Maximal Ideals Relative to a Filter on Posets and Some Applications

Guanghao Jiang

Department of Mathematics, Nanjing University
Nanjing 210093, P. R. China

Luoshan Xu

Department of Mathematics, Yangzhou University
Yangzhou 225002, P. R. China

Abstract

In this paper, the concept of maximal ideals relative to a filter on posets is introduced and examined. Notions of pseudo maximal elements and pseudo irreducible elements are also defined and investigated. Relations between various kinds of irreducible elements and primes in a (distributive) continuous lattice are given in diagrams. Some diagrams in the book “Continuous Lattices and Domains” (Camb. Univ. Press, 2003) are enriched and completed. A non-explicitly posed question in the book is also partially answered in two different ways.

Mathematics Subject Classification: 06A11; 06B35; 54H10

Keywords: poset; maximal ideals relative to a filter; pseudo maximal element; continuous lattice; topological generating set

1 Introduction

In 1972, Dana Scott introduced a class of lattices called continuous lattices in order to provide models for the type free calculus in logic (see [2] and [12]). A litter bit later, a more general notion of continuous domains, which are mathematical structures used in semantics as carriers of meaning, was introduced in [1] and [7]. Now, domain theory has received more and more attentions (see [7]-[11] and [13]-[14]) of both mathematicians and computer scientists. The

\[ \text{Supported by NSF of China (10371106, 60774073).} \]

\[ \text{Corresponding author: e-mail: luoshanxu@yahoo.com} \]
study of distinctive features of some special elements in continuous lattices, as well as in domains, is of fundamental importance. And the study of irreducible elements and primes in continuous lattices was begun in [4]. From then on, pseudo primes, weakly primes and weakly irreducible elements were also investigated by many authors (see [3]-[5] and [10]). Relations between different kinds of irreducible elements and primes in a continuous lattice were well summarized by a diagram in [3]. We note that there is a blank for the case of distributive continuous lattices in the diagram. And we note that there is a non-explicitly posed question after Proposition V-3.7 in [3] on page 406 about continuous lattices in which the set of all irreducible elements or the set of all primes is Lawson closed. To fill in the blank with suitable items and answer the non-explicitly posed question, we introduce and examine the concept of maximal ideals relative to a filter and define notions of pseudo maximal elements and pseudo irreducible elements. Then with detailed study, we see that pseudo irreducible elements are the exactly what we need to fill in the blank. As applications, we manage to answer the question partially in two ways: one uses the notions of pseudo maximal elements defined by maximal ideals relative to a filter; another way employs some topological generating subsets.

2 Maximal Ideals Relative to a Filter

An ideal on a poset $L$ means a lower set which is also directed, and a filter an ideal on the dual poset. All the ideals (resp., filters) of $L$ is denoted by $\text{Idl } L$ (resp., $\text{Filt } L$). A proper ideal $I$ is called an irreducible ideal if for any two ideals $J, K \in \text{Idl } L$, $I = J \cap K$ implies $J = I$ or $K = I$.

In [6], the notion of locally maximal ideals on posets was introduced: an ideal $M$ on a poset $L$ is a locally maximal ideal iff there is an element $x \in L$ s.t. $M$ is maximal among the ideals which do not contain $x$. We generalize this to the concept of maximal ideals relative to a filter.

**Definition 2.1.** Let $M$ be a proper ideal on a poset $L$. If there is a filter $F \in \text{Filt } L$ s.t. $M$ is maximal among the ideals which do not intersect $F$ (i. e., for an ideal $I$ on $L$, $I \cap F = \emptyset$ and $I \supseteq M$ implies $I = M$), then we say $M$ is a maximal ideal relative to the filter $F$ on poset $L$, or roughly, a maximal ideal relative to a filter.

**Remark 2.2.** Let $M$ be a proper ideal on a poset $L$ and $x \in L \setminus M$. Then it is easy to see that $M$ is a locally maximal ideal relative to $x$ iff $M$ is a maximal ideal relative to filter $\uparrow x$. Counterexamples can be found (cf: [6, Example 2] or the dull poset of the lattice given by the figure above Proposition I-3.3 in
to show that maximal ideals relative to a filter may not be locally maximal ideals.

We now show the existence of maximal ideals relative to a filter.

