Poisson Approximation to the Beta-Negative Binomial Distribution

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Abstract

The \( w \)-function and the Stein-Chen identity are used to determine two formulas of uniform and non-uniform upper bounds on Poisson approximation to the beta-negative binomial distribution.

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1 Introduction

The beta-negative binomial distribution is a negative binomial distribution whose probability of success parameter \( p \) follows a beta distribution with shape parameters \( \alpha \) and \( \beta \). In other words, if the probability of success parameter \( p \) of a negative binomial distribution (with parameters \( n \) and \( p \)) has a beta distribution with shape parameters \( \alpha \) and \( \beta \), then the resulting distribution is referred to as the beta-negative binomial distribution with parameters \( \alpha, \beta \) and \( n \), denoted by \( \mathcal{BN}(\alpha, \beta, n) \). For a standard negative binomial distribution, \( p \) is usually assumed to be fixed for successive trials, but the value of \( p \) changes for each trial for the beta-negative binomial distribution. The beta-negative binomial distribution is sometimes referred to as inverse Markov-Pólya distribution.

Let \( X \) be the beta-negative binomial random variable. Then, for \( k \in \mathbb{N} \cup \{0\} \), its probability distribution (Johnson et al. [4]) is defined by

\[
p(k) = \binom{-\beta}{k} \frac{(\beta + \alpha - 1)}{(-n-k)} = \frac{\Gamma(n+\alpha)\Gamma(k+\beta)\Gamma(n+k)\Gamma(\alpha+\beta)}{\Gamma(n+k+\alpha+\beta)\Gamma(n)\Gamma(k+1)\Gamma(\alpha)\Gamma(\beta)},
\]

(1.1)
where $\alpha$, $\beta$ and $n$ are all positive real numbers, $\Gamma$ gamma function, and mean and variance of $X$ are $n\beta$ and 
\[
\frac{n\beta(n + \alpha - 1)(\alpha + \beta - 1)}{(\alpha - 2)(\alpha - 1)^2},
\]
respectively.

Immediately from (1.1), we obtain the recurrence relation of probability function as follows:
\[
\frac{p(k - 1)}{p(k)} = \frac{k(n + k + \alpha + \beta - 1)}{(k + \beta - 1)(n + k - 1)}, \quad k = 1, 2, ..., \tag{1.2}
\]
where $p(0) = \frac{(n + \alpha - 1)!(\alpha + \beta - 1)!}{(n + \alpha + \beta - 1)!(\alpha - 1)!}$.

**Remarks.**

1. In the case of $n = 1$, the beta-negative binomial distribution is the beta-geometric distribution with parameters $\alpha$ and $\beta$.

2. If $\beta = a$ and $\alpha = c - a$, then the distribution is the so-called generalized Waring distribution, see Johnson et al. [4] on pp. 257.

It is well-known that the negative binomial distribution with parameters $n$ and $p$ can be approximated by the Poisson distribution with mean $\lambda$, denoted by $P(\lambda)$, under some conditions concerning parameters $n$, $p$ and $\lambda$. As mentioned above and the beta-negative binomial distribution is obtained from a negative binomial distribution, we expect that the beta-negative binomial distribution can also be approximated by the Poisson distribution. In this paper, we use the $w$-function associated with the random variable $X$ together with the Stein-Chen identity to determine two formulas of uniform and non-uniform upper bounds for approximating the beta-negative binomial distribution by the Poisson distribution.

## 2 Main results

We will prove our main results by using the $w$-function associated with the beta-negative binomial random variable $X$ and the Stein-Chen identity. For the $w$-function, Majsnerowska [5] adapted the relation of $w$-function associated with a non-negative integer-valued random variable $X$ (Cacoullos and Papathanasiou [2]) to be the recurrence relation of $w$-function in the form of
\[
w(k + 1) = \frac{p(k)}{p(k + 1)}w(k) - \frac{\mu - (k + 1)}{\sigma^2} \geq 0, \quad k = 0, 1, ..., \tag{2.1}
\]
where $w(0) = \frac{\mu}{\sigma^2}$ and $\mu$ and $\sigma^2$ are mean and variance of $X$.

