Int. J. Contemp. Math. Sciences, Vol. 3, 2008, no. 12, 575 - 582

On Some Properties of Regular Holomorphic Solutions of a Class of Higher Order Operator-Differential Equations

Rovshan Z. Gumbataliyev

Institute of Mathematics and Mechanics of NAS of Azerbaijan
9, F. Agayev str., AZ1141, Baku, Azerbaijan
rovshangumbataliev@rambler.ru

Abstract

In the paper we give definition of regular holomorfic solutions of a class of higher order operator-differential equations and Phragmen-Lindelof type theorem is proved for these solutions.

Mathematics Subject Classification: 46C05, 39B42, 32A10

Keywords: Hilbert space, operator-differential equations, holomorphic function

1. Introduction

In the paper [1] P.D.Lax gives definition of intrinsic compactness for some spaces of solutions in infinite interval and indicates its close connection with Phrag-men-Lindelof principles for the solutions of elliptic equations. Such theorems for abstract equations were obtained in the papers [2-3]. In our case the main difference from the above-indicated papers is that a principal part of the equation has a complicated- miltiple character and therefore our conditions essentially differ from the ones of the indicated papers. Notice that the found conditions are expressed by the operator coefficients of the equation.

2. Problem statement

On a separable Hilbert space H consider on operator differential equation

$$P\left(\frac{d}{d\tau}\right)u\left(\tau\right)\equiv\left(\frac{d^2}{d\tau^2}-A^2\right)^m u\left(\tau\right)+\sum_{i=0}^{2m}A_{2m-j}u^{(j)}\left(\tau\right)=0,\ \ \tau\in S_{\pi/2},\quad (1)$$

where $S_{\pi/2}$ is a corner vector

$$S_{\pi/2} = \{\tau / |\arg\tau| < \pi/2\},$$

and $u(t)$ is a vector –valued holomorphic function determined in $S_{\pi/2}$ with values from H, the operator A is positive- definite self-adjoint, $A_j \left(j = \overline{1,2m}\right)$ are linear operators in H, $A_j A^{-j} \left(j = \overline{1,2m}\right)$, are bounded in H. All the derivatives are understood in the sense of complex variable theory [4]. By $H_{2,\alpha}$ we denote a space of vector-functions $f(\tau)$ with values in H that are holomorphic in the sector $S_{\pi/2}$, moreover

$$\sup_{\varphi:|\varphi|<\frac{\pi}{2}} \int_0^\infty \left\| f\left(te^{i\varphi}\right)\right\|_H^2 dt < \infty, \quad \left(\tau = te^{i\varphi}\right).$$

Lets introduce the space $W_{2,\alpha}^{2m}(H)$ as a class of vector- functions $u(\tau)$ with values in H that are holomorphic in the sector $S_{\pi/2}$ and for which

$$\sup_{\varphi:|\varphi|<\frac{\pi}{2}} \int_0^\infty \left(\left\| \frac{d^{2m}}{dt^{2m}} u\left(te^{i\varphi}\right)\right\|_H^2 + \left\|A^{2m} u\left(te^{i\varphi}\right)\right\|_H^2 \right)^{1/2} dt < \infty, \quad \left(\tau = te^{i\varphi}\right).$$

Definition. *If the vector-function* $u(\tau) \in W_{2,\alpha}^{2m}(H)$ *satisfies the equation (1) in* $S_{\pi/2}$ *identically, it is said to be a regular holomorphic solution of the equation (1).*

By $U_\beta^{(\alpha)}$ *we denote a set of regular holomorphic solutions of the equation (1) for which* $e^{\beta\tau} u(\tau) \in H_{2,\alpha}$.

Obviously,

$$U_0^{(\alpha)} = Ker P(d/d\tau) = \{u / P(d/d\tau)\, u(\tau) = 0\},$$

and for $\tau \geq 0$

$$U_\tau^{(\alpha)} = \left\{ u/u \in U_0^{(\alpha)}, e^{\tau\alpha} u(\tau) \in H_{2,\alpha} \right\}.$$

3. Some auxiliary facts

Lets consider some facts that well need in future. It holds

Lemma 1. *The set* $U_0^{(\alpha)}$ *is close in the norm* $\|u\|_{w_{2,\alpha}^{2m}}$.