**Theorem 2.3.** Let $L$ be a poset, $I \in \text{Idl} L$, $F \in \text{Filt} L$ and $I \cap F = \emptyset$. Then there exists a maximal ideal $M$ relative to $F$ s.t. $M \cap F = \emptyset$ and $M \supseteq I$.

**Proof.** Define $\mathcal{A} = \{J : J$ is an ideal of $L$, $J \cap F = \emptyset$ and $J \supseteq I\}$. By the assumption, we see $I \in \mathcal{A} \neq \emptyset$ and $\mathcal{A}$ is a poset ordered by set inclusion $\subseteq$. Let $\mathcal{B}$ be a chain of $\mathcal{A}$. Let $K = \bigcup_{J \in \mathcal{B}} J$. Claim that $K \in \mathcal{A}$. It is clear that $K$ is a lower set. For $x, y \in K$, there are $J_1, J_2 \in \mathcal{B}$ s.t. $x \in J_1$ and $y \in J_2$. Since $\mathcal{B}$ is a chain of $\mathcal{A}$, $J_1 \subseteq J_2$ or $J_2 \subseteq J_1$. Suppose $J_1 \subseteq J_2$ without losing generality. Then $x, y \in J_2$ and there is $z \in J_2 \subseteq K$ s.t. $x, y \leq z$. This shows $K$ is an ideal of $L$. By the definition of $K$ and $\mathcal{A}$, we see that $I \subseteq K$ and $K \cap F = \emptyset$. Thus $K \in \mathcal{A}$. By Zorn’s Lemma, there is a maximal element $M \in \mathcal{A}$. This $M$ is indeed a maximal ideal relative to filter $F$ which is what we need. 

The following proposition gives characterization of maximal ideals relative to a filter on a sup semilattice.

**Proposition 2.4.** Let $L$ be a sup-semilattice. Let $I \in \text{Idl} L$, $F \in \text{Filt} L$ and $I \cap F = \emptyset$. Then $I$ is a maximal ideal relative to filter $F$ if and only if for all $x \in L \setminus I$, there are $y \in F$ and $a \in I$ s.t. $y \leq x \vee a$.

**Proof.** Necessity: Let $I$ be a maximal ideal relative to filter $F$. Then for all $x \in L \setminus I$, define $I_1 = \bigcup\{\downarrow(x \vee a) : a \in I\}$. It is easy to show that $I_1$ is an ideal and $I_1 \supseteq I \neq I_1$. By the maximality of $I$ and $I \cap F = \emptyset$, we have that $I_1 \cap F \neq \emptyset$. Pick $y \in I_1 \cap F$. Thus there is $a \in I$ s.t. $y \leq x \vee a$.

Sufficiency: By Theorem 2.3 and $I \cap F = \emptyset$, there is a maximal ideal $M$ relative to filter $F$ s.t. $M \cap F = \emptyset$ and $M \supseteq I$. We show that $M = I$. Suppose $M \neq I$. Then there is $x \in M$, $x \notin I$. By the assumption, there are $y \in F$ and $a \in I$ s.t. $y \leq x \vee a$. Since $I \subseteq M$ and $M$ is an ideal, $a \in M$, $x \vee a \in M$ and $y \in M$. This contradicts to $F \cap M = \emptyset$, as desired. 

A feature of maximal ideals relative to a filter is given as follows.

**Theorem 2.5.** Maximal ideals relative to a filter are all irreducible ideals.

**Proof.** Let $L$ be a poset. Let $M$ be a maximal ideal relative to a filter $F$ and $M \cap F = \emptyset$. Suppose there are ideals $I$ and $J$ s.t. $M = I \cap J$ and $I \supseteq M \neq I$, $J \supseteq M \neq J$, i.e., there are $a \in I \setminus M$ and $b \in J \setminus M$. Then by the maximality of $M$, it follows that $a \in F$ and $b \in F$. Since $F$ is a filter, there is $c \in F$ such that $c \leq a, b$. This gives that $c \in I \cap J = M$ and $M \cap F \neq \emptyset$, a contradiction! So, $M$ is an irreducible ideal.
Recall that for a semilattice \( L \), a proper ideal \( I \) of \( L \) is called a \emph{prime ideal} if for any two elements \( a, b \) of \( L \), \( a \wedge b \in I \) implies \( a \in I \) or \( b \in I \). The following two corollaries now can be immediately follows from Theorem 2.3 and 2.5:

**Corollary 2.6.** Let \( L \) be a lattice. Let \( I \in \text{Idl} \ L, F \in \text{Filt} \ L \) and \( I \cap F = \emptyset \). Then there is an irreducible ideal \( J \) of \( L \) s.t. \( J \cap F = \emptyset \) and \( J \supseteq I \).

