On the Convergence Properties of Power Similar Bases of Polynomials in Clifford Analysis

M. Abul-Ez
Department of Mathematics, Faculty for Girls
King Khalid University, Saudi Arabia
mabulez56@hotmail.com

M. Zayed
Department of Mathematics, Faculty for Girls
King Khalid University, Saudi Arabia
mohraza12@hotmail.com

Abstract. In the present paper the problem of taking the power of a base of special monogenic polynomials is studied, thus leading to a number of results under some additional conditions of associated infinite matrices, related essentially to the so-called algebraicness and Boes condition of these matrices. The obtained results are the extent of generalization of those given in ([7], [16] and [17]).

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1. Introduction

By the effectiveness of the base \( \{P_n(x)\} \) of special monogenic polynomials, \( x \) is the Clifford variable in the closed ball \( \overline{B}(r) \) we mean that the base \( \{P_n(x)\} \) forms a base for the class \( E \) of special monogenic functions (regular) monogenic in \( \overline{B}(r) \) with norm given by \( M(f, R), \ f \in E \). Then \( \{P_n(x)\} \) is a base for the vector space \( SM_{[x]} \) of special monogenic polynomials in \( x \).

When the two simple monic bases \( \{P_n(x)\} \) and \( \{Q(x)\} \) are both effective in the closed ball \( \overline{B}(r) \), or at the origin, the product base \( \{L_n(x)\} = \{P_n(x)\} \{Q_n(x)\} \) possesses the same properties. If one of the two bases \( \{P_n(x)\} \) and \( \{Q(x)\} \) is effective while the other is not, the product base \( \{L_n(x)\} \) is certainly not effective. But if two bases \( \{P_n(x)\} \) and \( \{Q_n(x)\} \) are both not
effective, the product base \( \{ L_n(x) \} \) may be effective. This is evident when \( \{ P_n(x) \} \) is taking to be the inverse base of \( \{ Q_n(x) \} \).

In [5] the bases of any positive integer power were defined where the effectiveness and the order of these power bases are deducible from the results on the product bases. The inverse base has been defined in [3] and in consequence the base of the integer powers has been considered where also the effectiveness and the order of these bases are studied. According to the definition of the similar infinite matrix given in [7]. In analogy with the complex case the bases of positive integer powers of the base of special monogenic polynomials are related to the positive integer powers, and inverse of Clifford matrices.

According to the definition of the similar infinite matrix given in [7], it is interesting to look at the power similar bases defined as

\[
\{ U_n(x) \}^{(s)} = \{ T_n(x) \} \{ Q_n(x) \} \{ P_n(x) \}
\]

and then study the effectiveness property of \( \{ U_n(x) \}^{(s)} \) in terms of effectiveness properties of the factor bases \( \{ P_n(x) \} \) and \( \{ Q_n(x) \} \).

2. Notations and Preliminaries

The polynomial bases problem in complex analysis has been already generalized to the case of Clifford analysis, namely by introducing the so-called “special monogenic polynomials” (see [1] and [2]). For the purpose of this work, we mention some notations and preliminaries of Clifford algebra and analysis as well as the terminology of the theory of polynomial bases. We refer the reader to ([1], [2], [9] and [10]).

Let \( e_1, e_2, \ldots, e_n \) be an orthonormal base of \( \mathbb{R}^n \). The Clifford algebra \( A \) constructed over \( \mathbb{R}^n \) has the base given by \( \{ e_A : A \subseteq \{ 1, \ldots, n \} \} \) where \( e_i = e_{(i)} \); \( i = 1, \ldots, n, \ e_0 = e_\phi = 1 \) (the identity of \( A \)) with the multiplication rule \( e_i e_j + e_j e_i = -2 \delta_{ij}, \ i, j = 1, \ldots, n \) and where \( e_A = e_{\alpha_1} \ldots e_{\alpha_h} \), \( A = \{ \alpha_1, \ldots, \alpha_h \} \) with \( \alpha_1 < \alpha_2 < \ldots < \alpha_h \). Every element of \( A \) may be represented in the form \( a = \sum_A a_A e_A \) where \( a_A \) are real numbers. By \( \bar{a} = \sum_A a_A e_A \) where \( e_A = e_{\alpha_h} \ldots e_{\alpha_1}, \ e_j = -e_j, \ j = 1, \ldots, n \) and \( e_0 = e_0 \) we define a conjugate element. A norm is defined by putting for any \( a, b \in A, |a|^2 = \sum_A |a_A|^2 \) with \( |ab| \leq 2 \sum |a| |b| \).

