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Poisson Approximation to the Beta-Negative Binomial Distribution

Kanint Teerapabolarn

Department of Mathematics, Faculty of Science
Burapha University, Chonburi 20131, Thailand
kanint@buu.ac.th

Abstract

The w-function and the Stein-Chen identity are used to determine two formulas of uniform and non-uniform upper bounds on Poisson approximation to the beta-negative binomial distribution.

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1 Introduction

The beta-negative binomial distribution is a negative binomial distribution whose probability of success parameter p follows a beta distribution with shape parameters α and β. In other words, if the probability of success parameter p of a negative binomial distribution (with parameters n and p) has a beta distribution with shape parameters α and β, then the resulting distribution is referred to as the beta-negative binomial distribution with parameters α, β and n, denoted by $\mathcal{BN}(\alpha,\beta,n)$. For a standard negative binomial distribution, p is usually assumed to be fixed for successive trials, but the value of p changes for each trial for the beta-negative binomial distribution. The beta-negative binomial distribution is sometimes referred to as *inverse Markov-Pólya distribution.*

Let X be the beta-negative binomial random variable. Then, for $k \in \mathbb{N}\cup\{0\}$, its probability distribution (Johnson et al. [4]) is defined by

$$p(k) = \frac{\binom{-\beta}{k}\binom{\beta+\alpha-1}{-n-k}}{\binom{\alpha-1}{-n}} = \frac{\Gamma(n+\alpha)\Gamma(k+\beta)\Gamma(n+k)\Gamma(\alpha+\beta)}{\Gamma(n+k+\alpha+\beta)\Gamma(n)\Gamma(k+1)\Gamma(\alpha)\Gamma(\beta)}, \tag{1.1}$$

where $\alpha,\ \beta$ and n are all positive real numbers, Γ gamma function, and mean and variance of X are $\dfrac{n\beta}{\alpha-1}$ and $\dfrac{n\beta(n+\alpha-1)(\alpha+\beta-1)}{(\alpha-2)(\alpha-1)^2}$, respectively.

Immediately from (1.1), we obtain the recurrence relation of probability function as follows:

$$\frac{p(k-1)}{p(k)}=\frac{k(n+k+\alpha+\beta-1)}{(k+\beta-1)(n+k-1)},\ k=1,2,..., \tag{1.2}$$

where $p(0)=\dfrac{(n+\alpha-1)!(\alpha+\beta-1)!}{(n+\alpha+\beta-1)!(\alpha-1)!}$.

Remarks. 1. In the case of $n=1$, the beta-negative binomial distribution is the beta-geometric distribution with parameters α and β.

2. If $\beta=a$ and $\alpha=c-a$, then the distribution is the so-called *generalized Waring distribution*, see Johnson et al. [4] on pp. 257.

It is well-known that the negative binomial distribution with parameters n and p can be approximated by the Poisson distribution with mean λ, denoted by $\mathcal{P}(\lambda)$, under some conditions concerning parameters $n,\ p$ and λ. As mentioned above and the beta-negative binomial distribution is obtained from a negative binomial distribution, we expect that the beta-negative binomial distribution can also be approximated by the Poisson distribution. In this paper, we use the w-function associated with the random variable X together with the Stein-Chen identity to determine two formulas of uniform and non-uniform upper bounds for approximating the beta-negative binomial distribution by the Poisson distribution.

2 Main results

We will prove our main results by using the w-function associated with the beta-negative binomial random variable X and the Stein-Chen identity. For the w-function, Majsnerowska [5] adapted the relation of w-function associated with a non-negative integer-valued random variable X (Cacoullos and Papathanasiou [2]) to be the recurrence relation of w-function in the form of

$$w(k+1)=\frac{p(k)}{p(k+1)}w(k)-\frac{\mu-(k+1)}{\sigma^2}\geq 0,\quad k=0,1,..., \tag{2.1}$$

where $w(0)=\dfrac{\mu}{\sigma^2}$ and μ and σ^2 are mean and variance of X.

