

On Dimension of the Manifold of Polyhedral G-Configuration Space

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Abstract

Let G be a finite subgroup of $O(3)$. By G -polyhedron we mean a convex 3-polytope with centroid the origin O in Euclidean space E^3 and symmetry group $G=G(P)$. denote by $F(P)$ the face lattice of P . Two polyhedra P and Q are called G -equivalent, if there exists a combinatorial isomorphism $\lambda:F(P)\rightarrow F(Q)$ such that $\lambda(gx)=g(\lambda x)$, $(g, x)\in G(P)\times F(P)$. Denote by $\langle P \rangle$ the set of all polyhedra Q which are G -equivalent to P . We first give $\langle P \rangle$ its natural Hausdorff metric topology and then we prove that the configuration space $\langle P \rangle$ is a smooth manifold. We also give a formula which determines explicitly the dimension of this configuration space.

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1. Introduction

A convex 3- polytope or **polyhedron** is a convex- hull of finitely many points in Euclidean space E^3 which are not all coplanar. For any polyhedron P we denote the set of all vertices, edges and faces of P by $F_0(P), F_1(P), F_2(P)$, respectively. These sets, together with empty set ϕ and P itself, form a lattice $F(P)$ under set

inclusion. Let us label the polygonal faces of P by A_1, A_2, \dots, A_s , and the vertices by v_1, v_2, \dots, v_r . Define $M(P) = (m_{i,j})$, an $r \times s$ matrix with $m_{i,j} = 1$ if $v_i \in A_j$ and 0 otherwise. The number of non-zero elements of $M(P)$ is called multiplicity of P and denoted by $\mu(P)$ [8]. Clearly we have $\mu(P) = 2e$, where e is the number of edges of the polyhedron P . Two polyhedra P and Q are said to be combinatorially equivalent, denoted by $P \approx Q$, if their face lattices are isomorphic. We denote by $[P]$ the set of all polyhedra that are equivalent to P .

Let $\Gamma = (V, E)$ be a graph with vertex set V and edge set E . A connected graph Γ is called 3-connected if the deletion of any two vertices in V together with their corresponding edges in E leaves the graph Γ connected. Γ is planar graph if it can be embedded on the plane without edge crossing. The following classical theorem, called "fundamental theorem of convex types", is due to Steinitz [9].

Theorem 1.1. (Steinitz) The edge graphs of the convex polyhedra are exactly the simple, planar and 3-connected graphs.

In the original proof of this theorem, called "fundamental theorem of convex types", Steinitz uses a combinatorial reduction technique (Steinitz operations: stellation, inverse stellation or their duals) that allows one to generate P from a tetrahedron (3-simplex) by adding new vertices and faces iteratively. A new demonstration of the above mentioned technique is given in Ziegler [10].

The following should also be taken as a part of the Steinitz's Fundamental Theorem.

Theorem 1.2 (Steinitz) [7, 10]. For every polyhedron P , the *configuration space* $[P]$ is a smooth manifold of dimension $\dim[P] = e + 6$, where e is the number of edges of P . Furthermore, $[P]$ has isotopy property in the sense that if $Q, Q' \in [P]$ there is a continuous one-parameter family of configurations that starts with Q and ends with Q' , i.e., a path in E^{3n} connecting Q and Q' such that each point of the path represented by a polyhedron combinatorially equivalent to P .

For each polyhedron P a symmetry of P is a rigid transformation of $f: E^3 \rightarrow E^3$ that preserves P set-wise. Any such symmetry maps vertices to vertices, edges to edges and faces to faces and preserves inclusions (incidences); hence induces an automorphism on $F(P)$. The set $G(P)$ of all symmetries of P is a finite subgroup of the Euclidean group $E(3)$, acting on $F(P)$ as a subgroup of automorphisms. By a theorem of Mani [6], there is, for each combinatorial type $[P]$, one polyhedron for which the symmetry group is equal to the automorphisms of the face lattice of the polyhedron.