Let \( \Omega \) be a connected open subset of \( \mathbb{R}^{m+1}, \ m \leq n \). The function \( f \) in \( \Omega \) and with values in \( A \) are considered. Points in \( \mathbb{R}^{m+1} \) denoted by \( x = (x_0, x_1, \ldots, x_m) = (x_0, \bar{x}) \) are identified with the hypercomplex scalars \( x = \sum_{k=0}^m x_k e_k \), which form the real subspace \( \mathcal{B} \) of \( A \). A function \( f = f(x) = \sum_A f_A(x)e_A \in C_1(\Omega, A) \) is called left monogenic in \( \Omega \) if it satisfies \( Df = \sum_{k=0}^m e_k \frac{\partial f}{\partial x_k} = 0 \). Also \( f \) is called right monogenic if \( Df = 0 \).

In the sequel we shall use monogenic to stand for either left monogenic or right monogenic.
**Definition 1.** *(Special monogenic polynomial).* A polynomial $P(x)$ is special monogenic if $DP(x) = 0$ (so $P(x)$ is monogenic) and there exist $a_{ij} \in A$ for which

$$P(x) = \sum_{i,j}^{\text{finite}} x^i x^j a_{ij}.$$ 

**Definition 2.** *(Special monogenic function).* Let $\Omega$ be a connected open subset of $\mathbb{R}^{m+1}$ containing $0$ and let $f$ be monogenic in $\Omega$. Then $f$ is called special monogenic in $\Omega$ iff its Taylor series near zero (which known to exist) has the form

$$f(x) = \sum_{n=0}^{\infty} P_n(x)$$

for certain special monogenic polynomial $P_n(x)$.

The main references for special monogenic functions are [11] and [18].

**Remark 1.** Note that a homogenous special monogenic polynomial $P_n(x)$ of degree $n$ is necessarily of the form $P_n(x) = z_n(x)c_n$. Hence $c_n \in A$ and $z_n(x)$ is given by the generating function

$$\frac{1-z}{|1-x|^m} \quad \text{(see [1])}.$$ 

**Effective Base of Special Monogenic Polynomials.** In complex analysis [19] the definition of the base for the vector space of complex polynomials $\mathbb{C}[z]$ has been adapted to the setting of Clifford analysis [1] by noticing that the set of special monogenic functions is a free module over $z_n(x)$. Let

$$\{P_n(x) = \sum_{k=0}^{\infty} z_k(x)P_{nk}, n \in \mathbb{N}\},$$

be a base for the set of special monogenic polynomials [1]; i.e. every $z_n(x)$ can be expressed as a right $A$-linear combination

$$z_n(x) = \sum_{i=0}^{\infty} P_i(x)P_{ni} : P_{ni} \in A,$$ 

where only a finite number of terms differ from zero, and for any sequence $(a_i)_{i=0}^{l} \in A$, $\sum_{k=0}^{l} P_k(x)a_k \to a_0 = a_1 = ... = a_l = 0$. $P = (P_{nk})$ is called the Clifford matrix of coefficients, and $\overline{P} = (P_{nk})$ is the inverse of $P$ and is called the Clifford matrix of operators. If $\deg P_n(x) = n$ for every $n \in \mathbb{N}$, the base is called *simple*. Moreover a simple base $\{P_n(x)\}$ is called *monic* if such that $|P_{nn}| = 1$ for all $n$. A base $\{P_n(x)\}$ is *algebraic* if its matrix of coefficients $P$ satisfies an algebraic equation of finite degree [4]. In analogy with the complex case [13], we introduce the notion of quasi-simple monic bases in Clifford setting as follows:

**Definition 3.** *(Quasi-simple monic base).* Let $P$ be a row-finite Clifford matrix in which every element in the leading diagonal is unity while every element below the leading diagonal is zero. Such a matrix is not necessarily associated with a base of sm-polynomials. But if such a matrix has a row-finite reciprocal $\overline{P}$, which is obviously of the same form, it may be associated with a base which we may call quasi-simple monic base.
In such case, it is clear that the power matrix \( P^n \), where \( n \) is a +ve or a −ve integer, is a matrix of the same form, and has a row-finite reciprocal of the same form, namely \( P^{-n} \). Thus \( P^n \) may be associated with a quasi-simple monic base of sm-polynomials. In a similar way of the complex case (see [13], p.323) one can show that, in case of algebraic quasi-simple monic bases, a matrix \( Q = P^s \), where \( s \) is any integer, may be defined so as to associated with a quasi-simple monic base of sm-polynomials.

