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The Neumann Conditions for Sturm-Liouville Problems with Turning Points

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Abstract

This paper presents the asymptotic eigenvalues of the Sturm-Liouville problems with Neumann condition. In this paper we apply the concept of turning points of second order differential equation. Note that the weight function in this equation has two zeros in domain. By making use of the solutions of equation we obtain the higher-order approximations to the eigenvalues for the Neumann boundary conditions.

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1 Introduction

We consider the second order differential equation

$$\frac{d^2w}{d\xi^2} = (u^2(\xi^2 - 1) + q(\xi))w, \qquad \xi \in [a, b] \quad a < -1, b > 1 \tag{1}$$

with the Neumann boundary conditions $w'(a) = w'(c) = 0$ where $a < c < b$. Here u is a large real-value parameter. The weight function, $\psi(\xi) = \xi^2 - 1$ has two zeros in 1 and -1 so they are called turning points of this equation.

Differential equations with turning points have various applications in mathematics, elasticity, optics, geophysics and other branches of natural sciences (see[2],[6]). Turning points appear in branches of natural sciences for example physic, optics, geophysics and etc. Moreover, a wide class of differential equations with Bessel-type singularities can be reduced to differential equations having turning points. Some aspect of the turning point theory and a number of applications are described in [6,4,5,7,8,10].

2 Approximation of the solutions

In [9], one may find the asymptotic expansion of solutions of equation (1).The differential equation (1) for each nonnegative and integer value of n, has a pair of infinitely differential solutions $w_1(u,\xi)$, $w_2(u,\xi)$ are given by their approximations $w_{2n+1,1}(u,\xi), w_{2n+1,2}(u,\xi)$

$$w_{2n+1,1}(u,\xi) \cong U_1(u,\xi)\sum_{s=0}^{n} A_s(\xi)u^{-2s} + \frac{\partial U_1(u,\xi)}{\partial \xi}u^{-2}\sum_{s=0}^{n-1} B_s(\xi)u^{-2s} \tag{2}$$

$$w_{2n+1,2}(u,\xi) \cong U_2(u,\xi)\sum_{s=0}^{n} A_s(\xi)u^{-2s} + \frac{\partial U_2(u,\xi)}{\partial \xi}u^{-2}\sum_{s=0}^{n-1} B_s(\xi)u^{-2s} \tag{3}$$

where $A_0(\xi) = 1$, and $B_s(\xi), A_s(\xi)$ are defined in [8] and $U_1(u,\xi), U_2(u,\xi)$ are two independent solution of equation

$$\frac{\partial^2 U}{\partial \xi^2} = u^2(\xi^2 - 1)U \tag{4}$$

are given by $(u \longrightarrow \infty)$

$$U_1(u,\xi) = \sqrt[4]{4\pi}\Gamma(\frac{1}{2}+\frac{u}{2})^{\frac{1}{2}} u^{-\frac{1}{12}}(\frac{\eta_\xi}{\xi^2-1})^{\frac{1}{4}} A_i(u^{\frac{3}{2}}\eta_\xi)(1+O(u^{-1})), \tag{5}$$

$$U_2(u,\xi) = \sqrt[4]{4\pi}\Gamma(\frac{1}{2}+\frac{u}{2})^{\frac{1}{2}} u^{-\frac{1}{12}}(\frac{\eta_\xi}{\xi^2-1})^{\frac{1}{4}} B_i(u^{\frac{3}{2}}\eta_\xi)(1+O(u^{-1})). \tag{6}$$

The Airy functions $A_i(u^{\frac{3}{2}}\eta_\xi)$, $B_i(u^{\frac{3}{2}}\eta_\xi)$ are two independent solutions of equation

$$\frac{d^2 W}{d\xi^2} = u^2 \xi W(\xi) \tag{7}$$

where

$$\eta_\xi = -\frac{3}{2}\int_\xi^1 (1-\tau^2)^{\frac{1}{2}} d\tau^{\frac{2}{3}} \qquad 0 \le \xi \le 1, \qquad \eta_\xi = -\frac{3}{2}\int_1^\xi (1-\tau^2)^{\frac{1}{2}} d\tau^{\frac{2}{3}} \qquad 1 < \xi. \tag{8}$$