We can take $G(P)$ as a finite subgroup of orthogonal group $O(3)$. Two polyhedra P and Q are said to be symmetry equivalent [8] if the action of $G(P)$ on $F(P)$ is equivalent to the action of $G(Q)$ on $F(Q)$. This means that there is an isomorphism

$\lambda: F(P) \rightarrow F(Q)$ of face lattices and self isometry $f: E^3 \rightarrow E^3$ such that for all $g \in G(P)$ and all $x \in F(P)$

$$\lambda g(x) = (f^{-1} \circ g \circ f)(\lambda(x)).$$

If we further assume that P and Q have the same (rather than conjugate) subgroups then $\lambda(g(x)) = g(\lambda(x))$. In this case we write $P \cong Q$, and say that P and Q are *G-equivalents*.

By a *configuration* we mean an indexed collection of planes and points in E^3 such that the points being indexed by vertices of P , and the planes indexed by polygonal faces contained on those planes. Hence there is an obvious correspondence between a *configuration* and a polyhedron P . We can topologize $[P]$ by Hausdorff metric and call it *Configuration space* or *C-space of P*.

Now, *C-space* $[P]$ of polyhedron P may be refined by taking the symmetry group $G(P)$ into account. By the proof of the Steinitz's theorem, we know that $[P]$ is a contractible manifold of dimension, $\dim[P] = e + 6$. Let $\langle P \rangle$ denote the set of all polyhedra, *G-equivalent* to P . Then, clearly $[P]$ is a union of these symmetry types, one of which is $\langle P \rangle$ itself. We call $\langle P \rangle$ the *G-configuration space* or *GC-space of P*.

2. Transformation Groups and Slice Theorem

A Lie group is a manifold and a group such that the operations (multiplication and inversion) of the group are continuous. The actions of Lie groups on manifolds result orbit spaces. The structure of these orbits is usually quite complicated. But sometimes it can be shown that they are stratified into smooth manifolds.

The stratification mainly is done by the help of a theorem, called **Slice Theorem**, which is fundamental in studying the structure of **Transformation Groups**. Hence we give a brief resumé of the relevant facts about this theorem and referring the reader to [1,2,3] for details.

Let M be a smooth manifold and G a compact subgroup of the orthogonal group $O(3)$ acting smoothly on M via:

$$\varphi: G \times M \rightarrow M.$$

Then, $(M, G) = (M, G, \varphi)$ is called a transformation group and M is called a G -manifold. If the action is transitive (having precisely one orbit) then M is called homogeneous space.

For each $x \in M$, $G(x) = \{g(x) : g \in G\}$ is an orbit and $G_x = \{g \in G : g(x) = x\}$ is the stability subgroup of G at x . If H is closed subgroup in G then the orbit type of H is the subset of M of those points $x \in M$ such that G_x is conjugate to H . So each orbit type is a union of G -orbits. Hence it is possible to write the orbit space M/G as disjoint union of orbit types. Thus a homogeneous space is a transformation group of the form $(G/H, G)$, $x \in M$. For any compact transformation group (M, G) the orbit $G(x)$ through x is a compact homogeneous space embedded in M .

The statements in the following theorem are among the consequences of the Slice Theorem.

Theorem 2.1. [3]. Let G be a compact Lie group and M a (smooth) G -manifold. Then we have:

1. The orbit $G(x)$ of x is a G -invariant submanifold of M , $x \in M$;
2. If every orbit in M has type G/G_x , $x \in M$, then the orbital space M/G is a smooth manifold;
3. If H is a closed subgroup of G , then the union $M(H)$ of the orbits of type G/H is a G -invariant submanifold of M . Furthermore, the orbit space $M(H)/G$ again is a smooth manifold. For the proof see [3, Theorems 4.10; 4.18 and 4.19]

Now consider P with centroid O and symmetry group $G \subseteq O(3)$ and, let $F_x = \text{Fix } G_x = \{y \in E^3 : \text{for all } g \in G_x, g(y) = y\}$ be the fixed point set of the stability group G_x . Thus F_x is linear subspace of E^3 . Define an equivalence relation \sim_G in E^3 by $x \sim_G y \Leftrightarrow F_x = F_y$. Now $y \in F_x$ implies that $G_x \subseteq G_y, F_y \subseteq F_x$ and the equivalence class $[x] \subseteq F_x$. Then the equivalence classes $[x]$, $x \in E^3$ stratify E^3 by finitely many such strata (orbit types) [3]. Hence $[x]$ is a union of G -orbits. In particular $G(x)$ partitions in E^3 into finitely many orbital types of G [8]. Now, consider $\langle P \rangle$ and let $Q \in \langle P \rangle$. Each vertex of Q may be moved along a line within a plan or in any direction in E^3 in a small neighbourhood of its original position in P itself, likewise each face of polyhedron close to the corresponding face A of P , according as A intersects a one-stratum in an exterior point of A (necessarily at right angle) or intersects 2-stratum in interior of A , or neither of these. Let v be a vertex of P . Then, $\delta(v) = k, k = 1, 2, 3$ if v lies on a k -stratum of G , and $\delta(A) = k, k = 1, 2, 3$ if the face A has k degrees of freedom within