Proof. Let $\{u_n(\tau)\}_{n=1}^\infty \subset U_0^{(\alpha)}$ and let $\|u_n(\tau) - u(\tau)\|_{W_{2,\alpha}^{2m}} \to 0$. Then, obviously $u(\tau) \in W_{2,\alpha}^{2m}$. Show that $u(\tau) \in U_0^{(\alpha)}$, i.e. $P(d/d\tau)\, u(\tau) = 0$. Since

$\|u_n(\tau)-u(\tau)\|_{W_{2,\alpha}^{2m}}\to 0$, by the theorem on intermediate derivatives [4] for each j $(0\le j\le 2m)$ a sequence $\left\{A_{2m-j}u_n^{(j)}(\tau)\right\}_{n=1}^{\infty}$ converges on the space $H_{2,\alpha}$. Indeed, as $n\to\infty$

$$\left\|A_{2m-j}u_n^{(j)}(\tau)-A_{2m-j}u^{(j)}(\tau)\right\|_{H_{2,\alpha}}\le const\,\|u(\tau)-u_n(\tau)\|_{W_{2,\alpha}^{2m}}\to 0.$$

Show that the sequence $\left\{A_{2m-j}u_n^{(j)}(\tau)\right\}_{n=1}^{\infty}$ uniformly converges is any compact $S\subset S_\alpha$. Really, since $A_{2m-j}u_n^{(j)}(\tau)$ and $A_{2m-j}u^{(j)}(\tau)\in H_{2,\alpha}$, here exists their boundary value in the sense $L_2(R_+;H)$, respectively $Z_{n,j}^{\pm}(t)$ and $Z_j^{\pm}(t)\in L_2(R_+;H)$. Since

$$A_{2m-j}u^{(j)}(\tau)=\frac{1}{2\pi i}\int_0^\infty\frac{z_{n,j}^{-}(\xi)}{\xi e^{-i\alpha}-\tau}e^{-i\alpha}d\xi-\frac{1}{2\pi i}\int_0^\infty\frac{z_{n,j}^{+}(\xi)}{\xi e^{i\alpha}-\tau}e^{i\alpha}d\xi,$$

$$A_{2m-j}u^{(j)}(\tau)=\frac{1}{2\pi i}\int_0^\infty\frac{z_{j}^{-}(\xi)}{\xi e^{-i\alpha}-\tau}e^{-i\alpha}d\xi-\frac{1}{2\pi i}\int_0^\infty\frac{z_{j}^{+}(\xi)}{\xi e^{i\alpha}-\tau}e^{i\alpha}d\xi$$

and

$$\left\|A_{2m-j}u_n^{(j)}(\tau)-A_{2m-j}u^{(j)}(\tau)\right\|_{H_{2,\alpha}}=$$

$$=\frac{1}{\sqrt{2}}\left(\left\|z_{n,j}^{-}(\xi)-z_j^{-}(\xi)\right\|_{L_2(R_+;H)}^2+\left\|z_{n,j}^{+}(\xi)-z_j^{+}(\xi)\right\|_{L_2(R_+;H)}^2\right)^{1/2}\to 0,$$

then

$$\left\|z_{n,j}^{-}(\xi)-z_j^{-}(\xi)\right\|_{L_2(R_+;H)}\to 0,\quad \left\|z_{n,j}^{+}(\xi)-z_j^{+}(\xi)\right\|_{L_2(R_+;H)}^2\to 0.$$