**Corollary 2.7.** Maximal ideals relative to a filter on a distributive lattice are prime ideals.

### 3 Pseudo Maximal (Irreducible) Elements

Book [3] gave a summary on various irreducible elements and primes of continuous lattices by diagrams (pp. 405-406). But in the case of distributive continuous lattices, there is a blank in the diagram on page 406. To fill in the blank with suitable items, we introduce the notions of pseudo maximal elements and pseudo irreducible elements as follows.

**Definition 3.1.** Let \( L \) be a poset. If there is a maximal ideal \( M \) relative to a filter (resp., an irreducible ideal \( J \)) s.t. \( p = \sup M \) (resp, \( p = \sup J \)), then element \( p \) is called a pseudo maximal element (resp, a pseudo irreducible element). All the pseudo maximal/irreducible elements of \( L \) is denoted by \( \psi \text{MAX} L / \psi \text{IRR} L \).

It follows from Theorem 2.5 that pseudo maximal elements are all pseudo irreducible elements. It follows from Corollary 2.7 that pseudo maximal elements are all pseudo prime elements in the sense of Definition I-3.24 in [3] in a distributive lattice.

The following three lemmas from [3] are very useful in exploring properties and relations of \( \psi \text{MAX} L, \psi \text{IRR} L \) and \( \psi \text{PRIME} L \).

**Lemma 3.2.** (see [3, Proposition I-3.3]) In a domain, the following hold.

1. If \( x \ll y \), then there is a Scott open filter \( U \) with \( y \in U \subseteq \uparrow x \);
2. If \( y \not\leq z \), then there is a Scott open filter \( U \) containing \( y \) but not \( z \).

**Lemma 3.3.** (see [3, Lemma I-3.4]) Let \( U \) be a Scott open set in a dcpo. Then for any \( x \in L \setminus U \), there is \( m \in L \setminus U \) with \( x \leq m \) and \( m \) maximal in \( L \setminus U \).

**Lemma 3.4.** (see [3, Proposition I-3.9]) For a subset \( X \) of any poset \( L \), the following statements are equivalent:

1. \( X \) is order generating in the sense that \( x = \inf (\uparrow x \cap X) \) for all \( x \in L \);
2. whenever \( y \not\leq x \), then there is \( p \in X \) with \( x \leq p \) but \( y \not\leq p \).
Theorem 3.5. Let $L$ be a domain. Let $x, y ∈ L$, $y ⊈ x$. Then there is a pseudo maximal element $p$ with $x ≤ p$ and $y ⊈ p$.

Proof. By Lemma 3.2, there is a Scott open filter $U$ of $L$ s.t. $y ∈ U$ and $x ∉ U$. By Lemma 3.3, there is $p ∈ L\setminus U$ s.t. $x ≤ p$ and $p$ is maximal in $L\setminus U$. Since $\downarrow p$ is an ideal and $U$ is a filter and $\downarrow p ∩ U = \emptyset$, there is a maximal idea $M$ relative to filter $U$ s.t. $M ∩ U = \emptyset$ and $M ⊇ \downarrow p$ by Theorem 2.3. Noticing that $U$ is Scott open, we have that if $\sup M ∈ U$, then $M ∩ U ≠ \emptyset$, a contradiction! So, $\sup M ∉ U$ and $\sup M ∈ L/U$. By that $p$ is maximal in $L\setminus U$ and $p ≤ \sup M$, we have that $y ∉ p$ and $p = \sup \downarrow p = \sup M$ is a pseudo maximal element.

The following corollary is immediately from Lemma 3.4 and Theorem 3.5.

Corollary 3.6. (1) For a domain $L$, the set $\psi \MAX L\setminus\{1\}$ is order generating.
   (2) For a continuous lattice $L$, the set $\psi \IRR L\setminus\{1\}$ is order generating.

Recall that a non-empty lattice $A$ of a semilattice generates a filter

$$F = \{\uparrow (x_1 ∧ x_2 ∧ \cdots ∧ x_n) : x_i ∈ A, i ∈ \{1, 2, \cdots, n\}, n = 1, 2, \cdots\}$$

which is called the filter generated by $A$.

