

# Stochastic Runge-Kutta Method with Weak and Strong Convergency

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## Abstract

Many deterministic systems are described by Ordinary differential equations and can often be improved by including stochastic effects, but numerical methods for solving stochastic differential equations (SDEs) are required, and work in this area is far less advanced than for deterministic differential equations. In this paper, we consider the weak and strong convergence of stochastic differential equations and obtain two new 3-stage explicit Runge-Kutta with order 2 in weak sense, and a two-stage stochastic Runge-Kutta (SRK) methods for computing strong solutions of Itô SDEs.

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## 1 Introduction

Many physical systems are modeled by SDEs, where random effects are being modeled by a Wiener process (see, for example, [6], [4], [3]) that is nowhere differentiable [2]. Because such differential equations cannot usually be solved analytically, so numerical methods are required and should be designed to perform with a certain order of accuracy.

Consider the following one-step approximation for  $d$ -dimensional equation

$$\bar{X}_{t,x}(t+h) = \bar{X}_{t,x}(t) + A(t, x, h, \xi), \quad (1)$$

where  $A$  is some  $\mathbb{R}^d$ -valued function and  $\xi$  is a random vector.

Suppose  $t_0 < t_1 < \dots < t_N = T$  is an equidistant partition of  $[t_0, T]$  with step

size  $\Delta = (T - t_0)/N$ . The discrete approximation of one-step approximation (1) is

$$\begin{aligned}\bar{X}_0 &= X_0 \\ \bar{X}_{n+1} &= \bar{X}_n + A(t_n, \bar{X}_n, \Delta, \xi_n).\end{aligned}\quad n = 0, \dots, N - 1 \quad (2)$$

Discrete approximation  $\bar{X} = \{\bar{X}_0, \bar{X}_1, \dots, \bar{X}_N\}$  converge weakly to  $X$  with order  $\beta$  if for each  $g \in \varphi_p^{2\beta+2}$ , there exist  $k_g \geq 0$  such that

$$|E[g(\bar{X}_N) - g(X_T)]| \leq k_g \Delta^\beta, \quad (3)$$

where  $\beta$  is the order of the scheme[4].

For strong solutions of SDE, the strong order of convergence of a numerical method is defined as follows:

Let  $y_N$  be the numerical approximation to  $y(t_N)$  after  $N$  steps with constant stepsize  $h = \frac{t_N - t_0}{N}$ . Then  $y_N$  is said to convergence strongly to  $y(t_N)$  with order  $p$  if there exist constants  $K_p, \delta > 0$  such that

$$\mathbb{E}[|y_N - y(t_N)|] \leq K_p h^p, \quad (4)$$

for all  $h < \delta$ .

An outline of this paper is as follows: In the next section, SRK methods for SDEs are introduced. In section three, one new classes of explicit SRK methods with strong order 1 are constructed. In continuation two new 3-stage explicit Runge-Kutta with order 2 in weak sense are obtained. Finally some numerical results will be presented in the last section.

## 2 Stochastic Runge-Kutta methods for SDEs

Consider the autonomous one-Wiener process Itô SDE given by

$$dy(t) = g_0(y(t)) dt + g_1(y(t)) dW(t), \quad y(t_0) = y_0, \quad t \in [t_0, T], \quad y(t) \in \mathbb{R}. \quad (5)$$

By successively application of the Itô's formula, the Itô-Taylor series ([8],[9]) expansion of (5) is given by