Proposition 2.1. *Let $w(X)$ be the w-function associated with the beta-negative binomial random variable X and $p(k)>0$ for every $k\in\mathbb{N}\cup\{0\}$. Then we have*

$$w(k)=\frac{(n+k)(\beta+k)}{(\alpha-1)\sigma^2}\ ,\ k=0,1,..., \tag{2.2}$$

where $\sigma^2 = \dfrac{n\beta(n+\alpha-1)(\alpha+\beta-1)}{(\alpha-2)(\alpha-1)^2}$ *and* $\alpha > 2$.

Proof. Following (1.2) and (2.1), we have $w(0) = \dfrac{n\beta}{(\alpha-1)\sigma^2}$ and

$$w(k) = \frac{n\beta}{(\alpha-1)\sigma^2} + w(k-1)\frac{k(n+k+\alpha+\beta-1)}{(k+\beta-1)(n+k-1)} - \frac{k}{\sigma^2},\ k = 1, 2, ...,$$

Therefore,

$$w(1) = \frac{(n+1)(\beta+1)}{(\alpha-1)\sigma^2}, w(2) = \frac{(n+2)(\beta+2)}{(\alpha-1)\sigma^2}, w(3) = \frac{(n+3)(\beta+3)}{(\alpha-1)\sigma^2}, ...,$$

which gives (2.2). □

For the Stein-Chen identity, Chen [3] adapted and applied idea of normal case to the Poisson setting. The Stein-Chen identity or the Stein identity for Poisson distribution with a parameter λ which, given h, is defined by

$$\lambda f(x+1) - xf(x) = h(x) - \mathcal{P}_\lambda(h), \tag{2.3}$$

where $\mathcal{P}_\lambda(h) = e^{-\lambda}\sum_{k=0}^{\infty} h(k)\dfrac{\lambda^k}{k!}$ and f and h are bounded real valued functions defined on $\mathbb{N} \cup \{0\}$.

For $A \subseteq \mathbb{N} \cup \{0\}$, let h_A be defined by

$$h_A(x) = \begin{cases} 1 & \text{if } x \in A, \\ 0 & \text{if } x \notin A. \end{cases} \tag{2.4}$$

Follows from Barbour et al. [1], the solution $f_A(x)$, when $h = h_A$, of (2.3) is of the form

$$f_A(x) = \begin{cases} (x-1)!\lambda^{-x}e^{\lambda}[\mathcal{P}_\lambda(h_{A\cap C_{x-1}}) - \mathcal{P}_\lambda(h_A)\mathcal{P}_\lambda(h_{C_{x-1}})] & \text{if } x \geq 1, \\ 0 & \text{if } x = 0, \end{cases} \tag{2.5}$$

where $C_x = \{0, ..., x\}$, and, for $x_0 \in \mathbb{N}\cup\{0\}$, the solutions $f_{C_{x_0}}$, when $h = h_{Cx_0}$, of (2.3) can be expressed in the form of

$$f_{C_{x_0}}(x) = \begin{cases} (x-1)!\lambda^{-x}e^{\lambda}[\mathcal{P}_\lambda(h_{C_{x_0}})\mathcal{P}_\lambda(1-h_{C_{x-1}})] & \text{if } x_0 < x, \\ (x-1)!\lambda^{-x}e^{\lambda}[\mathcal{P}_\lambda(h_{C_{x-1}})\mathcal{P}_\lambda(1-h_{C_{x_0}})] & \text{if } x_0 \geq x, \\ 0 & \text{if } x = 0, \end{cases} \tag{2.6}$$

The following proposition is established to prove the main results.

Proposition 2.2. *Let $x \in \mathbb{N}$ and $\Delta f_A(x) = f_A(x+1) - f_A(x)$. Then, for $A \subseteq \mathbb{N} \cup \{0\}$,*

$$\sup_{x \geq 1} |\Delta f_A(x)| \leq \lambda^{-1}(1 - e^{-\lambda}) \tag{2.7}$$

and, *for $x_0 \in \mathbb{N} \cup \{0\}$,*

$$\sup_{x \geq 1} |\Delta f_{C_{x_0}}(x)| \leq \frac{\lambda^{-1}(e^{\lambda} - 1)}{x_0 + 1}. \tag{2.8}$$