In order to compute the approximations of solutions we need different form of the Airy function. In the boundary conditions, $W'(a) = W'(c) = 0$, we have chosen c in the $(0, b)$, therefore the Airy functions have different asymptotic forms for $0 < c < 1$ and $1 < c < b$. We have the asymptotic forms of Airy functions on domain $[0, b]$. If we want to get the asymptotic forms of solutions on $[a, b]$, we must use the connection formulas of U_1, U_2

$$U_1(u, -\xi) = \cos(\frac{\pi u}{2})U_2(u, \xi) + \sin(\frac{\pi u}{2})U_1(u, \xi), \tag{9}$$

$$U_2(u, -\xi) = \cos(\frac{\pi u}{2})U_1(u, \xi) - \sin(\frac{\pi u}{2})U_2(u, \xi). \tag{10}$$

There is a relation between the function U_1, U_2 and the derivation of U_1, U_2. These relations are given by the following forms

$$U_1'(a, \xi) = \frac{\xi}{2}U_1(a, \xi) - U_1(a-1, \xi), \qquad U_2'(a, \xi) = \frac{\xi}{2}U_2(a, \xi) - U_2(a+1, \xi). \tag{11}$$

So, for $\xi > 0$ by inserting (11) in (3) and (2) we obtain the solutions W_1, W_2

$$W_1(u, \xi) \cong U_1(u, \xi)\sum_{s=0}^{n} A_s(\xi)u^{-2s} + u^{-2}(u\xi - \sqrt{2u})\sum_{s=0}^{n} B_s(\xi)u^{-2s} \tag{12}$$

$$W_2(u, \xi) \cong U_2(u, \xi)\sum_{s=0}^{n} A_s(\xi)u^{-2s} + u^{-2}(\sqrt{2u} - u\xi)\sum_{s=0}^{n} B_s(\xi)u^{-2s}. \tag{13}$$

By using the connection formulas (9),(10), we get the solutions equation (1) on $[a, b]$. In [8] Olver investigated the solutions of (1). He proved that the solutions of equation (1) are in the following forms

$$W_1(u, \xi) = U_1(u, \xi)(1 + O(u^{-1})), \qquad W_2(u, \xi) = U_2(u, \xi)(1 + O(u^{-1})). \tag{14}$$

3 Derivative of solutions

Now for $\xi > 0$, by using the derivative of $U_1(u, \xi)$ and $U_2(u, \xi)$ we will have

$$\begin{aligned}\frac{\partial W_1(u, \xi)}{\partial \xi} &= 2\sqrt{u}\pi^{\frac{1}{4}}\Gamma(\frac{1}{2} + \frac{u}{2})^{\frac{1}{2}} u^{-\frac{1}{12}}(\frac{\eta_\xi}{\xi^2 - 1})^{\frac{1}{4}} A_i(u^{\frac{3}{2}}\eta_\xi)(\frac{\xi\sqrt{2u}}{2} - e^{-\sigma(\xi)})(1 + O(u^{-1})), \qquad (15)\\ &= W_1(u, \xi)(\xi u - \sqrt{2u}e^{-\sigma(\xi)}), \qquad \sigma(\xi) = \frac{4}{3}\eta_\xi^{\frac{2}{3}}\end{aligned}$$

similarly, we can show that

$$\frac{\partial W_2(u, \xi)}{\partial \xi} = W_2(u, \xi)(\xi u - \sqrt{2u}e^{-\sigma(\xi)}). \tag{16}$$