Obviously

$$\sup_{\tau\in s}\left\|A_{2m-j}u_n^{(j)}(\tau)-A_{2m-j}u^{(j)}(\tau)\right\|\le\sup_{\tau\in s}\frac{1}{2\pi}\int_0^\infty\frac{\left\|z_{n,j}^{-}(\xi)-z_j^{-}(\xi)\right\|}{\left|\xi e^{-i\alpha}-\tau\right|}d\xi+$$

$$+\sup_{\tau\in s}\frac{1}{2\pi}\int_0^\infty\frac{\left\|z_{n,j}^{+}(\xi)-z_n^{+}(\xi)\right\|}{\left|\xi e^{-i\alpha}-\tau\right|}d\xi\le$$

$$\leq \frac{1}{2\pi}\left(\left\| z_{n,j}^{-}(\xi)-z_{j}^{-}(\xi)\right\|^{2}d\xi\right)^{1/2}\left(\int\limits_{0}^{\infty}\frac{1}{\left|\xi e^{-i\alpha}-\tau\right|}d\xi\right)^{1/2}+$$

$$+\frac{1}{2\pi}\left(\left\| z_{n,j}^{+}(\xi)-z_{j}^{+}(\xi)\right\|^{2}d\xi\right)^{1/2}\left(\int\limits_{0}^{\infty}\frac{1}{\left|\xi e^{-i\alpha}-\tau\right|}d\xi\right)^{1/2}. \qquad (2)$$

Since $\tau\in S\subset S_{\alpha}$ then

$$\sup_{\tau\in s}\int\limits_{0}^{\infty}\left|\xi e^{-i\alpha}-\tau\right|^{-2}d\xi\leq\int\limits_{0}^{\infty}\sup_{\tau\in s}\left|\xi e^{-i\alpha}-\tau\right|^{-2}d\xi\leq const.$$

Therefore, as $n\to\infty$ it follows from (2) that

$$\sup_{\tau\in s}\left\|A_{2m-j}u_{n}^{(j)}(\tau)-A_{2m-j}u^{(j)}(\tau)\right\|\to 0,$$

i.e. $\left\{A_{2m-j}u_{n}^{(j)}\right\}_{n=1}^{\infty}$ uniformly converges in the compact to the vector-function $A_{2m-j}u^{(j)}(\tau)$. On the other hand

$$\sup_{\tau\in s}\left\|P\left(d/d\tau\right)u_{n}(\tau)-P\left(d/d\tau\right)u(\tau)\right\|$$

$$=\sup_{\tau\in s}\left\|\sum_{j=0}^{2m}A_{2m-j}u_{n}^{(j)}(\tau)-\sum_{j=0}^{2m}A_{2m-j}u^{(j)}(\tau)\right\|\leq$$

$$\leq\sum_{j=0}^{2m}\left\|A_{2m-j}A^{-(2m-j)}\right\|\sup_{\tau\in s}\left\|A^{(2m-j)}u_{n}^{(j)}(\tau)-A^{(2m-j)}u^{(j)}(\tau)\right\|\leq$$

$$\leq const\sum_{j=0}^{2m}\sup_{\tau\in s}\left\|A^{(2m-j)}u_{n}^{(j)}(\tau)-A^{(2m-j)}u^{(j)}(\tau)\right\|.$$

On from $u_{n}(\tau)\in U_{0}^{(\alpha)}\left(P\left(d/d\tau\right)u(\tau)=0\right)$ we get

$$\sup_{\tau\in s}\left\|P\left(d/d\tau\right)u(\tau)\right\|\leq const\sum_{j=0}^{2m}\sup_{\tau\in s}\left\|A^{(2m-j)}u_{n}^{(j)}(\tau)-A^{(2m-j)}u^{(j)}(\tau)\right\|,$$

it follows the convergence in S

$$A^{(2m-j)}u_n^{(j)}(\tau) \to A^{(2m-j)}u^{(j)}(\tau)$$

that $P(d/d\tau)\,u(\tau) = 0$ i.e. $u(\tau) \in U_0^{(\alpha)}$. Lemma is proved.

Lemma 2. *Let A be a positive self-adjoint operator and one of the following conditions be fulfilled:*

1) $A^{-1} \in \sigma_p\,(0 < p < \infty)$, $A_j A^{-j}\,\left(j = \overline{1, 2m}\right)$ are bounded H and solvability condition hold;

2) $A^{-1} \in \sigma_p\,(0 < p < \infty)$. Then if $u(\tau) \in U_0^{(\alpha)}$, then for its Laplace transformation $\hat{u}(\lambda)$ estimation

$$\left\|\hat{u}(\lambda)\right\| \le const\,(|\lambda + 1|)^{-1}, \ \lambda \in \left\{\lambda/\,|\arg\lambda| < \frac{\pi}{2} + \alpha\right\}$$

is true.

