On the Underwriting Gain of a Whole Life Insurance in a Dual Random Environment

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Abstract

A dual random model of life insurance, where both the time until death and the rate of return on investment are random quantities, is considered. The paradox of the underwriting gain by Jewell [11], which states that the underwriting gain at death of a whole life insurance has a positive expectation in case of constant positive interest rates and level premium payments, is extended. It is shown that under a dual random Wiener or Ornstein-Uhlenbeck model, the expected retrospective underwriting gain of a whole life insurance with a single net premium plan is always non-negative. In Vadiveloo et al. [15] the authors have shown the optimality of the net single premium plan for the case of deterministic interest rates. In the dual random environment, Ramsay [13] has intuitively argued that this result will be satisfied under stable market conditions, say under a "low volatility" of interest rates and stock prices. A simple sufficient condition for this is derived under a dual random Wiener model.

Mathematics Subject Classification: Primary 62P05, Secondary 91B30

Keywords: random future lifetime, random return, underwriting gain, whole life insurance, single net premium plan

1. A dual random model of life insurance. Consider an insurance company, which has in her life portfolio groups of similar lives aged $x$ under the same type of insurance contract. Under dual randomness in life insurance, one understands those scientific methods, which permit to handle in an appropriate modelling framework both the time until death and the interest rate or/and rate of return as random quantities. This dual nature of the insurance environment is supposed to be reflected by the following notations and model assumptions:
(M1) The random variable $T=T(x)$ describes the future lifetime of a life aged $x$. Its distribution is denoted $F(t)$, $0 \leq t < \infty$. The corresponding survival function is $1-F(t) = p_x$ and the probability density is $f(t) = p_x \mu_{x+t}$, where $\mu_{x+t}$ is the force of mortality of an $x+t$ years old person.

(M2) The size of the death benefit, to be paid at time $t$ in case death occurs at age $x+t$, is denoted by $Y(t)$ and is considered as a stochastic process defined on the non-negative real numbers $\mathbb{R}^+$. 

(M3) The rate of return on investment (in particular interest rates) used for life insurance valuations, for example premium calculation, is a stochastic process defined on $\mathbb{R}^+$. The stochastic discount factor, which describes the discounted value of 1 to be paid at time $t$, is modelled by a monotone decreasing stochastic process $D(t) = \exp\left\{-\int_0^t R(s)ds\right\}$, where $R(s)$ is a stochastic process modelling the force of interest. The stochastic process $A(t) = D(t)^{-1}$ describes a stochastic accumulation factor. In a dual random environment the assumption is made that $T$ is independent of $D(t)$ and any other financial market conditions.

(M4) In general premiums are non-negative and are not paid after death. If $P(t)$ is a non-negative and non-decreasing function on the interval $[0,t)$, which represents the total premiums paid up to time $t$ excluding interest earned, then the present value of the aggregate premiums paid up to time $t$ is given by the non-negative and non-decreasing stochastic process

\begin{equation}
P(0,t) = \int_0^t D(s)dP(s), \quad 0 \leq t \leq T. \tag{1.1}
\end{equation}

The current value at any time $s$, $0 \leq s \leq t$, of these premiums equals

\begin{equation}
P(s,t) = A(s)P(0,t). \tag{1.2}
\end{equation}

The function $P(t)$ is called a premium payment plan if it is a specified deterministic function on $[0,t)$ for all $t \geq 0$.

(M5) The current value at time $s$ of the death benefit $Y(t)$ at time $t$ is given by

\begin{equation}
B(s,t) = A(s)D(t)Y(t), \quad 0 \leq s \leq t \leq T. \tag{1.3}
\end{equation}

Usually the present value $B(0,t)=D(t)Y(t)$ is assumed to be a non-negative and non-increasing stochastic process.

(M6) The simplifying assumption is made that the variance $\text{Var}[B(0,T)]$ do not depend on $P(0,T)$. 
The first and second moments with respect to time and time until death of the introduced stochastic processes are assumed to be finite.