Given a \( f \) such that \( f(x) = \sum_{n=0}^{\infty} z_n(x)c_n \) (near 0), \( c_n \in \mathcal{A}_m \), we can formally express it in terms of the \( P_k \) as

\[
(2.2) \quad f(x) = \sum_{k=0}^{\infty} P_n(x) \left( \sum_{n=0}^{\infty} \mathcal{P}_{nk}c_n \right).
\]

**Definition 4.** (Effective base). If for all special monogenic functions \( f \), defined in a neighborhood of \( \mathcal{B}(r) \), the series in (2.2) in terms of \( P_k \), converges normally to \( f \) in \( \mathcal{B}(r) \), the base \( \{P_n(x)\} \) is called effective in \( \mathcal{B}(r) \).

To characterize the effectiveness property of the base we shall use the following Cannon function which defined by

\[
(2.3) \quad \omega(r) = \lim_{n \to \infty} \sup \{\omega_n(r)\}^{\frac{1}{n}},
\]

where the expression,

\[
(2.4) \quad \omega_n(r) = \sum_{k=0}^{\infty} \sup_{|x|=r} \left| P_k(x)\mathcal{P}_{nk} \right|
\]

is called the Cannon sum of \( \{P_n(x)\} \). Then due to the generalized Cannon’s theorem ([1], Theorem 1) the simple base is effective in \( \mathcal{B}(r) \) if and only if \( \lambda(r) = r \).

### 3. Power similar base of sm\(^1\)-polynomials

By transforming a base of polynomials, another base of polynomials was generated in [7] and [16]. It was called a similar base of polynomials. Consequently, the power similar base of complex polynomials was introduced in [17]. The definition of the power similar base of complex polynomials, can be adapted to the setting of Clifford analysis as follows.

**Definition 5.** (Power similar base in Clifford setting). Let \( \{P_n(x)\} \) and \( \{Q_n(x)\} \) be two bases of sm-polynomials, where as usual we have

\[
(3.1) \quad P_n(x) = \sum_i z_i(x)P_{ni} \quad \text{and} \quad Q_n(x) = \sum_j z_j(x)Q_{nj}, \quad n \geq 0, x \in \mathbb{R}^{m+1}
\]

\(^1\text{(sm)- stands for (specil monogenic polynomials) for short, since it is frequently used.}\)
Then we have the representations

\[ z_n(x) = \sum_i P_i(x) P_{ni}, \quad \text{and} \quad z_n(x) = \sum_j Q_j(x) Q_{nj}. \]

Suppose that \( \{U_n(x)\} \) is the base similar to the base \( \{Q_n(x)\} \) with respect to the base \( \{P_n(x)\} \), then according to the definition of the product base (see [5]) we have

\[ U_n(x) = \sum_k z_k(x) U_{nk}, \quad n \geq 0 \]

where

\[ U_{nk} = \sum_{i,j} P_{jk} Q_{ij} P_{ni} \]

Now, consider the \( s \)-th power of the similar base \( \{U_n(x)\} \) in the form \( \{U_n(x)\}^{(s)} \) where

\[ U_n^{(s)}(x) = \sum_k z_k(x) U_{nk}^{(s)}, \quad n \geq 0. \]

Then the Clifford matrix \( U^{(s)} \) of coefficients of the power similar base \( \{U_n(x)\}^{(s)} \) is such that

\[ U^{(s)} = (TPQ)^{(s)} = TPQ^{(s)}P. \]

Therefore

\[ \{U_n(x)\}^{(s)} = \{\overline{P}_n(x)\} \{Q(x)\}^{(s)} \{P_n(x)\} \]

where \( \{\overline{P}_n(x)\} \) is the inverse base of the base \( \{P_n(x)\} \) defined in [3].

Thus, The power similar base \( \{U_n(x)\}^{(s)} \) is the base of sm-polynomials similar to the power base \( \{Q_n(x)\}^{(s)} \) of sm-polynomials with respect to the base \( \{P_n(x)\} \) of sm-polynomials. Thus we have the following relation

\[ U_{nk}^{(s)} = \sum_{i,j} P_{jk} Q_{ij}^{(s)} P_{ni} \]

\[ = \sum_{i,j,j_1,...,j_{s-1}} P_{jk} Q_{i j_1} Q_{j_1 j_2} ... Q_{j_{s-1} j} P_{ni}. \]

Since the power of a base is also a base (see [5]) and a base similar to a base is also a base (see [7]), then the power similar of a base is a base. Thus the base property of the power similar set follows i.e. \( \{U_n(x)\}^{(s)} \) is a base for \( SM_{[x]} \).
4. Effectiveness of power similar bases in a closed ball

Let \( \{P_n(x)\} \) and \( \{Q_n(x)\} \) be simple monic bases of sm-polynomials each of them is effective in the closed ball \( \overline{B}(r) \), then for any finite numbers \( r_1, r_2 \) greater than \( r \), we have

\[
\begin{align*}
\omega_n(r) &< kr_1^n, \\
\lambda_n(r) &< kr_2^n; \quad n \geq 0,
\end{align*}
\]

where \( \omega_n(r) \) and \( \lambda_n(r) \) are the Cannon sums of the bases \( \{P_n(x)\} \) and \( \{Q_n(x)\} \) respectively.

