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An Short Note on the Sequence $\Omega(n)$

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Abstract

We shall denote (see [1]) with $\Omega(n)$ the number of prime factors in the prime factorization of n. Hence

$$\Omega(1) = 0, \Omega(2) = 1, \Omega(3) = 1, \Omega(4) = 2, \Omega(5) = 1, \Omega(6) = 2, \ldots$$

In this note we study the sequence

$$S(n) = \frac{\Omega(n+r-1) + \ldots \Omega(n+1) + \Omega(n)}{\Omega(n-1) + \Omega(n-2) + \ldots \Omega(n-t)}$$

where $r \geq 1$ and $t \geq 1$ are fixed. In a short elementary proof we find that

$$lim\ inf\ S(n) = 0 \qquad lim\ sup\ S(n) = \infty$$

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1 Main Results

Theorem 1.1 *Let $r \geq 1$ and $t \geq 1$ fixed. Let us consider the sequence*

$$S(n) = \frac{\Omega(n+r-1) + \ldots \Omega(n+1) + \Omega(n)}{\Omega(n-1) + \Omega(n-2) + \ldots \Omega(n-t)}$$

Then the following limits hold

$$lim\ inf\ S(n) = 0 \qquad lim\ sup\ S(n) = \infty$$

In particular if $r = 1$ and $t = 1$ we have

$$lim\ inf\ \frac{\Omega(n)}{\Omega(n-1)} = 0 \qquad lim\ sup\ \frac{\Omega(n)}{\Omega(n-1)} = \infty$$

Proof. In the proof, p_n is the n-th prime number and $\pi(x)$ is the number of primes not exceeding x.

Let us consider the primes not exceeding k ($k \geq 2$), that is $p_1, p_2, \ldots p_h$ where $h = \pi(k)$.

Now, let us consider the following number N which depend of n and s (where n and s are large)

$$N = (p_1 p_2 \ldots p_h)^n p_{h+1} p_{h+2} \ldots p_s$$

We have

$$\Omega(N) = \pi(k)n + s - \pi(k)$$

On the other hand

$$\begin{aligned} \Omega(N-1) + \Omega(N-2) + \ldots + \Omega(N-k) &= \Omega((N-1)\ldots(N-k)) \\ = \Omega(k!) + \Omega\left((N-1)\left(\frac{N}{2}-1\right)\ldots\left(\frac{N}{k}-1\right)\right) &\leq \\ \Omega(k!) + \log_{p_{s+1}}\left((N-1)\left(\frac{N}{2}-1\right)\ldots\left(\frac{N}{k}-1\right)\right) &\leq \Omega(k!) + \log_{p_{s+1}} N^k \\ = k\frac{\log(p_1 p_2 \ldots p_h)}{\log p_{s+1}} n + k \log_{p_{s+1}}(p_{h+1}p_{h+2}\ldots p_s) &+ \Omega(k!) \end{aligned}$$

Clearly, if $M > 0$ then there exist s and n sufficiently large such that

$$\frac{\Omega(N)}{\Omega(N-1) + \Omega(N-2) + \ldots \Omega(N-k)} > M$$

Therefore

$$lim\ sup\ S(n) = lim\ sup\ \frac{\Omega(n+r-1) + \ldots \Omega(n+1) + \Omega(n)}{\Omega(n-1) + \Omega(n-2) + \ldots \Omega(n-t)} = \infty$$

In the same way we obtain if $\epsilon > 0$ then there exist s and n sufficiently large such that

$$\frac{\Omega(N+k) + \ldots \Omega(N+2) + \Omega(N+1)}{\Omega(N)} < \epsilon$$

Therefore

$$lim\ inf\ S(n) = lim\ inf\ \frac{\Omega(n+r-1) + \ldots \Omega(n+1) + \Omega(n)}{\Omega(n-1) + \Omega(n-2) + \ldots \Omega(n-t)} = 0$$

The theorem is thus proved.

Corollary 1.2 *Let $r \geq 1$ and $k \geq 1$ fixed. Then the inequality*

$$\Omega(n+r-1) + \ldots + \Omega(n+1) + \Omega(n) > \Omega(n-1) + \Omega(n-2) + \ldots + \Omega(n-k)$$

has infinitely many solutions. That is, it is true for infinite n. *Analogously, the contrary inequality*

$$\Omega(n+r-1)+\ldots+\Omega(n+1)+\Omega(n)<\Omega(n-1)+\Omega(n-2)+\ldots+\Omega(n-k)$$

has infinitely many solutions.

Definition 1.3 *Let us consider a set of* k *consecutive integer numbers* $n+1, n+2, \ldots, n+k \quad (k \geq 1)$. *We shall call this set of numbers an integer interval* $I=[n+1, n+k]$ *of amplitude* $A(I)=k$, *where* $n+1$ *is the first number of the interval and* $n+k$ *is the last number. On the other hand, the notation* $\Omega(I)$ *mean* $\Omega(n+1)+\ldots+\Omega(n+k)$. *Two integer intervals* I *and* J *in this order are consecutives when the first number of the second interval is the consecutive of the last number of the first interval. For example [1,8] and [9,24] are consecutive integer intervals.*

Let us consider a partition of the positive integers in consecutive integer intervals.

$$I_1=[1, n],\ I_2=[n+1, n+a_1],\ I_3=[n+a_1+1, n+a_2], \ldots$$

Clearly the inequality $\Omega(I_n)<\Omega(I_{n+1})$ has infinitely many solutions since *lim sup* $\Omega(n)=\infty$.

It is easy to build partitions such that

$$\Omega(I_1)<\Omega(I_2)<\Omega(I_3)<\ldots \tag{1}$$

That is, $\Omega(I_n)<\Omega(I_{n+1})$ for all n.

In the next theorem we shall prove that (1) is impossible if there exists B such that $A(I_n) \leq B$ for all n (for example if $A(I_n)=B$ for all n).

Theorem 1.4 *Let us consider a partition of the positive integers in consecutive integers intervals*

$$I_1=[1, n],\ I_2=[n+1, n+a_1],\ I_3=[n+a_1+1, n+a_2], \ldots$$

such that $A(I_n) \leq B$ *for all* n, *then the inequality* $\Omega(I_n)>\Omega(I_{n+1})$ *has infinitely many solutions.*

Proof. It is well known [1] that

$$\sum_{n \leq x} \Omega(n) \sim x \log \log x$$

Therefore

$$\sum_{x<n \leq 2x} \Omega(n) \sim x \log \log x$$

If $x = 2^{k-1}$ we have

$$\sum_{2^{k-1}<n\leq 2^k} \Omega(n) \sim 2^{k-1} \log k \tag{2}$$

The integer interval $[2^{k-1}, 2^k]$ include a number $C(k)$ of consecutive intervals I_s. Clearly

$$C(k) \geq \left[\frac{2^{k-1}}{N}\right] - 1$$

Suppose that from a certain n, $\Omega(I_n) \leq \Omega(I_{n+1})$, then for each of the $C(k)$ intervals I_s we have $\Omega(I_s) \geq k - 1$. Therefore

$$\sum_{2^{k-1}<n\leq 2^k} \Omega(n) \geq \left(\left[\frac{2^{k-1}}{N}\right] - 1\right)(k-1) \tag{3}$$

If k is large (2) and (3) are an evident contradiction. The theorem is thus proved.

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References

[1] G.H. Hardy and E. M. Wright, *An introduction to the theory of numbers,* Oxford, 1960.

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