Comments 1.1. Concerning practical implications the following remarks about these assumptions may be useful:

(M2) In modelling traditional life insurance products one assumes usually that \( Y(t) \) is deterministic, for example \( Y(t)=1 \) for a whole life insurance contract. However in modern life insurance \( Y(t) \) is stochastic, for example an equity-linked contract with an asset value guarantee, because the benefits depend on the fluctuating performance or rate of return of financial asset values.

(M3) There are several ways to model the uncertainty in future values of the interest rate, or more generally of the rate of return on investment. In the widespread stochastic approach the following popular choices are made. Beekman and Fuelling [1], [2], [3] and other authors assume that

\[
D(t) = \exp\{-\delta t - X(t)\}, \quad X(t) = \int_0^t W(s)ds, \quad R(s) = \delta + W(s), \quad \delta > 0,
\]

where \( X(t) \) is either a Wiener process or an Ornstein-Uhlenbeck process. In a more general modelling framework, one assumes that \( R(s) \) is an Itô process:

\[
dR(s) = \mu(R(s), s)ds + \sigma(R(s), s)dZ(s),
\]

where \( Z(s) \) is a Wiener process, and \( \mu, \sigma^2 \) represent the instantaneous expected value and variance of the process \( R(s) \). Several economic conditions, which restrict the choice of appropriate processes, as for example non-negative interest rates, non arbitrarily large values of interest rates, arbitrage-free modelling of the term structure of interest rate, etc., have led in the recent years to a quite complex topic. Another very general approach, starting with Buckley [5], applies the growing subject named "fuzzy theory", which has been initiated by Zadeh [16].

(M4) In the classical approach, where \( D(t)=\exp\{-\delta t\} \) is deterministic, the usual notation is (see Vadiveloo et al. [15] or Kling [12]) :

\[
\rho_t := P(0,t) = \int_0^t e^{-\delta s} dP(s), \quad t \geq 0.
\]

(M5) It can be argued that the assumption on \( B(0,t) \) is not very restrictive (see Ramsay [13], p. 501). In the literature, when \( D(t)=\exp\{-\delta t\} \), the usual notation is

\[
\beta_t := B(0,t) = e^{-\delta t} b_t, \quad t \geq 0.
\]

(M6) Note that life insurance contracts, which as benefit include a return of premiums at death, violate the assumption made.
Conventions 1.1. The following convenient assumptions are made:

(C1) To include those types of insurance contracts as endowment insurance, the convention is made that if there exists a non-negative integer $n$ such that $B(0,t)=0$ for $t>n$, then $dP(t)=0$ for $t>n$, that is no premium is being paid after time $n$. In this interpretation $T$ can be viewed as the time at which a benefit is paid, where an adequate change of the distribution function is necessary. For example the distribution of $T$ in a $n$-year endowment insurance is

\[
F_n(t) = \begin{cases} 
F(t), & 0 \leq t < n, \\
1, & t > n. 
\end{cases}
\]

In particular, the random variable $T$ is not defective, that is $\Pr(0 \leq T < \infty) = 1$.

(C2) The current value at time $s$ of the underwriting gain accumulated at time $t$ is denoted by $G(s,t)=P(s,t)-B(s,t)$. In the formal distinction introduced by Frees [8] the present value $G(0,t)$ corresponds to the prospective view of life insurance and the terminal value $G(T,T)$ to the retrospective view. The latter is the underwriting gain at death, that is the accumulated value of the profit on the net premium immediately after the payment of the death claim benefit.

(C3) Consider the following class of net premium payment plans

\[
\Phi = \{ P(t) \text{ is a premium payment plan such that } E[G(0,T)]=0 \}. 
\]

Let $\Psi \subseteq \Phi$ be a subclass of net premium payment plans. Then a particular net premium payment plan $P^*(t)$ is said to be $\Psi$-optimal if $\Var[G^*(0,T)]=\min.$, where the minimum is taken over the set $\Psi$.

2. Underwriting gain and the net single premium optimum plan. In Vadiveloo et al. [15] the authors have shown the optimality of the net single premium for the case of deterministic interest rates. An attempt to generalize their result to a dual random environment has been undertaken by Ramsay [13]. Our aim is to give a precise mathematical account of the main ideas in order to obtain new rigorous results concerning the underwriting gain of life insurance contracts.