Proof. It is easily seen that

$$\left\|\hat{u}(\lambda)\right\| = \frac{1}{\sqrt{2}}P^{-1}(\lambda)\sum_{j=0}^{2m-1} Q_q(\lambda)\,u^{(q)}(0),$$

where

$$Q_q(\lambda) = \sum_{j=0}^{2m-q-1} \lambda^{2m-q-j}\tilde{A}_j$$

and

$$\tilde{A}_j = \begin{cases} A_j, & j = 1, 3, .., 2k-1, k = \overline{1, m} \\ A_j + (-1)^{j/2}\,C_m^{j/2}A^j, & j = 2, 4, .., 2m-2, 2m. \end{cases}$$

Obviously, an operator pencil $p(\lambda)$ is represented in the form $P(\lambda) = P_0(\lambda) + P_1(\lambda)$, where

$$P_0(\lambda) = \left(-\lambda^2 E + A^2\right)^m, \quad P_1(\lambda) = \sum_{j=0}^{2m} \lambda^j A_{2m-j},$$

and on the rays $\Gamma_{\pm\left(\frac{\pi}{2}+\alpha\right)} = \left\{\lambda/\arg\lambda = \pm\frac{\pi}{2} + \alpha\right\}$ in case 1) from the solvability condition, in case 2) from Keldysh lemma [5] it follow that

$$\sup_{\tau\in\Gamma_{\pm\left(\frac{\pi}{2}+\alpha\right)}} \left\|P(\lambda)\,P_0^{-1}(\lambda)\right\| \le const. \tag{3}$$

Therefore, from these rays

$$\left\|\hat{u}(\lambda)\right\| = \left\|P^{-1}(\lambda)\left(\sum_{q=0}^{2m-1} Q_q(\lambda)\, u^{(q)}(0)\right\|\right.$$

$$= \left\|\left(P_0(\lambda) + P_1(\lambda)\right)^{-1} \sum_{q=0}^{2m-1} Q_q u^{(q)}(\lambda)\, u^q(0)\right\| =$$

$$= \left\|\left(P(\lambda)\, P_0^{-1}(\lambda)\, P_0(\lambda)\right)^{-1} \sum_{q=0}^{2m-1} Q_q(\lambda)\, u^{(q)}(0)\right\| =$$

$$= \left\|\left(P_0^{-1}(\lambda)\, P(\lambda)\, P_0^{-1}(\lambda)\right)^{-1} \sum_{q=0}^{2m-1} Q_q(\lambda)\, u^{(q)}(0)\right\|. \tag{4}$$

But on these identities it hold (3) and

$$\left\|P_0^{-1}(\lambda)\right\| \le const\left(|\lambda|^2 + 1\right)^{-1} \tag{5}$$

$$\left\|\sum_{q=0}^{2m-1} Q_q(\lambda)\, u^{(q)}(0)\right\| \le const\left|\lambda^{2m-1}\right|. \tag{6}$$

Thus, from (4) allowing for (3), (5) and (6) on these rays we got the estimation

$$\left\|\hat{u}(\lambda)\right\| \le const\,(|\lambda| + 1)^{-1}.$$

Notice that the angle between the rays $\Gamma_{\pm\left(\frac{\pi}{2}+\alpha\right)}$ equals $(\pi + 2\alpha)$. It follows from, conditions 1) and 2) that $P^{-1}(\lambda)$ is a meramorphic operator-function of order ρ and minimal type of order ρ, i.e. it is represented in the form of relations of two entire functions of order $\rho$and minimal type for order ρ. Since $\hat{u}(\lambda)$ is a holomorphic vector-function in the domain (see [6])

$$S_{\frac{\pi}{2}-\alpha} = \left\{\lambda / |\arg\lambda| < \frac{\pi}{2} + \alpha\right\}$$

and the angle between the rays equals $\pi + 2\alpha$, for $0 \le \rho \le \pi/(\rho + 2\alpha)$ it follows from the Phragmen-Lindelof theorem that in the sector $S_{\frac{\pi}{2}+\alpha}$ the estimation

$$\left\|\hat{u}(\lambda)\right\| \le const\,(|\lambda| + 1)^{-1}$$

holds.