Proof. (2.7) follows from Barbour et al. [1] and, by Lemma 2.1 (3) of Teerapabolarn and Neammanee [7], (2.8) is valid. □

For constructing the two theorems below, we first mention the relation of w-function associated with a non-negative integer-valued random variable X, which was stated by Cacoullos and Papathanasiou [2]. *If a function g satisfies $E|w(X)\Delta g(X)| < \infty$ and $E|(X-\mu)g(X)| < \infty$, then*

$$Cov(X, g(X)) = \sigma^2 E[w(X)\Delta g(X)], \tag{2.9}$$

where $\Delta g(X) = g(X+1) - g(X)$. Note that, follows from (2.9), $E[w(X)] = 1$.

Theorem 2.1. *Let X be defined as above, $\lambda = \dfrac{n\beta}{\alpha - 1}$ and $\alpha > 2$. Then, for $A \subseteq \mathbb{N} \cup \{0\}$,*

$$d_{TV}(\mathcal{BN}(\alpha, \beta, n), \mathcal{P}(\lambda)) \leq (1 - e^{-\lambda})\frac{\lambda + n + \beta + 1}{\alpha - 2}, \tag{2.10}$$

where $d_{TV}(\mathcal{BN}(\alpha, \beta, n), \mathcal{P}(\lambda)) = \sup_A |\mathcal{BN}(\alpha, \beta, n)(A) - \mathcal{P}(\lambda)(A)|$.

Proof. From (2.3), when $h = h_A$ and $f = f_A$, we have

$$\begin{aligned}
d_{TV}(\mathcal{BN}(\alpha, \beta, n), \mathcal{P}(\lambda)) &= |\lambda E[f(X+1)] - E[Xf(X)]| \\
&= |\lambda E[f(X+1)] - Cov(X, f(X)) - \mu E[f(X)]| \\
&= |\lambda E[\Delta f(X)] - Cov(X, f(X))| \\
&= |\lambda E[\Delta f(X)] - E[\sigma^2 w(X)\Delta f(X)]| \ \ \text{(by (2.9))} \\
&\leq E|[\lambda - \sigma^2 w(X)]\Delta f(X)| \\
&\leq \sup_{x \geq 1} |\Delta f(x)| E|\lambda - \sigma^2 w(X)| \\
&\leq \lambda^{-1}(1 - e^{-\lambda}) E|\lambda - \sigma^2 w(X)| \ \text{(by(2.7))} \\
&= \lambda^{-1}(1 - e^{-\lambda}) E[\sigma^2 w(X) - \lambda] \ \text{(by(2.2))} \\
&= \lambda^{-1}(1 - e^{-\lambda})(\sigma^2 - \lambda), \qquad (2.11)
\end{aligned}$$

this implies (2.10). □

Theorem 2.2. *For* $x_0 \in \mathbb{N} \cup \{0\}$, $\alpha > 2$ *and* $\lambda = \dfrac{n\beta}{\alpha - 1}$,

$$\left| \sum_{k=0}^{x_0} p(k) - \sum_{k=0}^{x_0} \frac{e^{-\lambda}\lambda^k}{k!} \right| \leq \frac{(e^{\lambda} - 1)(\lambda + n + \beta + 1)}{(\alpha - 2)(x_0 + 1)}. \tag{2.12}$$

Proof. From (2.3), when $h = h_{C_{x_0}}$ and $f = f_{C_{x_0}}$ and using the same argument detailed as in the proof of Theorem 2.1 together with (2.8), the theorem is also obtained. □

Remarks. 1. The upper bounds in (2.10) and (2.12) are small when α is large, or β and n are small, i.e. each result of the theorems yields a good Poisson approximation whenever α is large and/or β and n are small.

2. In the case of $x_0 = 0$, by applying Lemma 2.1 (3) of Teerapabolarn and Neammanee [6], we get the zero-point probability approximation as the following

$$\left| p(0) - e^{-\lambda} \right| \leq \frac{\lambda^{-1}(\lambda + e^{-\lambda} - 1)(\lambda + n + \beta + 1)}{(\alpha - 2)}. \tag{2.13}$$

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