2.1. A prospective optimality condition. Let $E_X$ be the net single premium of a life insurance contract with benefit payment $B(0,T)$. Under the net equivalence principle $E[G(0,T)]=0$ one has

\[
E_z = \int_0^\infty E[P(0,t)]dF(t) = \int_0^\infty E[B(0,t)]dF(t). 
\]

In the prospective view the variance of the underwriting gain is given by
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(2.2) \[ \text{Var}[G(0,T)] = E\left[G(0,T)^2\right] = \text{Var}[P(0,T)] + \text{Var}[B(0,T)] - 2\text{Cov}[P(0,T), B(0,T)] \]

By assumption (M6) the value of \( \text{Var}[B(0,T)] \) does not depend on \( P(0,T) \). It follows that \( \text{Var}[G(0,T)] \) is minimum if and only if \( \text{Var}[P(0,T)] - 2\text{Cov}[P(0,T), B(0,T)] \) is minimum. Consider the following condition of negative covariance

(A) \( \text{Cov}[P(0,T), B(0,T)] = \int_0^\infty E[(P(0,T) - E_x)(B(0,T) - E_x)]dF(t) \leq 0. \)

Clearly for the net single premium plan \( P^*(0,t)=E_x \) for all \( t \geq 0 \), one has \( P^*(0,t)=E_x \) for all \( t \geq 0 \), and thus \( \text{Var}[P^*(0,T)] = \text{Cov}[P^*(0,T), B(0,T)] = 0. \) From (2.2) it follows that to establish the optimality of the net single premium it suffices to require the negative covariance condition (A). In this situation one has

(2.3) \( \min[\text{Var}[G(0,T)]] = \text{Var}[G^*(0,T)] = \text{Var}[B(0,T)]. \)

2.2. Retrospective view of the underwriting gain. If \( P^*(0,t)=E_x \) for all \( t \geq 0 \) is the net single premium plan, one has for the expected terminal value of the underwriting gain:

(2.4) \( E[G^*(T,T)] = E[A(T) \cdot (E_x - B(0,T))] = -\text{Cov}[A(T), B(0,T)]. \)

The underwriting gain at death, as a random variable, was introduced in the context of a paradox by Jewell [11]. It was shown by Chan and Shiu [7] to have a positive expectation in case interest rates are positive constant and under level premium payments. This result was studied further by Ramsay [13] in the context of stochastic interest rates. Hürlimann [10] shows that the paradox of Jewell [11] can be resolved if stochastic tariffing rules are allowed.

Under (2.4) a non-negative expected retrospective underwriting gain is obtained if the following condition is fulfilled:

(B) \( \text{Cov}[A(T), B(0,T)] \leq 0. \)

On the other side let \( P(t) \in \Phi \) be any other net premium payment plan. Then one has \( G(T,T)=G^*(T,T)+A(T)-(P(0,T)-E_x) \) and

(2.5) \( E[G(T,T)] = E[G^*(T,T)] + \text{Cov}[A(T), P(0,T)]. \)

Under condition (B) this will be non-negative if the following condition holds:
In this case the net single premium minimizes the expected retrospective underwriting gain:

$$\min[E[G(T, T)]] = E[G^*(T, T)] \geq 0.$$  

**Comment 2.1.** Ramsay [13] has argued that intuitively the (sufficient) conditions (A), (B), (C) will be satisfied under **stable market conditions**, for example under a "low volatility" of interest rates or/and stock prices. However to be useful in practice such a "vague" point of view needs a precise mathematical content in form of decision rules, which guarantee that these conditions hold with certainty. In the subsequent Sections, our purpose is to show that such results, which render "rigorous" the former approach, can effectively be determined.