**Proposition 2.1.** Let $w(X)$ be the $w$-function associated with the beta-negative binomial random variable $X$ and $p(k) > 0$ for every $k \in \mathbb{N} \cup \{0\}$. Then we have
\[
w(k) = \frac{(n + k)(\beta + k)}{(\alpha - 1)\sigma^2}, \quad k = 0, 1, ..., \tag{2.2}
\]
where $\sigma^2 = \frac{n\beta(n + \alpha - 1)(\alpha + \beta - 1)}{(\alpha - 2)(\alpha - 1)^2}$ and $\alpha > 2$.

**Proof.** Following (1.2) and (2.1), we have $w(0) = \frac{n\beta}{(\alpha - 1)^2}$ and

$$w(k) = \frac{n\beta}{(\alpha - 1)^2} + w(k - 1)\frac{k(n + k + \alpha + \beta - 1)}{(k + \beta - 1)(n + k - 1)} - \frac{k}{\sigma^2}, \quad k = 1, 2, \ldots,$$

Therefore,

$$w(1) = \frac{(n + 1)(\beta + 1)}{(\alpha - 1)^2}, \quad w(2) = \frac{(n + 2)(\beta + 2)}{(\alpha - 1)^2}, \quad w(3) = \frac{(n + 3)(\beta + 3)}{(\alpha - 1)^2}, \ldots,$$

which gives (2.2). \qed

For the Stein-Chen identity, Chen [3] adapted and applied idea of normal case to the Poisson setting. The Stein-Chen identity or the Stein identity for Poisson distribution with a parameter $\lambda$, which, given $h$, is defined by

$$\lambda f(x + 1) - xf(x) = h(x) - \mathcal{P}_\lambda(h), \quad (2.3)$$

where $\mathcal{P}_\lambda(h) = e^{-\lambda} \sum_{k=0}^{\infty} h(k) \frac{\lambda^k}{k!}$ and $f$ and $h$ are bounded real valued functions defined on $\mathbb{N} \cup \{0\}$.

For $A \subseteq \mathbb{N} \cup \{0\}$, let $h_A$ be defined by

$$h_A(x) = \begin{cases} 1 & \text{if } x \in A, \\ 0 & \text{if } x \notin A. \end{cases} \quad (2.4)$$

Follows from Barbour et al. [1], the solution $f_A(x)$, when $h = h_A$, of (2.3) is of the form

$$f_A(x) = \begin{cases} (x - 1)! \lambda^{-x} e^\lambda \mathcal{P}_\lambda(h_{A \cap C_{x-1}}) - \mathcal{P}_\lambda(h_A) \mathcal{P}_\lambda(h_{C_{x-1}}) & \text{if } x \geq 1, \\ 0 & \text{if } x = 0. \end{cases} \quad (2.5)$$

where $C_x = \{0, \ldots, x\}$, and, for $x_0 \in \mathbb{N} \cup \{0\}$, the solutions $f_{C_{x_0}}$, when $h = h_{C_{x_0}}$, of (2.3) can be expressed in the form of

$$f_{C_{x_0}}(x) = \begin{cases} (x - 1)! \lambda^{-x} e^\lambda \mathcal{P}_\lambda(h_{C_{x_0}}) \mathcal{P}_\lambda(1 - h_{C_{x-1}}) & \text{if } x_0 < x, \\ (x - 1)! \lambda^{-x} e^\lambda \mathcal{P}_\lambda(h_{C_{x-1}}) \mathcal{P}_\lambda(1 - h_{C_{x_0}}) & \text{if } x_0 \geq x, \\ 0 & \text{if } x = 0. \end{cases} \quad (2.6)$$

The following proposition is established to prove the main results.
Proposition 2.2. Let \( x \in \mathbb{N} \) and \( \Delta f_A(x) = f_A(x + 1) - f_A(x) \). Then, for \( A \subseteq \mathbb{N} \cup \{0\} \),
\[
\sup_{x \geq 1} |\Delta f_A(x)| \leq \lambda^{-1}(1 - e^{-\lambda}) \quad (2.7)
\]
and, for \( x_0 \in \mathbb{N} \cup \{0\} \),
\[
\sup_{x \geq 1} |\Delta f_{c_{x_0}}(x)| \leq \frac{\lambda^{-1}(e^\lambda - 1)}{x_0 + 1}. \quad (2.8)
\]