For $\xi < 0$, the derivative of $W_1(u,\xi), W_2(u,\xi)$ respect to ξ, are in the following forms

$$\frac{\partial W_1(u,\xi)}{\partial \xi} = [\sin(\frac{\pi u}{2}) + O(u^{-1})]W_1(u,-\xi)(\xi u + \sqrt{2u}e^{-\sigma(\xi)})$$
$$+[\cos(\frac{\pi u}{2}) + O(u^{-1})]W_2(u,-\xi)(\xi u + \sqrt{2u}e^{\sigma(\xi)}). \tag{17}$$

$$\frac{\partial W_2(u,\xi)}{\partial \xi} = [\cos(\frac{\pi u}{2}) + O(u^{-1})]W_1(u,-\xi)(-\xi u - \sqrt{2u}e^{-\sigma(-\xi)})$$
$$+[-\sin(\frac{\pi u}{2}) + O(u^{-1})]W_2(u,-\xi)(-\xi u - \sqrt{2u}e^{\sigma(-\xi)}). \tag{18}$$

4 Asymptotic eigenvalues

Some aspects for asymptotic eigenvalues of the boundary value problems have been considered in [1] and [3]. In [1], Atkinson and Mingarelli had found the asymptotic representation of the eigenvalues of equations (1)

$$\lambda_n^{\pm} \sim \pm \frac{n^2\pi^2}{\int_a^b (1-\tau^2)_{\pm}^{\frac{1}{2}} d\tau^{\,2}}, \tag{19}$$

where $(1-\tau^2)_+$ denotes the positive part of $(1-\tau^2)$ and $(1-\tau^2)_-$ denotes the negative part of $(1-\tau^2)$. The eigenvalues of equation (1) with boundary conditions $W'(a) = W'(c) = 0$ are the zeros of $\Delta(u)$ where

$$\Delta(u) = \begin{vmatrix} W_1'(u,a) & W_2'(u,a) \\ W_1'(u,c) & W_2'(u,c) \end{vmatrix}. \tag{20}$$

If we put $\xi = a$ in the equation (17), $\xi = c$ in (18) and multiply them we will have

$$\frac{\partial W_1(u,a)}{\partial \xi} \times \frac{\partial W_2(u,c)}{\partial \xi} = \Big\{[\sin(\frac{\pi u}{2}) + O(u^{-1})]W_1(u,-a)(au + \sqrt{2u}e^{-\sigma(-a)})$$
$$+[\cos(\frac{\pi u}{2}) + O(u^{-1})]W_2(u,-a)(au + \sqrt{2u}e^{-\sigma(-a)})\Big\} W_2(u,c)(cu - \sqrt{2u}e^{-\sigma(c)}). \tag{21}$$

$$\frac{\partial W_2(u,a)}{\partial \xi} \times \frac{\partial W_1(u,c)}{\partial \xi} = \Big\{[\cos(\frac{\pi u}{2}) + O(u^{-1})]W_1(u,-a)(-au - \sqrt{2u}e^{-\sigma(-a)})$$
$$+[\sin(\frac{\pi u}{2}) + O(u^{-1})]W_2(u,-a)(au + \sqrt{2u}e^{-\sigma(-a)})\Big\} W_1(u,c)(cu - \sqrt{2u}e^{-\sigma(c)}). \tag{22}$$

From above calculation we get

$$\frac{\partial W_1(u,a)}{\partial \xi} \times \frac{\partial W_2(u,c)}{\partial \xi} = [\sin(\frac{\pi u}{2}) + O(u^{-1})]W_1(u,-a)W_2(u,c)T_1$$

$$+\cos(\frac{\pi u}{2})+O(u^{-1})]W_2(u,c)W_2(u,-a)T_2, \tag{23}$$

similarly we can write

$$\frac{\partial W_2(u,a)}{\partial \xi}\times\frac{\partial W_1(u,c)}{\partial \xi}=[\sin(\frac{\pi u}{2})+O(u^{-1})]W_1(u,c)W_2(u,-a)T_3$$
$$-\cos(\frac{\pi u}{2})+O(u^{-1})]W_2(u,c)W_2(u,-a)T_4 \tag{24}$$