the above restrictions. Let $\bar{v} = \{g(v) : g \in G\}$, $\bar{A} = \{g(A) : g \in G\}$, $F_0(P)/G = \{\bar{v} : v \in F_0(P)\}$ and $F_2(P)/G = \{\bar{A} : A \in F_2(P)\}$. We denote by $M(P)$ the set of all pairs $(v, A) \in F_0(P) \times F_2(P)$ for which $v \in A$, and $M(P)/G = \{(g(v), g(A)) : g \in G \text{ and } (v, A) \in M(P)\}$. Denote by $\mu_*(P)$ the cardinality of $M(P)/G$. Recall that $\mu(P)$ is the number of incidences between the vertices and faces of P . Hence $\mu_*(P)$ is the number of orbits of such incidences under the action of $G(P)$ on $F(P)$. For example, if P is square right pyramid (**Figure 1**), then $\mu(P) = 2e = 16$, but $\mu_*(P) = 1 + 2 = 3$.

Clearly, if $v, u \in F_0(P)$ and $\bar{v} = \bar{u}$ then the dimension of F_v is equal to the dimension of F_u , i.e., $\delta(v) = \delta(u)$. The same holds for the faces of P . The following part of the theorem will be sketchy, adapted from Robertson [8], we refer the reader there for further details.

Theorem 2.2. Let P be polyhedron in E^3 then $\langle P \rangle$, the GC -space of P , is a smooth manifold.

Proof. Let N_p be the set of all polyhedra in $[P]$ with centroid O and say radius one [6]. Then N_p is a submanifold of co-dimension 4, by simply factoring out the effects of dilations and translations. This is called the submanifold of normal polyhedra. Hence, $(N_p, O(3))$ is a compact transformation group. For each Q in N_p , the isotropy group $O(3)_Q$ is just the symmetry group $G(Q)$ of Q [8]. But $(N_p, O(3))$ is a differentiable transformation group and according to **Theorem 2.1.** the orbit space $N_p/O(3)$ is smooth manifold. Consequently, $\langle P \rangle$ is smooth manifold as well (see [8], p.42).

As an example, let P be a polyhedron combinatorially equivalent to cube, "cuboid" [8]. Then $[P]$ is an 18- manifold and the principal orbit type corresponds to the configuration of polyhedra $Q \approx P$ with $G(Q) = 1$ is an open submanifold of $[P]$ of dimension 18. All other configuration spaces have dimensions lower than 18.

Now, according to **Theorem 1.2.** the configuration space $[P]$ of polyhedron P is a smooth manifold of dimension $e + 6$ where e is the number of edges of P .

An intuitive derivation of the dimension of the GC -space $\langle P \rangle$ can be given as follows.

Suppose that the polyhedron P has n vertices, e edges and f faces. If the vertices of P were allowed to move independently in E^3 they would have $3n$ degrees of freedom. However, it requires k vertices to specify a k -gonal face $k \geq 3$, and consequently the d losses $k - 3$ degrees of freedom. Thus the whole configuration

space $[P]$ has $d = 3n - \sum_{k \geq 3} (k - 3)$ degrees of freedom; the sum being taken over all the faces of P .

Now, using x_k to denote the number of k -gonal faces of P , and taking into account that each edge is contained in two faces, we have $\sum kx_k = 2e$ and the dimension of the G -space $\dim[P] = 3n - \sum_{k \geq 3} (k - 3) = 3n - kx_k + 3f = 3(n + f) - 2e = 3(n + f) - \mu(P)$. By the Euler formula in convex polyhedra, we have $n - e + f = 2$. Hence $\dim[P] = e + 6$.