Proof. (2.7) follows from Barbour et al. [1] and, by Lemma 2.1 (3) of Teerapabolarn and Neammanee [7], (2.8) is valid. \( \Box \)

For constructing the two theorems below, we first mention the relation of \( w \)-function associated with a non-negative integer-valued random variable \( X \), which was stated by Cacoullos and Papathanasiou [2]. If a function \( g \) satisfies
\[
E|\text{w}(X)\Delta g(X)| < \infty \quad \text{and} \quad E|(X - \mu)g(X)| < \infty,
\]
then
\[
\text{Cov}(X,g(X)) = \sigma^2 E[\text{w}(X)\Delta g(X)], \quad (2.9)
\]
where \( \Delta g(X) = g(X + 1) - g(X) \). Note that, follows from (2.9), \( E[\text{w}(X)] = 1 \).

Theorem 2.1. Let \( X \) be defined as above, \( \lambda = \frac{n\beta}{\alpha - 1} \) and \( \alpha > 2 \). Then, for \( A \subseteq \mathbb{N} \cup \{0\} \),
\[
d_{TV}(\mathcal{B}N(\alpha, \beta, n), \mathcal{P}(\lambda)) \leq (1 - e^{-\lambda})\frac{\lambda + n + \beta + 1}{\alpha - 2}, \quad (2.10)
\]
where \( d_{TV}(\mathcal{B}N(\alpha, \beta, n), \mathcal{P}(\lambda)) = \sup_A |\mathcal{B}N(\alpha, \beta, n)(A) - \mathcal{P}(\lambda)(A)| \).

Proof. From (2.3), when \( h = h_A \) and \( f = f_A \), we have
\[
d_{TV}(\mathcal{B}N(\alpha, \beta, n), \mathcal{P}(\lambda)) = |\lambda E[f(X + 1)] - E[Xf(X)]| \\
= |\lambda E[f(X + 1)] - \text{Cov}(X, f(X)) - \mu E[f(X)]| \\
= |\lambda E[\Delta f(X)] - \text{Cov}(X, f(X))| \\
= |\lambda E[\Delta f(X)] - E[\sigma^2 w(X)\Delta f(X)]| \quad \text{(by (2.9))} \\
\leq E[|\lambda - \sigma^2 w(X)|\Delta f(X)] \\
\leq \sup_{x \geq 1} |\Delta f(x)| E|\lambda - \sigma^2 w(X)| \\
\leq \lambda^{-1}(1 - e^{-\lambda})E|\lambda - \sigma^2 w(X)| \quad \text{(by (2.7))} \\
= \lambda^{-1}(1 - e^{-\lambda})E[\sigma^2 w(X) - \lambda] \quad \text{(by (2.2))} \\
= \lambda^{-1}(1 - e^{-\lambda})(\sigma^2 - \lambda), \quad (2.11)
\]
this implies (2.10). \( \Box \)
Theorem 2.2. For \( x_0 \in \mathbb{N} \cup \{0\} \), \( \alpha > 2 \) and \( \lambda = \frac{n \beta}{\alpha - 1} \),

\[
\left| \sum_{k=0}^{x_0} p(k) - \sum_{k=0}^{x_0} \frac{e^{-\lambda} \lambda^k}{k!} \right| \leq \frac{(e^\lambda - 1)(\lambda + n + \beta + 1)}{(\alpha - 2)(x_0 + 1)}. \tag{2.12}
\]

Proof. From (2.3), when \( h = h_{C_{x_0}} \) and \( f = f_{C_{x_0}} \) and using the same argument detailed as in the proof of Theorem 2.1 together with (2.8), the theorem is also obtained. \( \Box \)

Remarks. 1. The upper bounds in (2.10) and (2.12) are small when \( \alpha \) is large, or \( \beta \) and \( n \) are small, i.e. each result of the theorems yields a good Poisson approximation whenever \( \alpha \) is large and/or \( \beta \) and \( n \) are small.

2. In the case of \( x_0 = 0 \), by applying Lemma 2.1 (3) of Teerapabolarn and Neammanee [6], we get the zero-point probability approximation as the following

\[
\left| p(0) - e^{-\lambda} \right| \leq \frac{\lambda^{-1}(\lambda + e^{-\lambda} - 1)(\lambda + n + \beta + 1)}{(\alpha - 2)}. \tag{2.13}
\]

References


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