Theorem 3.7. Let $L$ be a continuous semilattice and $1 ≠ p ∈ L$. If the ideal $\downarrow p$ and the filter $F$ generated by $L\setminus p$ are disjoint, then $p$ is pseudo maximal.

Proof. Let $F$ be the filter generated by $L\setminus p$, then $L\setminus p ⊆ F$ and $F ∩ \downarrow p = \emptyset$. By Theorem 2.3, there is a maximal ideal $M$ relative to filter $F$ s.t. $M ∩ F = \emptyset$ and $M ⊇ \downarrow p$. So, $M ⊆ L\setminus F ⊆ \downarrow p$. Since $L$ is continuous, $p = \sup \downarrow p ≤ \sup M ≤ \sup \downarrow p = p$ and $p = \sup M$ is pseudo maximal.

By Theorem 2.5, 3.7 and Proposition I-3.25 in [3], one immediately has

Corollary 3.8. (1) In continuous semilattices, $\psi \PRIME L\setminus\{1\} ⊆ \psi \MAX L\setminus\{1\}$.
   (2) In continuous lattices, $\psi \PRIME L ⊆ \psi \MAX L ⊆ \psi \IRR L$.
   (3) In distributive continuous lattices, $\psi \PRIME L = \psi \IRR L = \psi \MAX L$.

Recall that an element $p$ in a semilattice $L$ is said to be prime (resp. irreducible) iff for all $x, y ∈ L, x ∧ y ≤ p$ implies $x ≤ p$ or $y ≤ p$ (resp. $x ∧ y = p$ implies $x = p$ or $y = p$). The set of all primes (resp. irreducible elements) of $L$ is denoted by $\PRIME L$ (resp. $\IRR L$). We say that $≪$ is multiplicative iff for all $a, b, x, y ∈ L$ with $a ≪ x$ and $b ≪ y$ one can deduce that $a ∧ b ≪ x ∧ y$. A continuous semilattice having multiplicative $≪$ is called stable.

Theorem 3.9. In a continuous semilattice $L$, if $\PRIME L = \psi \MAX L$, then $≪$ is multiplicative.
Proof. Suppose $\ll$ is not multiplicative. Then there are elements $a, x, y \in L$ s.t. $a \ll x, a \ll y$ and $a \not\ll x \land y$. Thus $\downarrow (x \land y) \cap \uparrow a = \emptyset$. By Theorem 2.3, there is a maximal ideal $M$ relative to filter $\uparrow a$ s.t. $M \cap \downarrow (x \land y) \neq \emptyset$. Let $p = \sup M$. Then $p \in \psi \text{MAX } L = \text{PRIME } L$ and $p$ is a prime. Since $L$ is continuous, $x \land y = \sup \downarrow (x \land y) \leq \sup M = p$. This implies that $x \leq p$ or $y \leq p$. Say $x \leq \sup M = p$ without losing generality. It follows from $a \ll x$ that $a \in M$, a contradiction! So, $\ll$ is multiplicative. \hfill \Box

Recall that (see [3, Definition V-3.1]) in a continuous lattice $L$ and $p \in L$, an element $p$ is called a weakly irreducible element, if for all finite subset $X_1, X_2, \cdots, X_n$ of $L$, there is $k \in \{1, 2, \cdots, n\}$ s.t. $p \in X_k^-$ (the Lawson closure of $X_k$) whenever $p \in \text{int} \lambda(X_1 X_2 \cdots X_n)$, where $X_1 X_2 \cdots X_n = \{x_1 \land x_2 \land \cdots \land x_n : x_i \in X_i, i = 1, 2, \cdots, n\}$ denotes the pointwise infimum of $X_1, X_2, \cdots, X_n$. Call $p$ a weak prime, if there is $k \in \{1, 2, \cdots, n\}$ s.t. $p \in (\uparrow X_k) -$ whenever $p \in \text{int} \lambda(1 X_2 \cdots X_n)$. The set of all weakly irreducible elements /primes of $L$ is denoted by $\text{WIRR } L = \text{WPRIME } L$.