### 3. Condition (A) for a whole life insurance.

In traditional life insurance, concerned with non-random benefits \( Y(t) \), the most simple **special case** is a whole life insurance characterized by \( Y(t) = 1, \ t \geq 0 \). In the subsequent discussion a model of the rate of return as defined by (1.4) is considered. Furthermore the class \( \Phi \) is restricted to the subclass \( \Phi_2 \) consisting of only two net premium payment plans, namely the net single premium plan \( P^*(0, t) = \text{Ex} \) and the **level net premium payment plan** defined by

$$\text{(3.1)} \quad P(0, t) = P_{x, R} \overline{\tau}_{x, R} = P_{x, R} \cdot \int_0^\infty \exp(-\delta s - X(s)) ds,$$

where

$$\text{(3.2)} \quad P_{x, R} = \frac{\overline{A}_{x, R}}{\overline{\alpha}_{x, R}}, \text{ with } \overline{A}_{x, R} = E[D(T)], \overline{\alpha}_{x, R} = E[\overline{\tau}_{1, R}].$$

To evaluate (3.2) under the model (1.4), use the independence assumption (M3) and Fubini's theorem to get

$$\text{(3.3)} \quad \overline{A}_{x, R} = E_R E_T[D(T)|R] = E_R \left[ \int_0^\infty e^{-\delta X(t)} dF(t) \right]$$

$$= \int_0^\infty e^{-\delta t} E_X \left[ e^{-X(t)} \right] dF(t) = \int_0^\infty e^{-\delta t + A(0, t)} p_\delta \mu dt,$$

with

$$\text{(3.4)} \quad A(0, t) = \begin{cases} \sigma^2 t, & \text{Wiener process} \\ \sigma^2 \left(1 - e^{-2\alpha \tau} \right), & \text{Ornstein–Uhlenbeck process} \end{cases}$$

The assumption (M7) implies that the values of the model parameters \( \delta, \sigma, \alpha \) are such that
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(3.5) \[
\int_0^\infty e^{-\delta + \frac{1}{2} \xi(0,t)} \, dt < \infty, \quad \int_0^\infty e^{-2\delta + 2 \xi(0,t)} \, dt < \infty.
\]

In the following, detailed calculations are only given for the Wiener process. One has

(3.6) \[
\bar{A}_{x,R} = \int_0^\infty e^{-\delta + \frac{1}{2} \xi(t)} \, dt \mu_x(t) \, dt = E_T \left[ e^{-\delta T} \right] = E_T \left[ \left( T' \right)^T \right] = \bar{A}_x(\delta^*) =: \bar{A}_x^*,
\]

where \( \delta^* = \delta - \frac{1}{4} \sigma^2 \). This formula provides a link between the dual random environment and the classical mortality random environment. Similarly one obtains

(3.7) \[
\bar{A}_{x,R} = E[A(T)] = \left. E_T \left[ e^{\delta T} \right] \right|_{\delta^*} = \bar{A}_x(\delta^{**}) =: \bar{A}_x^{**}, \quad \delta^{**} = \delta + \sigma^2,
\]

(3.8) \[
\bar{A}_{x,R} = E[\bar{A}(T)] = \left. E_T \left[ e^{\delta T} \right] \right|_{\delta^{**}} = \bar{A}_x(\delta^{***}) =: \bar{A}_x^{***}.
\]

Now the condition (A) of Section 2 for the level net premium plan reads

(3.9) \[
C_{A,R} := \text{Cov}[P(0,T), B(0,T)] = P_{x,R} \text{Cov} \left[ \bar{A}_{x,R}, D(T) \right].
\]

Once a precise condition for negative covariance \( C_{A,R} < 0 \) has been formulated, it follows automatically from Section 2 that \( \text{Var}[G * (0,T)] < \text{Var}[G(0,T)] \), hence the net single premium plan beats (in variance) the level net premium plan. It remains to calculate (3.8) for the Wiener process. One gets

(3.10) \[
C_{A,R} := P_{x,R} \left. E \left[ (\bar{A}_{x,R} - \bar{A}_x) \right] \right| (D(T) - \bar{A}_{x,R}) \right)
\]

(3.11) \[
C_{A,R} := P_{x,R} \left. \left[ E \left[ \bar{A}_{x,R} D(t) \right] \right] \right| dF(t) - \bar{A}_{x,R} \bar{A}_{x,R}
\]