Write as usual \( A_n(r) = \sup_{|x|=r} |P_n(x)|, \quad B_n(r) = \sup_{|x|=r} |Q_n(x)| \) and \( C_n^{(s)}(r) = \sup_{|x|=r} |U_n^{(s)}(x)| \). Since these bases are monic, then for \( r > 1 \) we have

\[
\begin{align*}
1 &< \frac{(m)}{n!} r^n < A_n(r) < \omega_n(r) < kr_1^n \\
1 &< \frac{(m)}{n!} r^n < B_n(r) < \lambda_n(r) < kr_2^n
\end{align*}
\]

Making use Cauchy’s inequality, and in view of (3.2) and (4.2) one can get

\[
B_n^{(s)}(r) = \sup_{|x|=r} |Q_n^{(s)}(x)|
\leq \sup_{|x|=r} \left| \sum_k z_k(x) Q_{nk}^{(s)} \right|
\leq 2^{\frac{n}{2}} \sum_k \frac{(m)_k}{k!} r^k \cdot |Q_{nk}^{(s)}|
\leq 2^{\frac{n}{2}} \sum_k \frac{(m)_k}{k!} k^s r_n^2 k^{(s-1)} \left( \frac{n!}{(m)_n} \right)^{s}
\leq 2^{\frac{n}{2}} \left( \frac{n!}{(m)_n} \right)^{s} k^s r_n^2 \sum_k \frac{(m)_k}{k!} \left( \frac{r_2}{r} \right)^n
\leq k r_2^n (n+1)^s r_n^2 \left( \frac{r_2}{r} \right)^n, \quad s \text{ is finite.}
\]
From which and also using Cauchy’s inequality, (4.2) and the relation \(|\overline{P}_{jk}| \leq 2^{-m} \frac{\omega_j(r)}{A_k(r)} \leq 2^{-m} \frac{\omega_j(r)}{\lambda_k(r)}\), we get

\[
(4.3) C_n^{(s)}(r) = \sup_{|x|=r} \left| U_n^{(s)}(x) \right| 
\leq \sup_{|x|=r} \left| \sum_k z_k(x) U_n^{(s)} \right| 
\leq 2^{\frac{3m}{r}} \sqrt{\frac{n!}{(m)_i}} \sum_{i,j,k} (m)_k \frac{1}{r^j} \left( \frac{r_2}{r} \right)^{(s-1)} \left( \frac{1}{r} \right) 
\leq k A_n(r) \sum_{i,j} \left( 1 + n \right) \left( \frac{r_2}{r} \right)^{n+1} \left( \frac{r_1}{r} \right)^n 
\]

Writing the Cannon sum of the base \( \{ U_n(x) \}^{(s)} \) and using (3.5), (4.1), (4.2) and (4.3) with \( |Q_{ij}| \leq 2^{-m} \frac{1}{(m)_j} \),

\[
\Phi_n^{(s)}(r) \leq 2^{\frac{3m}{r}} \sum_k \left| C_{nk}^{(s)}(r) \right| \left| U_n^{(s)}(r) \right| 
\leq k [n+1] \left( \frac{r_2}{r} \right)^{2n+1} \left( \frac{r_1}{r} \right)^n. 
\]

Hence the Cannon function of the power similar base is such that

\[
\Phi^{(s)}(r) = \limsup_{n \to \infty} \left\{ \Phi_n^{(s)}(r) \right\}^{\frac{1}{n}} 
\leq \left( \frac{r_2}{r} \right)^{2(s+1)} \left( \frac{r_1}{r} \right)^2 r_1. 
\]

Taking \( r_2, r_1 \) close to \( r \), we see that \( \Phi^{(s)}(r) \leq r \) or \( \Phi^{(m)}(r) = r \), from which we derive that the base is effective in \( \overline{B}(r) \), and thus we have the following result:

**Theorem 4.1** Let \( \{ U_n(x) \}^{(s)} \) be the power base similar to the monic base \( \{ Q_n(x) \}^{(s)} \) with respect to the monic base \( \{ P_n(x) \} \) and suppose that each of the bases \( \{ P_n(x) \} \) and \( \{ Q_n(x) \} \) is monic and effective in \( \overline{B}(r) \), then the power similar base is effective there.
To show that the effectiveness of the base \( \{Q_n(x)\} \) in a closed ball \( \overline{B}(r) \) is not necessary holds for the power similar base to be effective there, we give the following example in which the second base \( \{Q_n(x)\} \) is the analogue of the complex case which was constructed by Nassif and Makar [15].

