An Short Note on the Sequence \( \Omega(n) \)

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Abstract

We shall denote (see [1]) with \( \Omega(n) \) the number of prime factors in the prime factorization of \( n \). Hence

\[
\Omega(1) = 0, \Omega(2) = 1, \Omega(3) = 1, \Omega(4) = 2, \Omega(5) = 1, \Omega(6) = 2, \ldots
\]

In this note we study the sequence

\[
S(n) = \Omega(n + r - 1) + \ldots + \Omega(n + 1) + \Omega(n) + \Omega(n - 1) + \Omega(n - 2) + \ldots + \Omega(n - t)
\]

where \( r \geq 1 \) and \( t \geq 1 \) are fixed. In a short elementary proof we find that

\[
\lim \inf S(n) = 0 \quad \lim \sup S(n) = \infty
\]

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1 Main Results

Theorem 1.1 Let \( r \geq 1 \) and \( t \geq 1 \) fixed. Let us consider the sequence

\[
S(n) = \frac{\Omega(n + r - 1) + \ldots + \Omega(n + 1) + \Omega(n)}{\Omega(n - 1) + \Omega(n - 2) + \ldots + \Omega(n - t)}
\]

Then the following limits hold

\[
\lim \inf S(n) = 0 \quad \lim \sup S(n) = \infty
\]

In particular if \( r = 1 \) and \( t = 1 \) we have

\[
\lim \inf \frac{\Omega(n)}{\Omega(n - 1)} = 0 \quad \lim \sup \frac{\Omega(n)}{\Omega(n - 1)} = \infty
\]
Proof. In the proof, \( p_n \) is the \( n \)-th prime number and \( \pi(x) \) is the number of primes not exceeding \( x \).

Let us consider the primes not exceeding \( k \) \((k \geq 2)\), that is \( p_1, p_2, \ldots, p_h \) where \( h = \pi(k) \).

Now, let us consider the following number \( N \) which depend of \( n \) and \( s \) (where \( n \) and \( s \) are large)

\[
N = (p_1 p_2 \ldots p_h)^n p_{h+1} p_{h+2} \ldots p_s
\]

We have

\[
\Omega(N) = \pi(k)n + s - \pi(k)
\]

On the other hand

\[
\begin{align*}
\Omega(N - 1) + \Omega(N - 2) + \ldots + \Omega(N - k) &= \Omega((N - 1) \ldots (N - k)) \\
&= \Omega(k!) + \Omega \left( (N - 1) \left( \frac{N}{2} - 1 \right) \ldots \left( \frac{N}{k} - 1 \right) \right) \\
&\leq \Omega(k!) + \log p_{s+1} N^k \\
&= k \frac{\log(p_1 p_2 \ldots p_h)}{\log p_{s+1}} n + k \log p_{s+1} (p_{h+1} p_{h+2} \ldots p_s) + \Omega(k!)
\end{align*}
\]

Clearly, if \( M > 0 \) then there exist \( s \) and \( n \) sufficiently large such that

\[
\frac{\Omega(N)}{\Omega(N - 1) + \Omega(N - 2) + \ldots + \Omega(N - k)} > M
\]

Therefore

\[
\lim sup S(n) = \lim sup \frac{\Omega(n + r - 1) + \ldots + \Omega(n + 1) + \Omega(n)}{\Omega(n - 1) + \Omega(n - 2) + \ldots + \Omega(n - t)} = \infty
\]

In the same way we obtain if \( \epsilon > 0 \) then there exist \( s \) and \( n \) sufficiently large such that

\[
\frac{\Omega(N + k) + \ldots + \Omega(N + 2) + \Omega(N + 1)}{\Omega(N)} < \epsilon
\]

Therefore

\[
\lim inf S(n) = \lim inf \frac{\Omega(n + r - 1) + \ldots + \Omega(n + 1) + \Omega(n)}{\Omega(n - 1) + \Omega(n - 2) + \ldots + \Omega(n - t)} = 0
\]

The theorem is thus proved.

**Corollary 1.2** Let \( r \geq 1 \) and \( k \geq 1 \) fixed. Then the inequality

\[
\Omega(n + r - 1) + \ldots + \Omega(n + 1) + \Omega(n) > \Omega(n - 1) + \Omega(n - 2) + \ldots + \Omega(n - k)
\]
On the sequence $\Omega(n)$ has infinitely many solutions. That is, it is true for infinite $n$. Analogously, the contrary inequality

$$\Omega(n + r - 1) + \ldots + \Omega(n + 1) + \Omega(n) < \Omega(n - 1) + \Omega(n - 2) + \ldots + \Omega(n - k)$$

has infinitely many solutions.

**Definition 1.3** Let us consider a set of $k$ consecutive integer numbers $n + 1, n + 2, \ldots, n + k$ ($k \geq 1$). We shall call this set of numbers an integer interval $I = [n + 1, n + k]$ of amplitude $A(I) = k$, where $n + 1$ is the first number of the interval and $n + k$ is the last number. On the other hand, the notation $\Omega(I)$ mean $\Omega(n + 1) + \ldots + \Omega(n + k)$. Two integer intervals $I$ and $J$ in this order are consecutives when the first number of the second interval is the consecutive of the last number of the first interval. For example $[1,8]$ and $[9,24]$ are consecutive integer intervals.

Let us consider a partition of the positive integers in consecutive integer intervals.

$$I_1 = [1, n], \ I_2 = [n + 1, n + a_1], \ I_3 = [n + a_1 + 1, n + a_2], \ldots$$

Clearly the inequality $\Omega(I_n) < \Omega(I_{n+1})$ has infinitely many solutions since $\lim \sup \Omega(n) = \infty$.

It is easy to build partitions such that

$$\Omega(I_1) < \Omega(I_2) < \Omega(I_3) < \ldots \quad (1)$$

That is, $\Omega(I_n) < \Omega(I_{n+1})$ for all $n$.

In the next theorem we shall prove that (1) is impossible if there exists $B$ such that $A(I_n) \leq B$ for all $n$ (for example if $A(I_n) = B$ for all $n$).

**Theorem 1.4** Let us consider a partition of the positive integers in consecutive integer intervals

$$I_1 = [1, n], \ I_2 = [n + 1, n + a_1], \ I_3 = [n + a_1 + 1, n + a_2], \ldots$$

such that $A(I_n) \leq B$ for all $n$, then the inequality $\Omega(I_n) > \Omega(I_{n+1})$ has infinitely many solutions.

Proof. It is well known [1] that

$$\sum_{n \leq x} \Omega(n) \sim x \log \log x$$

Therefore

$$\sum_{x < n \leq 2x} \Omega(n) \sim x \log \log x$$
If \( x = 2^{k-1} \) we have
\[
\sum_{2^{k-1} < n \leq 2^k} \Omega(n) \sim 2^{k-1} \log k
\] (2)

The integer interval \([2^{k-1}, 2^k]\) include a number \( C(k) \) of consecutive intervals \( I_s \). Clearly
\[
C(k) \geq \left\lceil \frac{2^{k-1}}{N} \right\rceil - 1
\]

Suppose that from a certain \( n \), \( \Omega(I_n) \leq \Omega(I_{n+1}) \), then for each of the \( C(k) \) intervals \( I_s \) we have \( \Omega(I_s) \geq k - 1 \). Therefore
\[
\sum_{2^{k-1} < n \leq 2^k} \Omega(n) \geq \left( \left\lceil \frac{2^{k-1}}{N} \right\rceil - 1 \right) (k - 1) \quad (3)
\]

If \( k \) is large (2) and (3) are an evident contradiction. The theorem is thus proved.

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References


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