But one has

(3.12) \[
E[\bar{A}_{x,R} D(t)] = e^{-(\delta - \frac{1}{2} \sigma^2)} \int_0^t e^{-(\delta - \frac{1}{2} \sigma^2)} \, ds = e^{-\delta T} \left( 1 - e^{-\delta^* T} \right) \left( \frac{1}{\delta^* - \sigma^2} \right).
\]

Using (3.2), (3.6) and (3.7) it follows from (3.9) and (3.10) that

(3.13) \[
C_{A,R} = P_{x,R} \left. \left[ \frac{1}{\delta^* - \sigma^2} \right. E_T \left[ e^{-\delta T} - e^{-2(\delta - \sigma^2)^2} \right] \right| dF(t) - \bar{A}_x^* \bar{A}_x^* \right)
\]

With \( \bar{A}_x^{***} := \bar{A}_x(\delta^{***}) = E_T \left[ e^{-2(\delta - \sigma^2)^2} \right], \ \delta^{***} = \delta - \sigma^2 \), this can be rewritten as
4. A negative covariance result. For a whole life insurance and a level net premium plan the covariance formula (3.12) holds under a dual random Wiener model (1.4). Using the inequality

\[2 \bar{A}_x^{***} = E_T \left[ e^{-2(\delta^+ - \sigma^2)T} \right] \geq E_T \left[ e^{-2\sigma^2T} \right] =: 2 \bar{A}_x^*,\]

it follows that

\[-C_{A,R} \geq \frac{\sigma^2}{\bar{a}_x^+} \cdot \left( 2 \bar{A}_x^* - \bar{A}_x^+ \cdot (\delta^+ + \sigma^2 \bar{a}_x^+). \right),\]

Using further the relation \( \bar{A}_x^+ = 1 - \delta^+ \cdot \bar{a}_x^+ \), one sees that to get \( -C_{A,R} \geq 0 \), it suffices to satisfy \( 2 \bar{A}_x^+ \geq \bar{A}_x^+ \cdot (\bar{A}_x^+ + \sigma^2 \bar{a}_x^+), \) or equivalently \( \sigma^2 \leq \frac{\bar{V}_x^+}{\bar{a}_x^+ \cdot \bar{A}_x}, \) where \( \bar{V}_x^+ := \text{Var}_T \left[ e^{-\delta^+ T} \right]. \)

This calculation suggests that without further assumptions it seems not possible to extend Proposition A.1 in Vadiveloo [15] to a dual random environment. The obtained condition of "low market volatility" is a precise mathematical statement that can be verified in practice.

**Proposition 4.1.** Given is a whole life insurance with a level net premium plan and a dual random Wiener model (1.4). Assume the following implicit inequality for the volatility parameter \( \sigma^2 \) holds:

\[ 2 \sigma^2 \leq \frac{\bar{V}_x^+ (\delta - \frac{1}{2} \sigma^2)}{\bar{a}_x^+ (\delta - \frac{1}{2} \sigma^2) \cdot \bar{A}_x^+ (\delta - \frac{1}{2} \sigma^2)}. \]

Then the negative covariance relation \( C_{A,R} = \text{P}_{x,R} \text{Cov}[^{\bar{A}_x^+}, D(T)] \leq 0 \) holds.

**Example 4.1.** Suppose \( T \) is exponentially distributed with expectation of life at age \( x \) given by \( e_x = E[T] = \mu^{-1}. \) Setting \( \epsilon = \frac{1}{2} \sigma^2, \) the condition (4.3) reads

\[ 2 \epsilon \leq \frac{(\delta - \epsilon)^2 \cdot e_x^0}{(1 + (\delta - \epsilon) \cdot e_x^0) \cdot (1 + 2(\delta - \epsilon) \cdot e_x^0)}. \]

If \( \delta = 0.05, \quad \sigma = 0.05, \quad e_x^0 = 75, \) this inequality is satisfied.
If $\delta = 0.05, \sigma = 0.10, e_x^0 = 75$, this inequality is not satisfied.