$$\begin{aligned}y(t) &= y_0 + g_0(y_0)I_0 + g_1(y_0)I_1 + g'_0(y_0)(g_0(y_0))I_{00} + g'_0(y_0)(g_1(y_0))I_{10} + g'_1(y_0)(g_0(y_0))I_{01} \\ &+ g'_1(y_0)(g_1(y_0))I_{11} + \frac{1}{2}g''_0(y_0)(g_1(y_0), g_1(y_0))I_{00} + \frac{1}{2}g''_1(y_0)(g_1(y_0), g_1(y_0))I_{01} \\ &+ g''_0(y_0)(g_0(y_0), g_0(y_0))I_{000} + g'_0(y_0)(g'_0(y_0)(g_0(y_0)))I_{000} + g''_0(y_0)(g_0(y_0), g_1(y_0))I_{100} \\ &+ g'_0(y_0)(g'_0(y_0)(g_1(y_0)))I_{100} + g''_0(y_0)(g_1(y_0), g_0(y_0))I_{010} + g'_0(y_0)(g'_1(y_0)(g_0(y_0)))I_{010} \\ &+ g''_0(y_0)(g_1(y_0), g_1(y_0))I_{110} + g'_0(y_0)(g'_1(y_0)(g_1(y_0)))I_{110} + g''_1(y_0)(g_0(y_0), g_0(y_0))I_{001} \\ &+ g'_1(y_0)(g'_0(y_0)(g_0(y_0)))I_{001} + g''_1(y_0)(g_0(y_0), g_1(y_0))I_{101} + g'_1(y_0)(g'_0(y_0)(g_1(y_0)))I_{101} \\ &+ g''_1(y_0)(g_1(y_0), g_0(y_0))I_{011} + g'_1(y_0)(g'_1(y_0)(g_0(y_0)))I_{011} + g''_1(y_0)(g_1(y_0), g_1(y_0))I_{111} \\ &+ g'_1(y_0)(g'_1(y_0)(g_1(y_0)))I_{111} + \text{higher order terms},\end{aligned}\quad (6)$$

where  $I_\alpha$  is the multiple Itô stochastic integral defined by:

$$I_{j_1 j_2 \dots j_\ell} = \int_{t_0}^t \int_{t_0}^{s_\ell} \dots \int_{t_0}^{s_2} dW_{j_1}(s_1) \dots dW_{j_\ell}(s_\ell),$$

where  $j_i \in \{0, 1, \dots, d\}$  for  $d$ -Wiener processes and  $dW_0(s_i) \equiv ds_i$ . By truncating different terms of (6), the different numerical methods can be constructed but the computational cost can be high due to the proliferation of the elementary derivatives. Thus it is important to be able to derive derivative-free numerical methods and this leads to the development of SRK methods. SRK methods with  $s$ -stage be given by

$$\begin{cases} Y_i &= y_n + \sum_{j=1}^s Z_{ij}^{(0)} g_0(Y_j) + \sum_{j=1}^s Z_{ij}^{(1)} g_1(Y_j), \quad i = 1, 2, \dots, s \\ y_{n+1} &= y_n + \sum_{j=1}^s z_j^{(0)} g_0(Y_j) + \sum_{j=1}^s z_j^{(1)} g_1(Y_j), \end{cases} \quad (7)$$

which can be represented in tableau form as

$$\left| \begin{array}{c|c} Z^{(0)} & Z^{(1)} \\ \hline z^{(0)T} & z^{(1)T} \end{array} \right|,$$

where  $Z^{(k)} = (Z_{ij}^{(k)})$  for  $i, j = 1, 2, \dots, s$  and  $z^{(k)T} = (z_1^{(k)}, \dots, z_s^{(k)})$  for  $k = 0, 1$ . Since (7) is generalization of the class of Runge-Kutta methods in the deterministic case, hence as in the deterministic case,  $Z^{(0)}$  and  $z^{(0)T}$  are a matrix and vector of coefficients scaled by the stepsize  $h$ , respectively, while the matrix  $Z^{(1)}$  and row vector  $z^{(1)T}$  have elements which are arbitrary random variables.

### 3 SRK methods with strong global order 1

A SRK method will have strong global order 1 if (see [1] for further details)

$$\begin{cases} z^{(0)T} e = h, & z^{(1)T} e = J_1, & z^{(1)T} Z^{(1)} e = I_{11}, \\ \mathbb{E}[z^{(0)T} Z^{(1)} e] = \mathbb{E}[z^{(1)T} Z^{(0)} e] = \mathbb{E}[z^{(1)T} (Z^{(1)} e)^2] = \mathbb{E}[z^{(1)T} Z^{(1)2} e] = 0. \end{cases} \quad (8)$$

These conditions can be satisfied with two-stage and

$$\begin{aligned} Z^{(0)} &= hA, & z^{(0)T} &= h\alpha^T, & Z^{(1)} &= \sqrt{h}B, \\ z^{(1)T} &= \left( J_1 - \frac{I_{11}}{b_{21}\sqrt{h}}, \frac{I_{11}}{b_{21}\sqrt{h}} \right), & z^{(0)T} Z^{(1)} e &= 0, \end{aligned} \quad (9)$$