By Prop. I-3.25, 3.28, V-3.3, 3.4, 3.6 in [3], Th. 2.5 and 3.7, we have

**Proposition 3.10.** (1) In cont. lattices, $\psi \text{PRIME } L \subseteq \text{WPRIME } L \subseteq \psi \text{MAX } L$.

(2) In stable continuous lattices, $\text{PRIME } L = \psi \text{PRIME } L = \text{WPRIME } L$.

(3) In distributive continuous lattices, $\psi \text{PRIME } L = \text{WPRIME } L = \psi \text{MAX } L = \psi \text{IRR } L$ and $\text{IRR } L \subseteq \text{WIRR } L = \text{WPRIME } L$.

A lattice is called a weakly distributive lattice if for all non-empty finite subsets $\{a_1, a_2, \cdots, a_n, x\}$, $(a_1 \lor x) \land (a_2 \lor x) \land \cdots \land (a_n \lor x) = x$ whenever $a_1 \land a_2 \land \cdots \land a_n \ll x$. It is easy to show that distributive lattices are weakly distributive lattices. But the counterexample of Exercise V-3.10 in [3] shows that the converse is not true.

**Lemma 3.11.** (see [3, Exercise I-3.38 and V-3.11]) Let $L$ be a weakly distributive continuous lattice. Then $\text{WIRR } L = \text{WPRIME } L$.

To sum up above, we have the following diagrams. The content with underline in Diagram (3) is what for filling the blank we mentioned in the introduction.

**Diagram (1)** In a continuous lattice $L$:

$Irr L \subseteq \text{IRR } L \subseteq (\text{IRR } L)^- = \text{WIRR } L$.

$\{1\} \subseteq \text{PRIME } L \subseteq \psi \text{PRIME } L \subseteq \text{WPRIME } L \subseteq \psi \text{MAX } L \subseteq \psi \text{IRR } L$. 
Diagram (2) In a weakly distributive continuous lattice $L$:

\[
\begin{align*}
\text{Irr } L & \subseteq \text{IRR } L \subseteq (\text{IRR } L)^- = \text{WIRR } L. \\
\{1\} & \subseteq \text{PRIME } L \subseteq \psi \text{PRIME } L \subseteq \text{WPRIME } L \subseteq \psi \text{MAX } L \subseteq \psi \text{IRR } L.
\end{align*}
\]

Diagram (3) In a distributive continuous lattice $L$:

\[
\begin{align*}
\text{Irr } L & \subseteq \text{IRR } L \subseteq (\text{IRR } L)^- = \psi \text{IRR } L = \text{WIRR } L. \\
\{1\} & \subseteq \text{PRIME } L \subseteq (\text{PRIME } L)^- = \psi \text{PRIME } L = \text{WPRIME } L = \psi \text{MAX } L.
\end{align*}
\]

By Diagram (2), we have immediately:

**Proposition 3.12.** In a weakly distributive continuous lattice, $\text{WPRIME } L = \text{IRR } L$ iff $\text{IRR } L$ is Lawson closed; if $\text{IRR } L = \psi \text{IRR } L$ or $\text{PRIME } L = \text{WPRIME } L$, then $\text{IRR } L$ is Lawson closed.

By Proposition 3.10 (2) and Diagram (2), we have:

**Proposition 3.13.** Let $L$ be a weakly distributive continuous lattice. If $L$ is stable continuous, then $\text{IRR } L = \text{PRIME } L$ is Lawson closed.

In [3], immediately after Proposition V-3.7, there is a non-explicitly posed question that “In the non-distributive case we do not know similar characterizations of those continuous lattices $L$ in which $\text{IRR } L$ or $\text{PRIME } L$ is closed.” Proposition 3.12 and 3.13 answer the question partially.

## 4 Lawson Closed Generating Sets

In this section, we give a partial answer in another way to the non-explicitly posed question mentioned above.

Recall that a subset $X$ of a topological semilattice $L$ is said to be topologically generating if the smallest closed subsemilattice containing $X$ and 1 is $L$ itself.

**Lemma 4.1.** (see [2, Theorem VI-3.4]) If $L$ is a continuous lattice, then $\Lambda L$ is a compact Hausdorff topological semilattice with small compact subsemilattices.