5. Condition (B) for the whole life insurance. Given is a whole life insurance with a single net premium plan under a dual random Wiener or Ornstein-Uhlenbeck model (1.4). The expected value of the underwriting gain at the time of death is given by

$$E\left[G'(T, T)\right] = -\text{Cov}[A(T), D(T)] = A_{x,R} \cdot S_{x,R}^{-1} = E_T \left[e^{-\delta T + \frac{1}{2} \mathbb{D}(0, T)} \cdot E_T \left[e^{\frac{1}{2} \mathbb{D}(0, T)}\right] - 1, \right.$$

where $A(0, T)$ is defined in (3.4). Let us show that condition (B) is always satisfied.

**Proposition 5.1.** Under a dual random Wiener or Ornstein-Uhlenbeck model, the expected retrospective underwriting gain of a whole life insurance with a single net premium plan is always non-negative.

**Proof.** Consider the real functions of one variable

$$f(x) = \sqrt{e^{-\delta x + \frac{1}{2} A(0, x)}}, \quad g(x) = \sqrt{e^{\delta x + \frac{1}{2} A(0, x)}}.$$

Applying Hölder's inequality one gets

$$E\left[G'(T, T)\right] = A_{x,R} \cdot S_{x,R}^{-1} = E_T \left[f(T)^2 \cdot E_T \left[g(T)^2\right] - 1 \right. \geq E_T \left[f(T) \cdot g(T)\right]^2 - 1 = E_T \left[e^{\frac{1}{2} A(0, T)}\right] - 1 \geq 0. \quad \square$$

6. Condition (C) for the whole life insurance. For the typical case of a whole life insurance with a level net premium plan, the covariance in condition (C) reads

$$C_{C,R} := \text{Cov}[P(0, T), A(T)] = P_{s,R} \cdot \text{Cov}[\alpha_T, A(T)].$$

In case this covariance is positive, the expected retrospective underwriting gain will be lower for the single net premium plan than for the level net premium plan. In the following some useful mathematical results related to this covariance function are derived and interpreted. The investigation is restricted to a dual random Wiener model of type (1.4). It is not difficult to show that

$$C_{C,R} = P_{s,R} \cdot \text{Cov}[\alpha_T, A(T)] = P_{s,R} \left[E_T \left[e^{\frac{1}{2} A(0, T)}\right] - E_T \left[A(T)\right] \cdot E_T \left[e^{\frac{1}{2} A(0, T)}\right]\right]$$

$$= \frac{A^*_s}{\overline{A}_s} \cdot \left\{ \overline{S}_s^* - \overline{S}_s^{**} \cdot \overline{A}_s^* \right\}. \quad \text{(6.2)}$$
Lemma 6.1. In a dual random Wiener environment of type (1.4), the following conditions are equivalent:

\( (6.3) \quad \text{Cov}[\overline{\alpha}_T, A(T)] \geq 0 \)

\( (6.4) \quad \overline{s}^{**} \geq \overline{S}^{**} \cdot \overline{a}^* \)

\( (6.5) \quad \frac{1}{\overline{a}^*} \geq \delta^{**} + \frac{1}{\overline{s}^{**}} \)

\( (6.6) \quad \delta^{**} \cdot \overline{A}^* + \delta^* \geq \frac{\delta^*}{\overline{s}^{**}} + \delta^{**} \)

Proof. Use the relations \( \delta^{**} \cdot \overline{s}^{**} = \overline{s}^{**} - 1 \) and \( \delta^* \cdot \overline{a}^* = 1 - \overline{A}^* \).

Remark 6.1. In practical calculations one often uses the well-known integral representation

\( (6.7) \quad \overline{a}_x = \int_0^x e^{-\delta t} p_x dt \).

Example 6.1. In terms of the force of mortality, Makeham's analytical law of mortality is given by

\( (6.8) \quad \mu_{x+t} = a + bc^{x+t}, \quad a + b > 0, \quad c > 1, \)

and the corresponding future lifetime distribution reads

\( (6.9) \quad F(t) = p_x = 1 - \exp\{ - \left( at - mc^x (c^y - 1) \right) \}, \quad m = \frac{b}{\ln[c]} \).