From conditions

$$z^{(0)T} e = h, \quad z^{(0)T} Z^{(1)} e = 0$$

for this two-stage method, follows that

$$\alpha_1 = 1, \quad \alpha_2 = 0,$$

We will try to choose  $a_{21}$  and  $b_{21}$  by minimizing the error constants method.

$$\mathbb{E}[I_{10}]^2 = \mathbb{E}[I_{01}]^2 = \frac{1}{3}h^3, \quad \mathbb{E}[I_{111}]^2 = \frac{1}{6}h^3,$$

these error constants are given by

$$\begin{cases} \mathbb{E}[I_{10} - z^{(0)T} Z^{(1)} e]^2 = \frac{1}{3}h^3, \\ \mathbb{E}[I_{01} - z^{(1)T} Z^{(0)} e]^2 = \left(\frac{1}{3} + \frac{1}{2}\left(\frac{a_{21}}{b_{21}}\right)^2\right)h^3, \\ \mathbb{E}[I_{111} - z^{(1)T} Z^{(1)2} e]^2 = \frac{1}{6}h^3, \\ \mathbb{E}[I_{111} + \frac{1}{2}I_{01} - \frac{1}{2}z^{(1)T} (Z^{(1)} e)^2 - \frac{1}{2}z^{(1)T} Z^{(0)} e]^2 = \frac{1}{4}\left(1 + \frac{1}{2}(b_{21} + \frac{a_{21}}{b_{21}})^2\right)h^3. \end{cases} \quad (10)$$

These four equations are minimized, if  $a_{21} = 0$  and  $b_{21}$  is chosen small. Consequently this class of methods can be written

$$\begin{cases} Y_1 = y_n + \sqrt{h} b_{21} g_1(y_n), \\ y_{n+1} = y_n + h g_0(y_n) + J_1 g_1(y_n) + \frac{\sqrt{h}}{2b_{21}} \left(\left(\frac{J_1}{\sqrt{h}}\right)^2 - 1\right) (g_1(Y_1) - g_1(y_n)). \end{cases}$$

With choosing  $b_{21} = 1$ , it leads to the Itô method, that is presented by the tableau

$$\begin{array}{c|cc} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \hline h & 0 & J_1 - \frac{\sqrt{h}}{2} \left(\left(\frac{J_1}{\sqrt{h}}\right)^2 - 1\right) \quad \frac{\sqrt{h}}{2} \left(\left(\frac{J_1}{\sqrt{h}}\right)^2 - 1\right) \end{array}$$

has the principal error constants

$$\frac{1}{3}h^3, \quad \frac{1}{3}h^3, \quad \frac{1}{6}h^3, \quad \frac{3}{8}h^3.$$

## 4 Three-stage SRK method with weak order 2

General form of 3-stage explicit Runge-Kutta for approximating stochastic differential equation be stated as follows:

$$\bar{Y}_{n+1} = \bar{Y}_n + \Delta \sum_{j=1}^3 \alpha_j g_0(\eta_j) + \Delta \hat{W}_n \sum_{j=1}^3 \beta_j g_1(\eta_j) + R, \quad (11)$$

where  $\mu_1 = 0$  ,  $\eta_1 = Y_n$  and

$$\eta_j = \bar{Y}_n + \Delta \sum_{i=1}^{j-1} \lambda_{ji} g_0(\eta_i) + \Delta \hat{W}_n \sum_{i=1}^{j-1} \gamma_{ji} g_1(\eta_i) \quad , \quad j = 1, \dots, 3$$

and  $R$  is the residual term. Numerical constants  $\alpha_j, \beta_j, \mu_j, \lambda_{ij}, \gamma_{ij}$  and  $R$  should be chosen.

Generalized Butcher array of coefficient in (11) will be:

0	0	0	0	0	0	0
$\mu_2$	$\lambda_{21}$	0	0	$\gamma_{21}$	0	0
$\mu_3$	$\lambda_{31}$	$\lambda_{32}$	0	$\gamma_{31}$	$\gamma_{32}$	0
$R$	$\alpha_1$	$\alpha_2$	$\alpha_3$	$\beta_1$	$\beta_2$	$\beta_3$

where the first matrix is deterministic coefficients and the rest matrices correspond to stochastic parts depends on the Wiener Process components.