**Example 1.** Let \( \{P_n(x)\} \) and be two simple monic bases of sm-polynomials where:

\[
P_{2n}(x) = z_{2n-1}(x) + z_{2n}(x),
\]

\[
P_{2n+1}(x) = z_{2n+1}(x),
\]

\[
P_{0}(x) = 1,
\]

for this base we have

\[
z_{2n+1}(x) = P_{2n+1}(x), \quad z_{2n}(x) = P_{2n}(x) - P_{2n-1}(x)
\]

and the Cannon sum will be

\[
\omega(r) = \limsup_{n \to \infty} \frac{\omega_{2n}(r)}{2n+1} = r,
\]

i.e. the base \( \{P_n(x)\} \) is effective in \( \overline{B}(r) \leq r, \, r > 0 \). The base \( \{Q_n(x)\} \) is given such that

\[
Q_{2n,2n-1} = 1
\]

\[
Q_{2n+1,2n} = 2n + 1
\]

\[
Q_{2n,2n-r} = (2n-1)(2n-3)\ldots(2n-2r+1)a_r, \quad n \geq 1, \, r \geq 1
\]

\[
Q_{2n+1,2n-2r-1} = Q_{2n,2n-2r}, \quad n \geq 2, \, r \geq 1
\]

\[
Q_{2n+1,2n-2r} = (2n+1)Q_{2n,2n-2r}, \quad n \geq 1, \, r \geq 1
\]

\[
Q_{2n+1,2n-2r-1} = (2n+1)(2n-1)(2n-3)\ldots(2n-2r+1)b_r, \quad n \geq 1, \, r \geq 1
\]

where the coefficients \( a_r \) and \( b_r \) are given by

\[
\sum_{r=0}^{\infty} a_r y^r = (1 + y)^{-\frac{1}{2}}, \quad b_r = \frac{2r+3}{2r+2}a_r, \quad r \geq 0.
\]

In this base \( \{Q_n(x)\} \) the Cannon sum will be

\[
\lambda_{2n}(r) > B_{2n}(r) > |Q_{2n,0}| = |a_n| (2n-1)(2n-3)\ldots1 = \frac{[(2n)!]^3}{2^{3n}(n!)^3},
\]

hence \( \lambda(r) = \infty \) for all \( r > 0 \), and the base \( \{Q_n(x)\} \) is never effective. Now take the power \( s = 2 \), then \( \{Q_n(x)\}^2 \) will be in the form
\[
Q_0^2(x) = 1 \\
Q_1^2(x) = 2 + z_1(x) \\
Q_2^2(x) = 2z_{2n-1}(x) + z_{2n}(x) \\
Q_{2n+1}^2(x) = 2(2n + 1)[2z_{2n-1}(x) + z_{2n}(x) + z_{2n+1}(x)], \ n \geq 1.
\]

By the definition of similar base we can construct the power similar base \(\{U_n(x)\}\) as follows

\[
\{U_n(x)\} = \{P_n(x)\} \{Q_n(x)\}^{(s)} \{P_n(x)\}.
\]

Then we can easily get

\[
U_{2n+1}^2(x) = z_{2n+1}(x) + 2(2n + 1)[z_{2n}(x) + z_{2n-1}(x)] \\
U_{2n}^2(x) = z_{2n}(x) + 2z_{2n-1}(x) + 2(2n - 1)[z_{2n-2}(x) + z_{2n-3}(x)]
\]

from which we have

\[
z_{2n}(x) = U_{2n}^2(x) - 2U_{2n-1}^2(x) + 2(2n - 1)\left[U_{2n-2}^2(x) - U_{2n-3}^2(x)\right], \\
z_{2n+1}(x) = U_{2n+1}^2(x) - 2(2n + 1)\left[U_{2n}^2(x) - U_{2n-1}^2(x)\right].
\]

Thus the Cannon sum is given by

\[
\omega_{2n}^2(r) = r^{2n} + 4r^{2n-1} + 4(2n - 1)(2r^{2n-2} + 3r^{2n-3}) \\
+ 8(2n - 1)(2n - 3)(r^{2n-4} + r^{2n-5}), \\
\omega_{2n+1}^2(r) = r^{2n+1} + 4(2n + 1)(r^{2n} + 2r^{2n-1}) \\
+ 8(2n + 1)(2n - 1)(r^{2n-2} + r^{2n-3}).
\]

From the fact that the Cannon function is increasing, we have: \(\omega_{(2)}(r) \leq r, \ \forall r > 0\). Thus the base \(\{U_n^2(x)\}\) is effective in \(\overline{B}(r)\), though the base \(\{Q_n(x)\}\) is never effective.