The special case \( b=0 \) is an exponential distribution and \( a=0 \) gives Gompertz's law. An illustrative life table is found in Bowers et al. [4], p. 72. Formulas for life annuities can be found in Thalmann [14], Chan [6], Kling [12]. One has

\( (6.10) \quad \overline{a}_x(\delta) = \frac{e^{\alpha x} \Gamma(z, \alpha)}{\ln[c]}, \quad \text{with} \quad \alpha = \frac{bc^z}{\ln[c]}, \quad z = -\frac{a + \delta}{\ln[c]} \).

One knows that margins in the force of interest \( \delta \) can be exchanged with margins in the constant part \( a \) of the force of mortality. Since \( F(t) \) increases stochastically in \( a \), an increasing interest is equivalent to a stochastic increase in mortality. Viewing the life annuity as a function of two variables \( \overline{a}_x(a, \delta) \), this means that one has the invariant property

\( (6.11) \quad \overline{a}_x(a, \delta + \epsilon) = \overline{a}_x(a + \epsilon, \delta) \) for all \( \epsilon \geq 0 \).
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Using the identity \( \delta \mathcal{A}_x(\delta) = 1 - \mathcal{A}_x(\delta) \) and the inequality \( \mathcal{A}_x(a, \delta) \leq \mathcal{A}_x(a + \varepsilon, \delta) \) (stochastic decrease of \( T \) in \( a \), i.e. \( T_{a+\varepsilon} \leq T_a \)), one gets

\[
\delta \mathcal{A}_x(a, \delta + \varepsilon) = \delta \mathcal{A}_x(a + \varepsilon, \delta) = 1 - \mathcal{A}_x(a + \varepsilon, \delta) \leq 1 - \mathcal{A}_x(a, \delta) = \delta \mathcal{A}_x(a, \delta),
\]

which shows that the life annuity \( \mathfrak{P}_x(\delta) \) is a monotone decreasing function of the force of interest.

In general it seems difficult to validate without further assumption one of the above equivalent conditions. However a precise sufficient condition of "low market volatility" can be derived. This renders rigorous (in our special situation) the intuitive statement made earlier by Ramsay [13], p. 505.

**Proposition 6.1.** Given is a whole life insurance with a level net premium plan and a dual random Wiener model. Assume the following inequality holds:

\[
\frac{1}{2} \sigma^2 \leq \delta \cdot \left( \frac{\mathcal{S}_x(\delta) \mathcal{A}_x(\delta) - 1}{\mathcal{S}_x(\delta) - 1 + \mathcal{S}_x(\delta)(1 - \mathcal{A}_x(\delta))} \right).
\]

Then one has \( \text{Cov}[\mathcal{R}_T, A(T)] \geq 0 \) and condition (C) is satisfied.

**Proof.** (6.6) is equivalent to the inequality \( \mathcal{S}_x^{**} \cdot (\delta^{**} \cdot \mathcal{A}_x^{**} - (\delta^{**} - \delta^{**})) \geq \delta^{**} \). Setting \( \varepsilon = \frac{1}{2} \sigma^2 \) one has \( \delta^{**} = \delta - \varepsilon, \ \delta^{**} = \delta + \varepsilon \). The inequalities

\[
\mathcal{A}_x^{**} = \mathcal{A}_x(\delta - \varepsilon) \geq \mathcal{A}_x(\delta), \ \mathcal{S}_x^{**} = \mathcal{S}_x(\delta + \varepsilon) \geq \mathcal{S}_x(\delta),
\]

imply that it is sufficient to satisfy the stronger inequality

\[
\mathcal{S}_x(\delta) \cdot ((\delta + \varepsilon) \cdot \mathcal{A}_x(\delta) - 2 \varepsilon) \geq \delta - \varepsilon.
\]

