x^2} (x,t), \quad 0 < x < 1, \quad t > 0, \]
\[ \tilde{U}(x,0) = \tilde{f}(x) = \tilde{k} \sin(\pi x), \quad 0 < x < 1. \] (5.1)

and \( \tilde{k}[\alpha] = [k(\alpha), \tilde{k}(\alpha)] = [\alpha, 1, 1 - \alpha] \).

The exact solution are \( \tilde{U}(x,t;\alpha) = k(\alpha) e^{-x^2 t} \sin(\pi x) \) and \( \tilde{U}(x,t;\alpha) = \tilde{k}(\alpha) e^{-x^2 t} \sin(\pi x) \) for \( \alpha \in [0,1] \). It is clear that the partial derivatives of \( \frac{\partial \tilde{U}}{\partial t} \) and \( \frac{\partial^2 \tilde{U}}{\partial x^2} \) as the form (2) exist, so the general terms of the equations;

\[ \frac{\partial \tilde{U}}{\partial t} = \frac{\partial^2 \tilde{U}}{\partial x^2}, \quad 0 < x < 1, \quad t > 0, \]
\[ \frac{\partial \tilde{U}}{\partial t} = \frac{\partial^2 \tilde{U}}{\partial x^2}, \quad 0 < x < 1, \quad t > 0, \] (5.2)

can be recognized by using (4.13), (4.14);

\[ \tilde{U}_n(x,t;\alpha) = (-1)^n \tilde{k}(\alpha) \frac{\pi^{2n}}{n!} t^n \sin(\pi x), \quad n = 0, 1, 2, \ldots \] (5.3)
\[ \tilde{U}_n(x,t;\alpha) = (-1)^n k(\alpha) \frac{\pi^{2n}}{n!} t^n \sin(\pi x), \quad n = 0, 1, 2, \ldots \] (5.4)

Solution of Eq. (5.1) will be derived from (4.15). In this example we have also derived the exact solution.

Example 5.2 Consider the following partial differential equation, with specified initial conditions.

\[ \frac{\partial \tilde{U}}{\partial t} = \frac{\partial^2 \tilde{U}}{\partial x^2}, \quad 0 < x < 1, \quad t > 0, \]
\[ \tilde{U}(x,0) = \tilde{f}(x) = \tilde{k} \cos(\pi x - \frac{\pi}{2}), \quad 0 < x < 1. \] (5.5)

and \( \tilde{k}[\alpha] = [k(\alpha), \tilde{k}(\alpha)] = [\alpha, 1, 1 - \alpha] \)
The exact solution for \( \alpha \in [0,1] \) are 
\[
U(x,t;\alpha) = k(\alpha) \ e^{-x^2} \cos(\frac{\pi x}{2}) \text{ and } \]
\[
\bar{U}(x,t;\alpha) = \bar{k}(\alpha) \ e^{-x^2} \cos(\frac{\pi x}{2}).
\]
Both \( \frac{\partial U}{\partial t} \) and \( \frac{\partial^2 U}{\partial x^2} \) are in the form (2), so the general terms would be as:
\[
\bar{U}_n(x,t;\alpha) = (-1)^n \bar{k}(\alpha) \ \pi^2n \frac{t^n}{n!} \cos(\frac{\pi x}{2}), \quad n = 0,1,2,... \quad (5.6)
\]
\[
U_n(x,t;\alpha) = (-1)^n k(\alpha) \ \pi^2n \frac{t^n}{n!} \cos(\frac{\pi x}{2}), \quad n = 0,1,2,... \quad (5.7)
\]
Substituting from (5.6) and (5.7) into (4.17) we will derive the exact solution.

Example 5.3 Consider the fuzzy parabolic equation:
\[
\frac{\partial \bar{U}}{\partial t}(x,t;\alpha) = \frac{1}{2} x \ \frac{\partial^2 \bar{U}}{\partial x^2}(x,t), \quad 0 < x < 1, \quad t > 0 \quad (5.8)
\]
with the boundary conditions:
\[
\bar{U}(x,0) = \bar{k}(x^2), \quad 0 < x < 1
\]
and \( \bar{k}[\alpha] = \{ k(\alpha) \ \bar{k}(\alpha) \} = [\alpha - 1,1 - \alpha]. \)
The exact solution are 
\[
\bar{U}(x,t;\alpha) = \bar{k}(\alpha) \ e^{-x^2} \text{ and } \bar{U}(x,t;\alpha) = \bar{k}(\alpha) \ e^{-x^2} \text{ for } \alpha \in [0,1].
\]
It is clear that the partial derivative of \( \frac{\partial \bar{U}}{\partial t} \) is in the form (2) and \( \frac{\partial^2 \bar{U}}{\partial x^2} \) is in the form(1),so the general terms by using (4.16) and (4.17) would be as:
\[
\bar{U}_n(x,t;\alpha) = \bar{k}(\alpha) \ \pi^2n \frac{t^n}{n!} x^2, \quad n = 0,1,2,... \quad (5.9)
\]
\[
U_n(x,t;\alpha) = k(\alpha) \ \pi^2n \frac{t^n}{n!} x^2, \quad n = 0,1,2,... \quad (5.10)
\]
Substituting from (5.9) and (5.10) into (4.20) we will derive the exact solution.

6. Conclusions

The main goal of this article has been to derive an analytical parametric solution for the fuzzy heat equation. We have achieved this goal by applying Adomian decomposition method.
This method has a useful feature in that it provides the solution in a rapid convergent power series with elegantly computable convergence of the solution.
References


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