where

$$T_1=acu^2-au\sqrt{2u}e^{\sigma(c)}+cu\sqrt{2u}e^{-\sigma(-a)}-2u\sqrt{2u}e^{\sigma(c)-\sigma(-a)},$$
$$T_2=acu^2-au\sqrt{2u}e^{\sigma(c)}+cu\sqrt{2u}e^{\sigma(-a)}-2u\sqrt{2u}e^{\sigma(c)+\sigma(-a)},$$
$$T_3=acu^2-au\sqrt{2u}e^{-\sigma(c)}+cu\sqrt{2u}e^{\sigma(-a)}-2u\sqrt{2u}e^{-\sigma(c)+\sigma(-a)},$$
$$T_4=acu^2-au\sqrt{2u}e^{-\sigma(c)}+cu\sqrt{2u}e^{-\sigma(-a)}-2u\sqrt{2u}e^{-\sigma(c)-\sigma(-a)}. \tag{25}$$

From $\Delta(u)=0$, we know that (23) is equal with (24). So we will have

$$[\sin(\frac{\pi u}{2})+O(u^{-1})]W_1(u,-a)W_2(u,c)T_1+\cos(\frac{\pi u}{2})+O(u^{-1})]W_2(u,-a)W_2(u,c)T_2$$
$$=[\sin(\frac{\pi u}{2})+O(u^{-1})]W_1(u,-a)W_2(u,c)T_3-\cos(\frac{\pi u}{2})+O(u^{-1})]W_2(u,-a)W_2(u,c)T_4,$$

Now, after calculations we find

$$\tan(\frac{\pi u}{2})=\frac{-W_2(u,-a)W_2(u,c)T_2-W_1(u,-a)W_1(u,c)T_4}{W_1(u,-a)W_2(u,c)T_1-W_2(u,-a)W_1(u,c)T_3}(1+O(u^{-1})). \tag{26}$$

From (2) and (3) we know that $W_1(u,\xi)\cong U_1(u,\xi)K_1(n,\xi)$ and $W_2(u,\xi)\cong U_2(u,\xi)K_2(n,\xi)$ where

$$K_1(n,\xi)=\sum_{s=0}^{n}A_s(\xi)u^{-2s}+u^{-2}(u\xi-\sqrt{2u})\sum_{s=0}^{n-1}B_s(\xi)u^{-2s}, \tag{27}$$

$$K_2(n,\xi)=\sum_{s=0}^{n}A_s(\xi)u^{-2s}+u^{-2}(\sqrt{2u}-u\xi)\sum_{s=0}^{n-1}B_s(\xi)u^{-2s}. \tag{28}$$

If we insert (27),(5) and (6) in (26), then we obtain

$$\tan(\frac{\pi u}{2})=$$

$$\frac{U_1(u,-a)U_1(u,c)K_1(n,-a)K_1(n,c)T_4+U_2(u,-a)U_2(u,c)K_2(n,-a)K_2(n,c)T_2}{U_1(u,c)U_2(u,-a)K_1(n,c)K_2(n,-a)T_3-U_1(u,-a)U_2(u,c)K_1(n,-a)K_2(n,c)T_1}. \tag{29}$$