Now, we can conclude, that the following method is 2-equivalence [7]:

$$\begin{aligned} \bar{Y}_{n+1} &= \bar{Y}_n + (\alpha_1 + \alpha_2 + \alpha_3)a \Delta + (\beta_1 + \beta_2 + \beta_3)b \Delta W \\ &+ (\alpha_2\gamma_{21} + \alpha_3(\gamma_{31} + \gamma_{32}))a_{01}b \Delta \Delta W + (\alpha_2\mu_2 + \alpha_3\mu_3)a_{10} \Delta^2 \\ &+ (\alpha_2\lambda_{21} + \alpha_3(\lambda_{31} + \lambda_{32}))aa_{01} \Delta^2 + \frac{1}{2}(\alpha_2\gamma_{21}^2 + \alpha_3(\gamma_{31} + \gamma_{32})^2)a_{02}b^2 \Delta^2 \\ &+ \gamma_{21}(\alpha_3\gamma_{32} + \beta_3\lambda_{32})a_{01}bb_{01} \Delta^2 + (\beta_2\gamma_{21} + \beta_3(\gamma_{31} + \gamma_{32}))bb_{01}(\Delta W)^2 \\ &+ (\beta_2\mu_2 + \beta_3\mu_3)b_{10} \Delta \Delta W + (\beta_2\lambda_{21} + \beta_3(\lambda_{31} + \lambda_{32}))ab_{01} \Delta \Delta W \\ &+ \frac{3}{2}(\beta_2\gamma_{21}^2 + \beta_3(\gamma_{31} + \gamma_{32})^2)b^2b_{02} \Delta \Delta W \\ &+ (\beta_2\mu_2\gamma_{21} + \beta_3\mu_3(\gamma_{31} + \gamma_{32}))b(b_{11} + \frac{b^2}{2}b_{03}) \Delta^2 \\ &+ (\beta_2\lambda_{21}\gamma_{21} + \beta_3(\lambda_{31} + \lambda_{32})(\gamma_{31} + \gamma_{32}))abb_{02} \Delta^2 + 3\beta_3\gamma_{21}\gamma_{32}bb_{01}^2 \Delta \Delta W \\ &+ \beta_3\mu_2\gamma_{32}b_{01}b_{10} \Delta^2 + \beta_3\lambda_{21}\gamma_{32}ab_{01}^2 \Delta^2 + \frac{3}{2}\beta_3\gamma_{21}^2\gamma_{32}b^2b_{01}b_{02} \Delta^2 \\ &+ \frac{3}{2}\beta_3\gamma_{21}\gamma_{32}(\gamma_{31} + \gamma_{32})b^2b_{01}b_{02} \Delta^2 + R, \end{aligned}$$

where the coefficients of the method satisfy with the following system of equations, then it coincide with truncated Taylor expansion [9] with weak order

2.

$$\begin{aligned}
 \alpha_1 + \alpha_2 + \alpha_3 &= 1 & , & & \beta_1 + \beta_2 + \beta_3 &= 1 \\
 \alpha_2\mu_2 + \alpha_3\mu_3 &= \frac{1}{2} & , & & \alpha_2\gamma_{21} + \alpha_3(\gamma_{31} + \gamma_{32}) &= \frac{1}{2} \\
 \alpha_2\lambda_{21} + \alpha_3(\lambda_{31} + \lambda_{32}) &= \frac{1}{2} & , & & \alpha_2\gamma_{21}^2 + \alpha_3(\gamma_{31} + \gamma_{32})^2 &= \frac{1}{2} \\
 \beta_3\gamma_{21}\gamma_{32} &= 0 & , & & \gamma_{21}(\alpha_3\gamma_{32} + \beta_3\lambda_{32}) &= 0 \\
 \beta_3\mu_2\gamma_{32} &= 0 & , & & \beta_2\mu_2 + \beta_3\mu_3 &= \frac{1}{2} \\
 \beta_3\lambda_{21}\gamma_{32} &= 0 & , & & \beta_2\lambda_{21} + \beta_3(\lambda_{31} + \lambda_{32}) &= \frac{1}{2} \quad (12) \\
 \beta_3\gamma_{21}^2\gamma_{32} &= 0 & , & & \beta_2\gamma_{21}^2 + \beta_3(\gamma_{31} + \gamma_{32})^2 &= \frac{1}{6} \\
 \beta_3\gamma_{21}\gamma_{32}(\gamma_{31} + \gamma_{32}) &= 0 & , & & \beta_2\mu_2\gamma_{21} + \beta_3\mu_3(\gamma_{31} + \gamma_{32}) &= 0 \\
 \beta_2\lambda_{21}\gamma_{21} + \beta_3(\lambda_{31} + \lambda_{32})(\gamma_{31} + \gamma_{32}) &= 0, & & & & 
 \end{aligned}$$

