

Hartogs Extension

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Abstract

We prove in this paper that if X is almost homogeneous compact complex surface, then every holomorphic mappings defined on the Hartogs domain T of \mathbb{C}^n into X can be extended meromorphically from $\Delta^n \setminus Z$ into X , where Z is an analytic subset of Δ^n such that codimension $Z \geq 2$.

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1 Introduction

We denote by T the open subset of \mathbb{C}^n defined by:

$$T = T_{\rho,\tau} = \{z \in \mathbb{C}^n : |z_i| < \rho, i = 1, \dots, n-1; |z_n| < 1\} \quad (1)$$

$$\bigcup \{z \in \mathbb{C}^n : |z_i| < 1, i = 1, \dots, n-1; \tau < |z_n| < 1\}. \quad (2)$$

Where $0 < \rho < 1$ and $0 < \tau < 1$.

By Hartogs theorem the envelope of holomorphy of T is equal to the polydisk Δ^n of \mathbb{C}^n . In this paper we studies extension of holomorphic and meromorphic maps from T into various compact complex manifolds. If we are given a compact complex manifold and a holomorphic (or meromorphic) map of a Hartogs domain T to the manifold, we consider the problem whether the domain of the map can be extends to the polydisk Δ^n or not. By the classical results of Hartogs any function defined on T holomorphically extends on Δ^n . By Levi-Oka's the theorem holds for meromorphic function, see [16]. This problem was treated by several authors, Ivashkovich [6], Kiernan [8], Griffiths [4], Shiffman [15], Y.T.Sui [16]. In their study, they are assuming that the target space are Kähler manifolds. We studies the problem in several cases, it

is needless to assume that the target space is Kählerien. To study this problem, it is introduced in [9] the notions of holomorphically (resp. meromorphically) extensifier manifold. By using these notions, it is solved the problem in several cases where the target space are parallelisable compact, homogeneous compact manifolds, almost homogeneous Kähler manifolds, in [10] it is solved the problem of elliptic surfaces. Ivashkovich in [7] solve the problem conjectured by Griffiths [4] for compact Kähler manifold, more precisely he proves that if X is a compact complex Kähler manifold any meromorphic map f from T into X extends meromorphically to Δ^n .

Section 2 is devoted to solve the problem when X is a compact complex almost homogeneous surface. It is shown in [9] that if X is a compact homogeneous manifold, then any holomorphic map from T into X extends holomorphically to $\Delta^n \setminus Z$ into X where Z is an analytic subset of Δ^n of codimension at least 2. Hence it is a natural question to ask, is this result still true if X is a complex compact almost homogeneous manifold. In [9] it is proved that if X is also Kählerien then X is meromorphically extensifier, if X is not supposed Kählerien, the problem is difficult and not solved until today. Using Potters classification [12], we prove that if X is complex compact almost homogeneous surface, then every holomorphic map from T into X extends meromorphically to $\Delta^n \setminus Z$ where Z is an analytic subset of codimension at least 2.

2 Almost Homogeneous compact complex analytic surfaces

In [9] it is proved that if X is homogeneous compact complex manifold, then any holomorphic map $f : T \rightarrow X$ extends holomorphically to $\Delta^n \setminus Z$ into X , where Z is an analytic subset of Δ^n of codimension at least 2. In this part we study extension of holomorphic mapping from T into a compact almost homogeneous surface.

Definition 1 *A compact complex surface X is called almost homogeneous if the groupe of automorphism of X has an open orbit.*

This is equivalent with: the group G operates trasetively outside an analytic subset S of X . If S is void X is called homogeneous.

Surfaces means a complex manifold of dimension 2. Note that if X is compact G is a complex Lie group.