If we calculate the right hand of (29), then we will have,

$$\tan(\frac{\pi u}{2})=\frac{L_1T_4+L_2T_2}{L_3T_3+L_4T_1}(1+O(u^{-2})). \tag{30}$$

where, $L_i (i = 1, 2, 3, 4)$ are in following forms, if we consider $\mu(\xi) = (-\eta_\xi)^{-\frac{3}{2}}$,then

$$L_1 = u_0^2(1+\sin\beta) + u_0^2(1+\sin\beta)(cB_0(c) - aB_0(-a)) - \frac{3}{2}u_0u_1\cos\beta(\mu(-a)+\mu(c))\frac{1}{u} + O(u^{-2}),$$
$$L_2 = u_0^2(1-\sin\beta) + u_0^2(1-\sin\beta)(aB_0(-a) - cB_0(c)) + \frac{3}{2}u_0u_1\cos\beta(\mu(-a)+\mu(c))\frac{1}{u} + O(u^{-2}),$$
$$L_3 = u_0^2\cos\beta + \{\frac{3}{2}u_0u_1(\mu(-a)+\mu(c)) + \sin\beta(\mu(c)+\mu(-a)) + (aB_0(-a) + cB_0(c))\}\frac{1}{u} + O(u^{-2}),$$
$$L_4 = -u_0^2\cos\beta + \frac{3}{2}u_0u_1(\mu(-a)+\mu(c)) - \sin\beta(\mu(-a)+\mu(c)) + (aB_0(-a) + cB_0(c))\frac{1}{u} + O(u^{-2}). \quad (31)$$

If we suppose $x = \frac{L_1T_4+L_2T_2}{L_3T_3+L_4T_1}(1 + O(u^{-2}))$, then we can write $\tan(\frac{\pi u}{2}) = x$. By expression the following theorem, we can acquire eigenvalues of the equation (1).

Theorem 4.1 *The equation (1) with Neumann condition, $W'(a) = W'(c) = 0$,when $0 < c < 1$, $a < -1$, has asymptotic eigenvalues in the following form $u_m = \frac{4m\pi+\pi}{4\int_{-1}^{1}\sqrt{(1-\varphi^2)d\varphi}} - \frac{2}{\pi}x_m$ where x_m is in form of*

$$x_m = \frac{A_1}{B_1}\left\{1 + \frac{A_2B_1 - A_1B_2}{\sqrt{2}A_1B_1(2m)^{1/2}} + \frac{A_1B_2^2 - A_2B_1B_2}{4A_1B_2^2 m} + \frac{A_1B_2^3 - A_2B_1B_2^2}{2\sqrt{2}A_1B_1^2(2m)^{3/2}}\right\} + O(u^{-2}). \quad (32)$$

Proof : We know, $x = \frac{L_1T_4+L_2T_2}{L_3T_3+L_4T_1}$, now we must calculate the quantity of x. We can calculate $L_1T_4 + L_2T_2$ and $L_3T_3 + L_4T_1$ in the following form

$$L_3T_3 + L_4T_1 = u_0^2\cos\beta(ae^{-\sigma(c)} + ce^{\sigma(-a)} + ce^{-\sigma(c)} - ce^{\sigma(-a)})u\sqrt{2u} + u(-2u_0^2\cos\beta e^{\sigma(-a)-\sigma(c)}$$
$$+2u_0^2\cos\beta e^{\sigma(c)-\sigma(-a)} + acG_3 + acG_4) + 2(-G_3e^{\sigma(-a)-\sigma(c)} - G_4e^{\sigma(c)-\sigma(-a)}). \quad (33)$$

$$L_1T_4 + L_2T_2 = 2acu_0^2 + \{(1+\sin\beta)(ce^{-\sigma(-a)} - ae^{-\sigma(c)}) + (1-\sin\beta)(ce^{\sigma(-a)} - ae^{\sigma(c)})\}u_0^2u\sqrt{2u}$$
$$+\{ac(G_1+G_2) - 2u_0^2(1+\sin\beta)e^{-\sigma(-a)-\sigma(c)} - 2u_0^2(1-\sin\beta)e^{\sigma(-a)+\sigma(c)}\}u$$
$$-2(G_1e^{-\sigma(-a)-\sigma(c)} + G_2e^{\sigma(-a)+\sigma(c)}). \quad (34)$$