**Proposition 4.2.** Let $L$ be a continuous lattice. For any order generating subset $K$ of $L$ which is closed with respect to the Lawson topology, we have $\text{WIRR } L \subseteq K \cup \{1\}$. 
Proof. Let $p \in L \setminus K$ and $p \neq 1$. Then by Hausdorffness, every $x \in K$ has a neighborhood $U(x)$ not containing $p$. By Lemma 4.1, we may suppose that all the $U(x)$ are closed subsemilattices of $L$. As the interiors of all the $U(x)$ cover the compact space $K$, we find finite $x_i \in K (i = 1, 2, \cdots, n)$ such that $K \subseteq \bigcup_{i=1}^{n} U(x_i)$. Let $X_1 = U(x_1) \cup \{1\} (i = 1, 2, \cdots, n)$. Then $X_1, X_2, \cdots, X_n$ are compact subsemilattices which do not contain $p$ with $K \subseteq X_1 \cup \cdots \cup X_n$. Define $\bigwedge^{(n)} : X_1 \times X_2 \times \cdots \times X_n \longrightarrow L$ such that $(x_1, x_2, \cdots, x_n) \longmapsto x_1 \wedge x_2 \wedge \cdots \wedge x_n$. Then $\bigwedge^{(n)}$ is continuous, and $\bigwedge^{(n)}(X_1 \times X_2 \times \cdots \times X_n) = X_1 \wedge \cdots \wedge X_n$, the pointwise infimum of $X_1, X_2, \cdots, X_n$. Since $X_1, X_2, \cdots, X_n$ are compact, $X_1 \wedge \cdots \wedge X_n$ is compact and closed in $\Lambda L$. Note that $X_1, X_2, \cdots, X_n$ are subsemilattices of $L$. It is easy to show that $X_1 \wedge \cdots \wedge X_n$ is a subsemilattices of $L$. Then the pointwise infimum $X_1 \wedge \cdots \wedge X_n$ is a closed subsemilattice of $L$ containing $K$. As $K$ is order generating in $L$, for each $x \in L$, there is $A_i \subseteq X_i \cap K (i = 1, 2, \cdots, n)$ s.t. $x = \bigwedge_{i=1}^{n} (\inf A_i)$. Since $X_1, X_2, \cdots, X_n$ are compact, $\inf A_i \in X_i (i = 1, 2, \cdots, n)$ and $x \in X_1 \wedge \cdots \wedge X_n$. We conclude that $L = X_1 \wedge \cdots \wedge X_n$. In particular, $p \in \text{int}_L(X_1 \wedge \cdots \wedge X_n)$. As $p \notin X_k$ for each $k$, we conclude that $p \notin \text{WIRR} L$. Then $\text{WIRR} L \subseteq K \cup \{1\}$. \qed

Lemma 4.3. (see [3, Theorem V-2.1]) Among the order generating subsets of a continuous lattice $L$ which are closed with respect to the Lawson topology there is a unique smallest one: the closure $(\text{IRR} L \setminus \{1\})^-$ of the set of irreducible elements $< 1$ in $L$.

Theorem 4.4. In a continuous lattice $L$, $\text{WIRR} L = (\text{IRR} L \setminus \{1\})^- \cup \{1\}$.

Proof. By Proposition 4.2 and Lemma 4.3, $\text{WIRR} L \subseteq (\text{IRR} L \setminus \{1\})^- \cup \{1\}$. By Proposition V-3.3 in [3], $(\text{IRR} L \setminus \{1\})^- \cup \{1\} \subseteq \text{WIRR} L$, $\text{WIRR} L = (\text{IRR} L \setminus \{1\})^- \cup \{1\}$. \qed

Corollary 4.5. In a continuous lattice $L$, $(\text{WIRR} L \setminus \{1\})^- = (\text{IRR} L \setminus \{1\})^-$. Proof. By Theorem 4.4, we have $\text{WIRR} L = (\text{IRR} L \setminus \{1\})^- \cup \{1\}$ and $\text{WIRR} L \setminus \{1\} \subseteq (\text{IRR} L \setminus \{1\})^-$. Thus, $(\text{WIRR} L \setminus \{1\})^- \subseteq (\text{IRR} L \setminus \{1\})^-$. Since $\text{WIRR} L \setminus \{1\}$ is order generating, $(\text{WIRR} L \setminus \{1\})^-$ is Lawson closed order generating. By Lemma 4.3, we get $(\text{WIRR} L \setminus \{1\})^- = (\text{IRR} L \setminus \{1\})^-$. \qed

Lemma 4.6. (see [3, I-3.15, V-2.4, V-3.2]) Let $L$ be a continuous lattice. Then

1. $L$ is distributive iff $\text{PRIME} L$ is order generating.
2. A subset $X$ is topologically generating if its closure $X^-$ with respect to the Lawson topology is order generating.
3. The sets $\text{WPRIME} L$ and $\text{WIRR} L$ are Lawson closed.