where  $R = \frac{1}{2}bb_{01}((\Delta W)^2 - \Delta)$ .

The above system in Maple environment be solved and observed that the system has the following answers:

$$\begin{array}{c|ccc|ccc}
 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
 1 & 1 & 0 & 0 & 1 & 0 & 0 \\
 1/5 & 1/5 & 0 & 0 & 1/3 & 0 & 0 \\
 \hline
 & 1/2 & 1/2 & 0 & -5/2 & -1/4 & 15/4
 \end{array}$$

and in particular case  $\mu_2 = 1/2, \lambda_{32} = 1$ , Butcher's array is:

$$\begin{array}{c|ccc|ccc}
 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
 1/2 & 1/2 & 0 & 0 & 0 & 0 & 0 \\
 0 & -1 & 1 & 0 & 1 & 0 & 0 \\
 \hline
 & -1/2 & 1 & 1/2 & -1/6 & 1 & 1/6
 \end{array}$$

## 5 Numerical Experiment

In this section, numerical results from the implementation of the above methods are presented. A two-stage stochastic Runge-Kutta method with strong global error 1 and two 3-stage stochastic Runge-Kutta method with weak order 2. Here, We only consider the first 3-stage stochastic Runge-Kutta method and the numerical result of this method will be shown in Table 2.

**Example 1:** Consider

$$dy = -a^2y(1 - y^2)dt + a(1 - y^2)dW(t), \quad y(0) = 0, \quad t \in [0, 1],$$

$h$	$Error$
$\frac{1}{25}$	0.17316
$\frac{1}{50}$	0.10971
$\frac{1}{100}$	0.84066e-1
$\frac{1}{200}$	0.62554e-1
$\frac{1}{400}$	0.46519e-1

Table 1: Global errors for  $a = 1, \epsilon = 0.001$ . of Example 1

with the exact solution

$$y(t) = \tan h(aW(t) + \arctan h(y_0)).$$

This problem was solved numerically for  $a = 1$  and  $\epsilon = 0.001$  and the numerical result be presented in Table 1.

$k_g$	$St.dev$	$Error$	$\Delta$
21.7424	6.138	5.4355	$2^{-1}$
35.9489	7.7198	2.24681	$2^{-2}$
38.1766	8.74103	0.596509	$2^{-3}$
6.50332	9.38097	0.0254036	$2^{-4}$
173.413	9.92609	0.169348	$2^{-5}$

Table 2: Numerical results for approximating  $E[X_t]$

**Example 2:** Consider nonlinear stochastic differential equation

$$dX_t = \left(\frac{1}{3}X_t^{\frac{1}{3}} + 6X_t^{\frac{2}{3}}\right)dt + X_t^{\frac{2}{3}}dW_t \quad X_0 = 1$$

with exact solution  $X_t = (2t + 1 + \frac{W_t}{3})^3$ . We want to estimate  $E[X_1] = 28$ .

Numerical result with the new 3-stage Runge-Kutta method is presented. In the Example 2, we consider one-dimensional nonlinear differential equation. Our aim is to estimate  $E[X_t]$ , where  $X_t$  is the solution of differential equation. Here,  $N = 5000$  simulation paths with step size  $\Delta = 2^{-1}, \dots, 2^{-5}$  be applied for approximation of the mathematical expectation.

The mean, the standard deviation of the errors are presented in Table 2. All results have been calculated with the same conditions in Mathematica and Matlab. We also approximate the constant  $k_g$ , from (3), with the Richardson's extrapolation in Table 2.

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