We consider

$$A_1 = \{(1+\sin\beta)(ce^{-\sigma(-a)} - ae^{-\sigma(c)}) + (1-\sin\beta)(ce^{\sigma(-a)} - ae^{\sigma(c)})\}u_0^2$$
$$A_2 = \{ac(G_1+G_2) - 2u_0^2(1+\sin\beta)e^{-\sigma(-a)-\sigma(c)} - 2u_0^2(1-\sin\beta)e^{\sigma(-a)+\sigma(c)}\}$$
$$A_3 = 2acu_0^2 - 2(G_1e^{-\sigma(-a)-\sigma(c)} + G_2e^{\sigma(-a)+\sigma(c)})$$
$$B_1 = u_0^2\cos\beta(ae^{-\sigma(c)} + ce^{\sigma(-a)} + ce^{-\sigma(c)} - ce^{\sigma(-a)})$$
$$B_2 = u(-2u_0^2\cos\beta e^{\sigma(-a)-\sigma(c)} + 2u_0^2\cos\beta e^{\sigma(c)-\sigma(-a)} + acG_3 + acG_4)$$
$$B_3 = -2(G_1e^{-\sigma(-a)-\sigma(c)} + G_2e^{\sigma(-a)+\sigma(c)})$$
$$G_1 = u_0^2(1+\sin\beta)(cB_0(c) - aB_0(-a)) - \frac{3}{2}u_0u_1\cos\beta(\mu(-a)+\mu(c))$$
$$G_2 = u_0^2(1-\sin\beta)(aB_0(-a) - cB_0(c)) + \frac{3}{2}u_0u_1\cos\beta(\mu(-a)+\mu(c))$$
$$G_3 = \{\frac{3}{2}u_0u_1(\mu(-a)+\mu(c)) + \sin\beta(\mu(c)+\mu(-a)) + (aB_0(-a) + cB_0(c))\}$$

$$G_4 = \{\frac{3}{2}u_0u_1(\mu(-a)-\mu(c))-\sin\beta(\mu(-a)+\mu(c))-(aB_0(-a)+cB_0(c))\} \quad (35)$$

Now by using A_i,$B_i(I=1,2,3)$,we can write ,$x=\frac{A1}{B_1}(\frac{1+\frac{A_2}{\sqrt{2}A_1}u^{-\frac{1}{2}}-\frac{A_3}{\sqrt{2}A_1}u^{-\frac{3}{2}}}{1+\frac{B_2}{\sqrt{2}B_1}u^{-\frac{1}{2}}-\frac{B_3}{\sqrt{2}B_1}u^{-\frac{3}{2}}})$. By dividing we can show that, x is in the following form

$$x=\frac{A1}{B_1}\{1+\frac{A_2B_1-A_1B_2}{\sqrt{2}A_1B_1}u^{-\frac{1}{2}}+B_2\frac{A_1B_2-A_2B_1}{2A_1B_2^2}u^{-1}+B_2^2(\frac{A_1B_2-A_2B_1}{2\sqrt{2}A_1B_1^2})u^{-\frac{3}{2}}\}+O(u^{-2}).$$

Because the values of u are large parameters, then the values of m are large numbers, therefore the m^{th} asymptotic of eigenvalues

$$u_m=\frac{m\pi+\frac{\pi}{4}}{\int_{-1}^{1}\sqrt{(1-\varphi^2}d\varphi}-\frac{2}{\pi}x_m, \qquad u_m\approx 2m, \qquad ,\int_{-1}^{1}\sqrt{(1-\varphi^2}d\varphi=\frac{\pi}{2}$$

where the quantity of x_m is in the following form

$$x_m=\frac{A_1}{B_1}\left\{1+\frac{A_2B_1-A_1B_2}{\sqrt{2}A_1B_1(2m)^{1/2}}+\frac{A_1B_2^2-A_2B_1B_2}{4A_1B_2^2m}+\frac{A_1B_2^3-A_2B_1B_2^2}{2\sqrt{2}A_1B_1^2(2m)^{3/2}}\right\}+O(u^{-2}).$$