If one assumes that \( \varepsilon \leq \delta \cdot \left( \frac{\mathcal{A}_x(\delta)}{2 - \mathcal{A}_x(\delta)} \right) \), one gets the inequality (6.13). However since \( \mathcal{A}_x(\delta) \leq 1 \) one has

\[
\frac{\mathcal{S}_x(\delta) \mathcal{A}_x(\delta) - 1}{\mathcal{S}_x(\delta) - 1 + \mathcal{S}_x(\delta)(1 - \mathcal{A}_x(\delta))} \leq \frac{\mathcal{A}_x(\delta)}{2 - \mathcal{A}_x(\delta)},
\]

which shows that the assumption about \( \varepsilon \) is fulfilled. \( \Box \)
In the context of Arbitrage Pricing Theory, especially Option Pricing Theory, one requires the relation
\[ \delta = \delta_f - \frac{1}{2} \sigma^2, \]
where \( \delta_f \) is the risk-free rate, in order to satisfy the condition of no-arbitrage. This relation is equivalent to the relations
\[ \delta^* = \delta_f - \sigma^2, \quad \delta^{**} = \delta_f. \]
In this arbitrage-free framework, let us derive an equivalent but simpler sufficient condition of "low market volatility", which is required to guarantee a positive covariance \( \text{Cov}[\bar{A}_T, A(T)] \geq 0 \).

**Theorem 6.1.** Let \( \delta \) be the risk-free rate and set \( \delta^* = \delta - \sigma^2, \quad \delta^{**} = \delta \). Suppose the volatility satisfies the condition

\[ \sigma^2 \leq \delta \cdot \left( \frac{\bar{S}_x(\delta) \bar{A}_x(\delta) - 1}{\bar{S}_x(\delta) - 1} \right). \tag{6.14} \]

Then the following inequality holds:

\[ \frac{1}{\bar{a}_x(\delta^*)} \geq \delta + \frac{1}{\bar{S}_x(\delta)}. \tag{6.15} \]

**Proof.** With \( \delta x = \bar{S}_x - 1 \) and \( \delta^* \cdot \bar{a}_x^* = 1 - \bar{A}_x^* \), the inequality (6.15) is equivalent to
\[ \bar{S}_x \cdot (\delta \cdot \bar{A}_x^* - \sigma^2) \geq \delta - \sigma^2, \]
which is satisfied provided
\[ \sigma^2 \leq \delta \cdot \left( \frac{\bar{S}_x \bar{A}_x^* - 1}{\bar{S}_x - 1} \right) \leq \delta \bar{A}_x. \]

To check this, apply the same method as in the proof of Proposition 6.1. □

**Comment 6.1.** In the classification introduced by Bühlmann, editorial to ASTIN Bulletin 17.2, (1987), the inequality (6.15), together with its sufficient condition (6.14), may be viewed as a typical statement of actuarial mathematics of the *third kind*. The corresponding assertion of the *second kind* is the inequality

\[ \frac{1}{\bar{a}_x(\delta)} \geq \delta + \frac{1}{\bar{S}_x(\delta)}, \tag{6.16} \]

obtained setting \( \sigma = 0 \) in (6.15), and which is thus always satisfied (no random fluctuations of the rate of return on investment). The inequality (6.16) is further equivalent to the inequality (validated by the inequality of Schwartz)

\[ \bar{S}_x(\delta) \cdot \bar{A}_x(\delta) = E_T \left[ e^{\sigma T} \right] \cdot E_T \left[ e^{-\sigma T} \right] \geq 1, \tag{6.17} \]
Equality holds only if $\delta=0$, or if $T=\text{const}$. In the latter deterministic case, one recovers the classical "compound interest theory" identity of the first kind between continuous annuities certain

$$\frac{1}{a_t} = \delta + \frac{1}{s_t},$$

of which there exists a nice interpretation (e.g. Gerber(1986), p. 52). Setting $t = e^{\delta t}$, the expectation of life at age $x$, one gets a further classical actuarial relation of the first kind. To summarize the discussion, Theorem 6.1 can be viewed as a third kind generalization of an elementary result, which has useful interpretations of all three actuarial kinds.

References


Received: November 6, 2007