Proposition 4.7. Let $L$ be a continuous lattice with $\text{PRIME} L$ being topologically generating. Then $\text{WIRR} L = (\text{PRIME} L)^- = \text{WPRIME} L$. 

Proof. By Lemma 4.6(2), we have that \((\text{PRIME } L)^-\) is order generating with respect to the Lawson topology. By Proposition 4.2, \(\text{WIRR } L \subseteq (\text{PRIME } L)^- \cup \{1\} = (\text{PRIME } L)^-\). By Diagram (1) in Section 3, \(\text{PRIME } L \subseteq \text{WPRIME } L \subseteq \text{WIRR } L\). It follows from Lemma 4.6(3) that \(\text{WPRIME } L \text{ and } \text{WIRR } L \) are Lawson closed and \((\text{PRIME } L)^- \subseteq \text{WPRIME } L \subseteq \text{WIRR } L\). So, \(\text{WIRR } L = (\text{PRIME } L)^- = \text{WPRIME } L\).

\[\text{Lemma 4.8.} \text{ (see [3, Proposition V-3.7]) In a distributive continuous lattice } L, \text{ PRIME } L \text{ is closed iff } \ll \text{ is multiplicative, that is, iff } L \text{ is stably continuous.}\]

\[\text{Theorem 4.9. Let } L \text{ be a continuous lattice in which PRIME } L \text{ is topologically generating. Then PRIME } L \text{ is closed iff } \ll \text{ is multiplicative, that is, iff } L \text{ is stably continuous.}\]

Proof. Suppose that \(L\) is stably continuous. By Theorem 3.10(2), we have \(\text{PRIME } L = \text{WPRIME } L\). By Proposition 4.7, \(\text{WPRIME } L = (\text{PRIME } L)^-\). So, \(\text{PRIME } L = (\text{PRIME } L)^-\) and \(\text{PRIME } L\) is Lawson closed.

Conversely, let \(\text{PRIME } L = (\text{PRIME } L)^-\) be Lawson closed. By Proposition 4.7, \(\text{PRIME } L = \text{WIIR } L\). Using Diagram (1) in last section, we have \(\text{IRR } L = \text{PRIME } L\) and \(\text{PRIME } L\) is order generating. By Lemma 4.6(1), \(L\) is distributive. It follows from Lemma 4.8 that \(L\) is stably continuous.

By Theorem 4.9 and its proof, we have the following:

\[\text{Corollary 4.10. Let } L \text{ be a continuous lattice in which the set PRIME } L \text{ is topologically generating. If the set PRIME } L \text{ is closed (or } L \text{ is stable), then } L \text{ is distributive.}\]

\[\text{Lemma 4.11.} \text{ ([4, Proposition 3.8]) If } L \text{ is a continuous lattice with PRIME } L \text{ being topologically generating, then}\]

\[L \text{ is distributive iff } T = \{x \in L : x = \inf (\uparrow x \cap \text{PRIME } L)\} \text{ is a sublattice.}\]

\[\text{Remark 4.12.} \text{ (1) Let } L \text{ be a continuous lattice in which the set PRIME } L \text{ is topologically generating. Then by Proposition 4.7 and Lemma 3.11, } L \text{ is weakly distributive continuous.}\]

\[\text{ (2) By Lemma 4.11, if the set } T = \{x \in L : x = \inf (\uparrow x \cap \text{PRIME } L)\} \text{ is not a sublattice of } L, \text{ then } L \text{ is not distributive. Exercise V-3.10 in [3] gives an example of a nondistributive continuous lattice in which the set PRIME } L \text{ is topologically generating.}\]

\[\text{ (3) By (2), Theorem 4.9 is indeed a characterization in the non-distributive case of those continuous lattices } L \text{ in which PRIME } L \text{ is Lawson closed. Thus Theorem 4.9 answers partially the non-explicitly posed question in [3] mentioned at the end of last section.}\]
References


Received: October 16, 2007