Consider a square right pyramid (**Figure 1**).

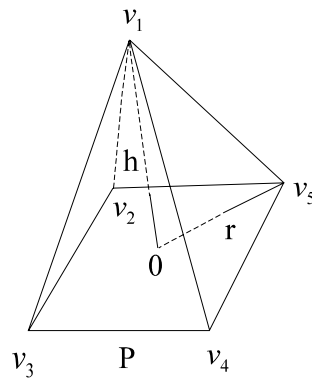


Figure 1.

Now $G(P)$, the symmetry group of P , is a dihedral group. Suppose we fix the group $G = G(P)$. Then vertex v_1 can be chosen only on the axis of G . Therefore it has one degree of freedom. Likewise v_2 must lie on reflection plane, hence has only two degrees of freedom. Having chosen v_2 , the vertices v_3, v_4 and v_5 which are on the same orbit of v_2 , have no degree of freedom at all, since they are determined by our choice of G and v_2 . Hence the vertices have a total of $1 + 2 = 3$ degrees of freedom. Similarly for the faces, each triangular face or equivalently the plane that contains it has only two degrees of freedom in the space of affine plane in E^3 , since each plane is invariant under a reflection element of G . But the square face has only one degree freedom, because it is orthogonal to the axis of rotation of G . Therefore the faces have just $2 + 1 = 3$ degrees of freedom. Of course the faces and vertices can not be chosen independently of one another. The incidence of v_1 with respect to any of the four triangular faces adjacent to it, determines the incidence of that vertex to the other three faces under the action of G . Hence v_1 has only one "independent" incidence. The vertices v_2, v_3, v_4 and v_5 are in the same G -orbit and each one is incident with

two triangles and one square faces. Take one of them say v_2 . There is a reflection which fixes v_2 and sends adjacent triangular faces each one to the other. Thus the number of independent incidences $\mu_*(P) = 1+2=3$. Each such incidence relation in the form of the condition that a vertex lies in a particular face, reduces the dimension of the GC -space by one. Hence we get

$$\dim\langle P \rangle = (1 + 2) + (1 + 2) - (1 + 2) = 3.$$

The idea of this example can be applied in general to find the dimension of the symmetry type of any polyhedron P . Indeed, we can factor out the relation $\dim[P] = 3(n + f) - \mu(P)$ by the action of $G(P)$ on $F(P)$ to find the dimension of the GC -space $\dim\langle P \rangle = \sum [\delta(\bar{v})] + \sum [\delta(\bar{A})] - \mu_*(P)$, where the first summation Σ is extended over all vertex orbits $\bar{v} \in F_0(P)/G$ and the second Σ over all face orbits $\bar{A} \in F_2(P)/G$.

Remark 1. If the polyhedron P has trivial group then $\dim[P] = \dim\langle P \rangle = 3(n + f) - 2e$ and, by Euler theorem for polyhedra, $n - e + f = 2$, we have $\dim(P) = \dim\langle P \rangle = e + 6$, where n , f and e , as usual, are the number of vertices and faces and edges of P , respectively. Since, $\dim[P] = 3(n + f) - 2e = e + 6$ is equivalent to the Euler formula, we can say that, in some extent our dimension formula, $\dim\langle P \rangle = \sum [\delta(\bar{v})] + \sum [\delta(\bar{A})] - \mu_*(P)$ is an extension of the Euler formula as well. Indeed, the equation $\dim[P] = 3(n + f) - 2e = e + 6$ is equal to $\dim P = e + 6 - 6 = e$ (modulo congruence transformations) and it is possible to prove that

$$\dim\langle P \rangle = \sum [\delta(\bar{v})] + \sum [\delta(\bar{A})] - \mu_*(P) = \varepsilon \text{ (modulo congruence),}$$

where ε is the number of edge orbits of the action of symmetry group of P on the face lattice $F(P)$. Note that the latter represents a relation between vertex orbits, face orbits and edge orbits of P , and in case if the symmetry group of polyhedron P has only the identity operation, then the formula reduces to Euler formula for P . Hence we have an extension of the Euler famous Theorem for convex polyhedra.

Remark 2. Polyhedral configuration spaces are key components in robot motion planning. For more study, we refer the interested reader to [4, 5].

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