Hence, restrictions must be imposed on \(\{Q_n(x)\}^{(s)}\) to be effective in the same region as the base \(\{Q_n(x)\}\). This case may be deduced if we take the power base \(\{Q_n(x)\}^{(s)}\) to be of certain type of bases called algebraic simple monic bases i.e. the matrix of coefficients satisfies an algebraic matrix equation of finite degree. For this case we give the following result

**Theorem 4.2** Let \(\{U_n(x)\}^{(s)}\) be a power similar base and suppose that the base \(\{P_n(x)\}\) is a simple monic base and effective in \(\overline{B}(r)\), then the power similar base is effective there, if and only if, the simple monic base \(\{Q_n(x)\}\) is effective in \(\overline{B}(r)\) and its power base is algebraic.
Proof. From the definition of the power similar base we have
\[ \{U_n(x)\}^{(s)} = \{P_n(x)\} \{Q_n(x)\}^{(s)} \{P_n(x)\}. \]

Since \( \{Q_n(x)\} \) is a simple monic and effective in \( \mathcal{B}(r) \); then by Theorem 4.1 above the power similar base \( \{U_n(x)\}^{(s)} \) is effective there.

Now, suppose that \( \{U_n(x)\}^{(s)} \) is effective in \( \mathcal{B}(r) \), and write:
\[ \{Q_n(x)\}^{(s)} = \{P_n(x)\} \{U_n(x)\}^{(s)} \{P_n(x)\}. \]

Invoking theorem 2.1 of ([7] p. 1060), we may infer that the base \( \{Q_n(x)\}^{(s)} \) is effective in \( \mathcal{B}(r) \). To complete the proof let us quote the following useful result of Abul-Ez and Constales [6].

Lemma 1 The algebraic simple monic base \( \{U_n(x)\} \) of sm-polynomials and all its root bases \( \{U_n(x)\}^{1, s = 2, 3, 4, \ldots} \) have the same region of effectiveness.

Thus the base \( \{Q_n(x)\} \) is effective in \( \mathcal{B}(r) \) as required and this complete the proof of Theorem 4.2.

But of course if we put the restriction on \( \{Q_n(x)\} \) to be algebraic quasi-simple monic base, then relying on the results in [6] it is easy to establish the following result:

Theorem 4.3 If \( \{P_n(x)\} \) is a simple monic base of sm-polynomials in \( \mathcal{B}(r) \), the power similar base \( \{U_n(x)\}^{(s)} \) will be effective in the same domain if and only if the base \( \{Q_n(x)\} \) is algebraic quasi-simple monic base effective there.

5. Effectiveness of power similar bases when the constituents are algebraic bases

Now we can also investigate the effectiveness of the power similar base in a closed ball by considering some conditions such as when its constituents are algebraic.

Let the two simple bases \( \{P_n(x)\} \) and \( \{Q_n(x)\} \) be algebraic and satisfy the following condition

\[ \begin{cases} 
\mu(r) \leq r \\
\nu(r) \leq r, \quad \text{respectively.}
\end{cases} \]

(5.1)

So that, for the power base \( \{Q_n(x)\}^{(s)} \) (cf. [6] and [7]), we have

\[ \begin{cases} 
\mu^{(s)}(r) = \limsup_{n \to \infty} \left[ B_n^{(s)}(r) \right]^\frac{1}{n} \leq r \\
\mu(r) = \limsup_{n \to \infty} \left[ A_n(r) \right]^\frac{1}{n} \leq r \\
\mu^{(s)}(r) = \limsup_{n \to \infty} \left[ A_n^{(s)}(r) \right]^\frac{1}{n} \leq r
\end{cases} \]

(5.2)
where $B_n^{(s)}(r) = \sup_{|x|=r} |Q_n^{(s)}(x)|$, $A_n(r) = \sup_{|x|=r} |P_n(x)|$ and $A_n^{(s)}(r) = \sup_{|x|=r} |P_n^{(s)}(x)|$.

From the relations (5.2) we have

$\begin{cases} 
B_n^{(s)}(r) < k R_1^n, & n \geq 0, \ R_1 > r \\
A_n^{(s)}(r) < k R_2^n, & n \geq 0, \ R_2 > r \\
A_n(r) < k R_3^n, & n \geq 0, \ R_3 > r.
\end{cases}$

(5.3)

Since the base $\{Q_n(x)\}$ is algebraic there is a matrix equation in the form (see [4]).

$$\mathcal{Q} = \sum_{i=1}^{N} c_i Q^{N-i} \ , \ c_i \in \mathbb{R},$$

(5.4)

where $N$ is a finite positive integer which, together with the coefficients $(c_i)^{N-1}_{0}$, is independent of the indices $n, i$. Using (5.3), (5.4) and Cauchy’s inequality, we obtain

$$\left|\mathcal{Q}_{nk}^{(s)}\right| < \sum_{i=1}^{N} |c_i| \left|Q_{nk}^{(s)N-i}\right|$$

< \sqrt{n! \left(\frac{B_n^{(s)}(r)}{r^{k}}\right)} \sum_{i=1}^{N} |c_i|$$

< k \frac{R_1^n}{r^{k}},$$

(5.5)

where $B_n^{(s)}(r) = \sup_{1 \leq j \leq n-1} B_j^{(s)}(r)$.