Theorem 4.2 *The equation* (1) *with Neumann condition,* $W'(a)=W'(c)=0$ *, when,*$1<c<b,\quad b>1,\quad a<-1$*, has asymptotic eigenvalues in the following* $u_m=\frac{4m\pi+\pi}{4\int_{-1}^{1}\sqrt{(1-\varphi^2}d\varphi}-\frac{2}{\pi}x_m$ *, where* x_m *is*

$$x_m=-\{1+\frac{2\sqrt{2}}{a}\sinh(\sigma(-a))\frac{1}{\sqrt{2m}}+(\frac{3u_1}{u_0}\eta_b^{-\frac{3}{2}}+2aB_0(-a))\frac{1}{2m}$$
$$+(\frac{6\sqrt{2}u_1\sinh(\sigma(-a))}{au_0}-\eta_b^{-\frac{3}{2}}+\sqrt{2}B_0(-a))\frac{1}{2m\sqrt{2m}}\}+O(m^{-2}).$$

Proof: We know for $c\geq 1$,

$$\tan(\frac{\pi u}{2})x=\frac{-M_4K_{21}T_2}{M_2K_{11}T_1}(1+O(u^{-1})). \quad (36)$$

In order to compute the eigenvalues we use the following notations

$$\begin{array}{ll} M_1=\sum_{s=0}^{\infty}(-1)^s u_{2s}(\frac{2}{3}u\mu^{-1}(-a))^{-2s}, & M_2=\sum_{s=0}^{\infty}(-1)^s u_{2s}(\frac{2}{3}u\mu^{-1}(c))^{-2s}, \\ M_3=\sum_{s=0}^{\infty}(-1)^s u_{2s+1}(\frac{2}{3}u\mu^{-1}(-a))^{-2s-1}, & M_4=\sum_{s=0}^{\infty}(-1)^s u_{2s+1}(\frac{2}{3}u\mu^{-1}(c))^{-2s-1} \\ K_1(n,-a)=K_{11},\quad K_1(n,c)=K_{12}, & K_2(n,-a)=K_{21},\quad K_2(n,c)=K_{22}. \end{array} \quad (37)$$

Now by calculation we get

$$\frac{M_4}{M_2}=1+\frac{3u_1}{u_0}\eta_b^{-\frac{3}{2}}u^{-1}+O(u^{-2}) \quad (38)$$

$$\frac{T_2}{T_1}=1+\frac{\sqrt{2}}{a\sqrt{u}}(e^{\sigma(-a)}-e^{-\sigma(-a)})+O(u^{-1}), \quad (39)$$

$$\frac{K_{21}}{K_{11}} = 1 + 2(ua + \sqrt{2u})B_0(-a)\frac{1}{u^2} + O(u^{-2}).$$

If we multiply (39) by (38) and put it in (36) then we will have

$$x = -\{1 + \frac{2\sqrt{2}}{a} sinh(\sigma(-a))\frac{1}{\sqrt{u}} + (\frac{3u_1}{u_0}\eta_b^{-\frac{3}{2}} + 2aB0(-a))\frac{1}{u}$$
$$+(\frac{6\sqrt{2}u_1 \sinh(\sigma(-a))}{au_0}\eta_b^{-\frac{3}{2}} + \sqrt{2}B_0(-a)\frac{1}{u\sqrt{u}}\} + O(u^{-2}). \qquad (40)$$

Because the values of u are large parameters, then the values of m are large numbers, therefore the m^{th} asymptotic of eigenvalues are in form of

$$u_m = \frac{m\pi + \frac{\pi}{4}}{\int_{-1}^{1} \sqrt{(1-\varphi^2}d\varphi} - \frac{2}{\pi}x_m.$$

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