In case 2) M.V. Keldysh theorem [5] fields that there exist the rays between which the angles are less than π/ρ and the estimation (3), (5), (6) and the estimation

$$\left\|\hat{u}(\lambda)\right\| \leq const\,(|\lambda|+1)^{-1}$$

hold. The lemma is proved.

4. The main result

Now, lets prove a theorem on Phragmen-Lindelof principle.

Theorem. *Let A be a positive-definite self-adjoint operator, $A_j A^{-j}$ $\left(j=\overline{1,2m}\right)$ be bounded in H and one of the two conditions hold:*

1) $A^{-1}\in\sigma_p$ $(0<p<\pi/(\pi+2\alpha))$ and solvability condition holds;

2) $A^{-1}\in\sigma_p$ $(0<p<\infty)$. Then, if a regular holomorphic solution $u(\tau)\in\bigcap\limits_{\tau\geq 0} u_\tau^{(\alpha)}$, then $u(\tau)=0$.

Proof. For $u(\tau)\in U_\tau^{(\alpha)}$ it follows that the Laplace transformation $\hat{u}(\lambda)$ admits holomorphic continuation to the domain $\left\{\lambda / |\arg(\lambda+\tau)|<\frac{\pi}{2}+\alpha\right\}$. Then inclusion $u(\tau)\in\bigcap\limits_{\tau\geq 0}\left(U_\tau^{(\alpha)}\right)$ implies that $\hat{u}(\lambda)$is an entire function, i.e. it is an entire function and $\hat{u}(\lambda)=P^{-1}(\lambda)\sum\limits_{q=0}^{2m-1}Q_q(\lambda)\,u^{(q)}(0)$, moreover $\hat{u}(\lambda)$ is an entire function of order ρ and of minimal type for order ρ. Further, in the second case, we can use the Keldysh lemma and obtain that $\hat{u}(\lambda)$ is an entire function of order ρ and of minimal type for order ρ on all the complex half-plane

$$\left\|\hat{u}(\lambda)\right\| \leq const\,(|\lambda|+1)^{-1}.$$

In the first case, since the angle between $\Gamma_{\pm\left(\frac{\pi}{2}+\alpha\right)}$ in the left half-plane $\pi-2\alpha$ and $\pi-2\alpha<\pi+2\alpha$, then again from the Phragmen-Lindelof theorem is follow, that $\hat{u}(\lambda)$ is an entire function of order ρ and of minimal type for order ρ and on the complex plane

$$\left\|\hat{u}(\lambda)\right\| \leq const\,(|\lambda|+1)^{-1}.$$

Thus, as $n\to\infty$, $\hat{u}(\lambda)=0$. Hence, it follows that $u(\tau)=0$. The theorem is proved.

References

[1] Lax P.D. Phragmen-Lindelof theorem in harmonic analysis and its application to some questions in the elliptic equations. Comm. Pure Appl. Math., 1957, v.10.

[2] Agmon S., Nirenberg L. Properties of solutions of ordinary linear differential equations in Banach space. Comm. Pure Appl. Math., 1963, v.16.

[3] Yakubov S.Ya. Linear differential- operator equations and their applications. Baku, "Elm", 1985, 22pp.

[4] Lions J.-L., Majenes E. Inhomogeneous boundary value problems and their applications. M., "Mir", 1971, 371 pp.

[5] Keldysh M.V. On completeness of eigen functions of some classes not self-adjoints operators. Uspekhi Math. Nauk, 1971, No4.

[6] Keldysh M.V. On eigen- functions of some class of not self-adjoint equations DAN SSSR, 1951, v.77, No 1.

Received: February 5, 2008