Also it is easy to establish analogue relations concerning the matrix of operators of the base $\{P_n(x)\}$ in the form

$$\left|\mathcal{P}_{nk}\right| < k \frac{R_3^n}{r^{k}}.$$  

(5.6)

Write

$$C_n^{(s)}(r) = \sup_{|x|=r} \left|U_n^{(s)}(x)\right|$$

and

$$\{U_n(x)\}^{(s)} = \{\mathcal{P}_n(x)\} \{Q_n(x)\}^{(s)} \{P_n(x)\} ,$$

then by using (5.3) and Cauchy’s inequality one can get
\[ C_n^{(s)}(r) = \sup_{|x|=r} \left| \sum_{i,j,k} z_k(x) P_{jk} Q_{ij} P_{ni} \right| \]

\[ \leq 2^{\frac{3m}{2}} \frac{(m)_n}{n!} \sqrt{\frac{i!j!}{(m)_i(m)_j}} \left( \frac{R_1}{r} \right)^n \left( \frac{R_2}{r} \right)^n \left( \frac{R_3}{r} \right)^n. \]

If we choose \( R_1, R_2 \) and \( R_3 \) as near to \( r \) as we can, then

\[ C_n^{(s)}(r) \leq k r^n \]

i.e. the power similar base \( \{ U_n(x) \}^{(s)} \) satisfies a similar conditions as its constituents, in the form

\[ \lim_{n \to \infty} \sup \left[ \frac{C_n^{(s)}(r)}{n^2} \right] \frac{1}{r} \leq r. \]

Carrying out the product \( \{ P_n(x) \} \{ Q_n(x) \}^{(s)} \{ P_n(x) \} \) we get a relation like (5.5) and (5.6) concerning \( \{ U_n(x) \}^{(s)} \) in the form

\[ \left| U_{nk}^{(s)} \right| = \left| \sum_{i,j,k} P_{jk} Q_{ij}^{(s)} P_{ni} \right| \]

\[ < k \frac{R_n}{r^k} \left( \frac{R_1 R_2}{r^2} \right)^n. \]

Introducing (5.7) and (5.9) in the Cannon sum of \( \{ U_n(x) \}^{(s)} \) we obtain

\[ \Phi_n^{(s)}(r) \leq 2^{\frac{3m}{2}} \sum_k \sup_{|x|=r} \left| U_k^{(s)}(x) \right| \left| \bar{T}_{nk}^{(s)} \right| \]

\[ \leq 2^{3m} k \left( \frac{R_1 R_2 R_3}{r^3} \right)^n. \]

So that \( \Phi^{(s)}(r) \leq r \), from which we infer that \( \Phi_n^{(s)}(r) = r \). The following result is therefore established.

**Theorem 5.1** Let \( \{ U_n(x) \}^{(m)} \) be a power similar base of sm-polynomials and let the two bases \( \{ P_n(x) \} \) and \( \{ Q_n(x) \} \), be algebraic simple monic bases effective in \( \mathcal{B}(r) \) satisfying conditions (5.1), then the power similar base is effective in \( \mathcal{B}(r) \), and satisfies (5.8).

6. Effectiveness of the power similar base when its constituents satisfying Boas condition

Let \( \{ P_n(x) \} \) and \( \{ Q_n(x) \} \) be two bases of sm-polynomials, where
\[ P_n(x) = z_n(x) + \sum_{i=0}^{n-1} z_i(x) P_{ni}, \quad \text{and} \quad Q_n(x) = z_n(x) + \sum_{j=0}^{n-1} z_j(x) Q_{nj}, \quad n \geq 0 \]
each of them accords to Boas condition (cf. [14]) in the form
\[
\begin{align*}
&|P_{ni}| \leq Ma^{n-i}, \quad 0 \leq i \leq n-1 \\
&|Q_{nj}| \leq Nb^{n-j}, \quad 0 \leq j \leq n-1
\end{align*}
\]
where \(a, b, M\) and \(N\) are any finite positive numbers.

In what follows we give an independent proof of the following theorem:

**Theorem 6.1** Let \(\{P_n(x)\}\) and \(\{Q_n(x)\}\) be simple monic bases of sm-polynomials satisfying the two inequalities in (6.1) respectively, then the power similar base \(\{U_n(x)\}\) will be effective in \(B(r); \ r \geq \max [a (1 + M); \ b (1 + N)]\).