Potters in [12] proves the following theorem:

Theorem 1 [12] *A almost homogeneous compact complex analytic surface X is biregularly equivalent with a surfaces of one the followings types:*

(i) *a rational surfaces,*

- (ii) $\mathbb{P}^2(\mathbb{C})$ or X is obtained from holomorphic $\mathbb{P}^1(\mathbb{C})$ -bundle over $\mathbb{P}^1(\mathbb{C})$,
- (iii) a topologically trivial $\mathbb{P}^1(\mathbb{C})$ -bundle over a one-dimensional torus T_1 ,
- (iv) a Hopf surface with abelian fundamental group,
- (v) a complex two-dimensional torus T_2 .

Using this classification, we studies the extension of mapping into almost homogeneous surface.

Proposition 1 *Let X a holomorphic $\mathbb{P}^1(\mathbb{C})$ -bundle over $\mathbb{P}^1(\mathbb{C})$, then any holomorphic mapping defined on T into X can be extended meromorphically to $\Delta^n \setminus Z$ into X where Z is an analytic subset of Δ^n such that $\text{codim } Z \geq 2$.*

Proof: Put $H_1 = \{[z_1 : z_2]; z_1 = 0\}$ and $H_2 = \{[z_1 : z_2]; z_2 = 0\}$; we have $\mathbb{P}^1(\mathbb{C}) \setminus H_1$ is isomorphic with \mathbb{C} by $[z_1 : z_2] \rightarrow \frac{z_2}{z_1}$ and $\mathbb{P}^1(\mathbb{C}) \setminus H_2$ isomorphic with \mathbb{C} by $[z_1 : z_2] \rightarrow \frac{z_1}{z_2}$.

Denote $\Phi : X \rightarrow \mathbb{P}^1(\mathbb{C})$ the canonical projection of the $\mathbb{P}^1(\mathbb{C})$ -bundle. Let $f : T \rightarrow X$ a holomorphic map, because $\mathbb{P}^1(\mathbb{C}) \setminus H_i$ ($i = 1, 2$) is holomorphically extensifier and the fiber of X are $\mathbb{P}^1(\mathbb{C})$ (so meromorphically extensifier) $f : T \setminus f^{-1}(\Phi^{-1}(H_i)) \rightarrow X \setminus \Phi^{-1}(H_i)$ extends meromorphically to $T \setminus \widetilde{f^{-1}(\Phi^{-1}(H_i))}$ the envelope of holomorphy of $T \setminus f^{-1}(\Phi^{-1}(H_i))$. Let E_i and F_i the parts of $f^{-1}(\Phi^{-1}(H_i))$ of $\text{codim } E_i = 1$ and $\text{codim } F_i \geq 2$.

We have

$$T \setminus \widetilde{f^{-1}(\Phi^{-1}(H_i))} \cong \widetilde{T \setminus E_i}.$$

But E_i is hypersurface, by [2] $\widetilde{T \setminus E_i} \cong \Delta^n \setminus \widetilde{E_i}$ where $\widetilde{E_i}$ is an hypersurface of Δ^n such that $E_i = \widetilde{E_i} \frown T$.

Put $Z = \widetilde{E_1} \cap \widetilde{E_2}$; f extends meromorphically to $\Delta^n \setminus Z$, we have

$$Z \cap T = E_1 \cap E_2 \subset f^{-1}(\Phi^{-1}(H_1 \cap H_2)),$$

but $H_1 \cap H_2 = \emptyset$ so T does not meets Z and so $\text{codim } Z \geq 2$.

Proposition 2 *Let X be a Hopf surface then every holomorphic mapping $f : T \rightarrow X$ extends holomorphicaly in the complementary of an analytic subset of Δ^n of codimension at least 2 into X .*

Proof: Let $f : T \rightarrow X$ a holomorphic mapping, the universal covering of X is $\mathbb{C}^2 \setminus \{0\}$. We put $\pi : \mathbb{C}^2 \setminus \{0\} \rightarrow X$ the canonical projection; T is simply connected so there exists an holomorphic mapping $g : T \rightarrow \mathbb{C}^2 \setminus \{0\}$ such that $\pi \circ g = f$; g extends to holomorphic mapping $\widetilde{g} : \Delta^n \rightarrow \mathbb{C}^2$; put $Z := \widetilde{g}^{-1}(0)$; Z is an analytic subset of Δ^n , but $T \cap \widetilde{g}^{-1}(0) = \emptyset$ so $\text{codim } Z \geq 2$ and $\pi \circ (\widetilde{g} | \Delta^n \setminus Z)$ is an holomorphic extension of f to $\Delta^n \setminus Z$.