**Proof.** Let \(r \geq \max [a (1 + M); \ b (1 + N)]\). For any positive integer \(s \geq 1\), the power base \(\{Q_n(x)\}^{(s)}\) of sm-polynomials is defined by
\[
Q^{(s)}_n(x) = z_n(x) + \sum_{j=0}^{n-1} z_j(x) Q^{(s)}_{nj}
\]
where
\[
Q^{(s)}_{nj} = \sum_{j_1,j_2,...,j_{s-1}} Q_{nj_1} Q_{j_1j_2} ... Q_{j_{s-1}j}.
\]

Using (6.1) we obtain after certain reduction
\[
\left| Q^{(s)}_{nj} \right| < 2^{m_s} s b^{n-j} n^{s-1} (1 + N)^s, \quad 0 \leq j \leq n-1
\]

Now since the coefficients \(\overline{QQ} = I\), where \(Q = (Q_{nj}), \overline{Q} = (Q_{nj})\) and \(I\) is the unite matrix, then due to the recurrence relation,
\[
Q_{n+k,n+k} = -\sum_{j=0}^{k-1} Q_{n+k,n+j} \overline{Q}_{n+j,n},
\]
we can obtain by mathematical induction
\[
\left| \overline{Q}_{nj} \right| < [b (1 + N)]^{n-j}, \quad 0 \leq j \leq n-1.
\]

Similarly for the base \(\{P_n(x)\}\) we have
\[
\left| \overline{P}_{ni} \right| < [a (1 + M)]^{n-i}, \quad 0 \leq i \leq n-1.
\]

Proceeding used to derive (6.2) can be applied using (6.4) to obtain
\[
\left| \overline{Q}^{(s)}_{nj} \right| < 2^{m_s} 2 s (2n)^{s-1} [b (1 + N)]^{n-j}, \quad 0 \leq j \leq n-1.
\]
Then in view of (6.1), (6.2) and (6.4), one can easily obtain

\[
C_n^{(s)}(r) \leq 2^{\frac{3m}{n}} \sum_{i,j,k} \frac{(m)_k}{k!} r^k |P_{jk}| |Q_i^{(s)}| |P_{ni}| + \frac{(m)_n}{n!} r^n \\
\leq 2^{\frac{3m}{n}} snM(1+N)^s r^n \sum_i k \left(\frac{a}{r}\right)^{n-i} i^{s-1} \sum_j \left(\frac{b}{r}\right)^{-j} \\
\leq 2^{\frac{3m}{n}} sn^{s+2} M(1+N)^s r^n.
\]

Let \(\Phi_n^{(s)}(r) = \sum_k \sup_{|x|=r} |U_k^{(s)}(x)\overline{u}_n^{(s)}|\) be the Cannon sum of the power similar base \(\{U_n(x)\}^{(s)}\), then a substitution from (6.1), (6.4), (6.5) and (6.6) yields

\[
\Phi_n^{(s)}(r) \leq 2^{\frac{3m}{n}} r^n 2^s M^2 (1+N)^s \sum_{k,i} \left(\frac{a}{r}\right)^{n-i} (2i)^{s-1} k^{s+3} \\
\leq 2^{\frac{3m}{n}} r^n \left[ s^2 M^2 (1+N)^s 2^s n^{2s+4} \right].
\]

It follows that the Cannon function \(\Phi^{(s)}(r)\) for \(\{U_n(x)\}^{(s)}\) will be such that

\[
\Phi^{(s)}(r) = \lim sup_{n \to \infty} \left[ \Phi_n^{(s)}(r) \right]^\frac{1}{n} \leq r
\]

for \(r \geq \max [a (1+M) ; b (1+N)]\), from which the theorem is completely proved.

Consequently we have the following.

**Theorem 6.2** Suppose that \(\{P_n(x)\}\) is a simple monic base effective in \(\overline{B}(r); r \geq b(1+N)\), and suppose that \(\{Q_n(x)\}\) satisfies the conditions in (6.1), then the power similar base \(\{U_n^{(s)}(x)\}\) is effective in \(\overline{B}(r); r \geq b(1+N)\).

**Proof.** This easily verified using the relation

\[
\{U_n(x)\}^{(s)} = \overline{P}_n(x) \{Q_n(x)\}^{(s)} \{P_n(x)\}.
\]

Since the base \(\{Q_n(x)\}\) fulfills (6.1), then its power base \(\{Q_n(x)\}^{(s)}\) will be effective in \(\overline{B}(r); r \geq b(1+N)\) and the theorem follows directly by appealing (Theorem 2.1) of [7].

**References**


Convergence properties


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