Proposition 3 *If X is $\mathbb{P}^1(\mathbb{C})$ -bundle over one dimensional torus T_1 then every holomorphic map from T into X extends meromorphically to Δ^n into X .*

Proof: The base T_1 is holomorphically extensifer, the fibre are $\mathbb{P}^1(\mathbb{C})$ so meromorphically extensifer.

A rational map of a complex manifold X to a complex projective space $\mathbb{P}^n(\mathbb{C})$, is map $\varphi : z \rightarrow [1 : \varphi_1(z) : \dots : \varphi_n(z)]$ given by n meromorphic global function on X . And we say that a rational map $\varphi : X \rightarrow \mathbb{P}^n(\mathbb{C})$ is birational if there exists a rational map $\Psi : \mathbb{P}^n(\mathbb{C}) \rightarrow X$ such that $\varphi \circ \Psi$ is the identity [5]. A rational surfaces is algebraic surfaces birationally isomorphic to $\mathbb{P}^2(\mathbb{C})$.

Proposition 4 *Every rational surface X is meromorphically extensifer.*

Proof: Let $f : T \rightarrow X$ be a holomorphic map. There exists a meromorphic functions φ_1, φ_2 on X , such that $g : z \rightarrow [1 : \varphi_1 \circ f(z) : \varphi_2 \circ f(z)]$ from $T \rightarrow \mathbb{P}^2(\mathbb{C})$ is meromorphic map. So there exists an analytic subset Z of T of codimension two such that $g : T \setminus Z \rightarrow \mathbb{P}^2(\mathbb{C})$ is holomorphic mapping, g extends to a meromorphic map \tilde{g} from Δ^n into $\mathbb{P}^2(\mathbb{C})$ (because $\mathbb{P}^2(\mathbb{C})$ is meromorphically extensifer, and $\widetilde{T \setminus Z} = \Delta^n$).

Moreover, there exists $\Psi : \mathbb{P}^2(\mathbb{C}) \rightarrow X$, such that $\varphi \circ \Psi$ is the identity. Let $\Gamma_{\Psi \circ \tilde{g}} \subset \Delta^n \times X$ be the graphe of $\Psi \circ \tilde{g}$ and let $\Gamma_f \subset T \times X$ be the graphe of f , by definition of rational mapping there exist an analytic subset S of $\mathbb{P}^2(\mathbb{C})$ such that $\varphi : X \setminus \varphi^{-1}(S) \rightarrow \mathbb{P}^2(\mathbb{C}) \setminus S$ is biholomorphic, put $U = \mathbb{P}^2(\mathbb{C}) \setminus S$ we have $\Gamma_{\Psi \circ \tilde{g}} \upharpoonright g^{-1}(U) = \Gamma_f \upharpoonright g^{-1}(U)$, but this graphe are irreducible so $\Gamma_{\Psi \circ \tilde{g}} = \Gamma_f$, finally the map $\Psi \circ \tilde{g}$ is meromorphic extension of f from Δ^n into X .

A torus T_2 is holomorphically extensifer (because the universal covering is \mathbb{C}^2) and $\mathbb{P}^2(\mathbb{C})$ meromorphically extensifer.

Corollary 1 *Let X be almost homogeneous analytic complex compact surface, then every holomorphic mappings defined on the Hartogs domain T of \mathbb{C}^n with values in X extends meromorphically to $\Delta^n \setminus Z$ into X , where Z is an analytic subset of Δ^n such that $\text{codim } Z \geq 2$.*

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