Approximation of Stochastic $\alpha$-Integrals 
in the Generalized Random Processes Algebra

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Abstract

In this paper the problem of stochastic $\alpha$-integrals approximation by
the finite sums of the elements from the algebra of generalized random
processes is investigated.

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1. Introduction

The theory of distributions by Swartz [23] is the principle instrument for
investigations in modern theory of differential equations. Unfortunately, it is
not appropriate to solve the nonlinear equations. Nevertheless, in the first
works by Dirac, in which the distributions were introduced, the necessity for
definition of product of distributions has appeared [5].

Many authors tried to solve this problem, using their own approaches for
the definition of generalized functions. Let us note, in particular, the re-
sults by Mikusinski [16]. For the first time the positive solution was found by
Colombeau [4]. By him and his followers [18,24] the various algebra of objects
were investigated and were named as "new generalized functions". The most
extensive one of such kind of algebras was offered and investigated by Egorov
[6].

In the paper [2] the general approach for construction of the algebra of
generalized functions was offered. In the parallel, with the development of the
theory of generalized functions there was a development of the theory of gen-
eralized random processes [1,3]. So on the basis of the theory of distributions
by Swartz the theory of generalized random processes was created by Gelfand [7].

In the paper [25] the own treatment of generalized random processes, using the sequential approach by Mikusinski, was offered by Urbanik. It is necessary to emphasize, that despite of the importance of the given theories in the modern theory of random processes, they have not found wide uses in the differential equations with stochastic functions because they are not appropriate to solve the nonlinear equations. In the work [12] on the basis of algebra of generalized functions by Egorov the algebra of generalized random processes was introduced. According to the concept of generalized stochastic differential in the paper [13] is shown that both stochastic integrals by Ito and stochastic integrals by Stratonovich can be approximated by the finite sums of elements from this algebra.

It should be noted, that the stochastic differential equations theory was based initially on the concepts of stochastic integrals by Ito and Stratonovich. The first integral is more often used by mathematicians, and second one is used by natural scientists. Nevertheless, many authors have not refused attempts of investigation of these integrals and attempt to solve the stochastic differential equations by the usual method from the non-stochastic analysis point of view at the approximation level [14,15,17,26].

Currently, the numerous generalizations of stochastic integrals by Ito and Stratonovich are known $\alpha$-integrals, stochastic integrals by Ogawa and others [19-22].

In this paper the problem of stochastic $\alpha$- integrals approximation by the finite sums of the elements from the algebra of generalized random processes is investigated. This work consists of two parts. In the first part the conditions of convergence are found, and in the second part the estimations of the convergence rate of the elements of algebra of generalized random processes to stochastic $\alpha$- integrals are investigated.

Let us remind some concepts from work [11], which we shall use further.

Let $(\Omega, A, P)$ be absolute stochastic space, $T = [0, a] \subset R$, $\{F_t\}_{t \in T}$ - standard flux of $\sigma$-algebra, $F_a \subset A$.

Let us consider set of functions

$$f_n(t, \omega) : N \times T \times \Omega \rightarrow R,$$

such that

1) $f_n(t, \omega)$ is the random variable on the $(\Omega, A, P)$ for any $t \in T, n \in N$;
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2) $f_n(t, \omega) \in C^\infty(R)$ for any $n \in N$ and for almost all $\omega \in \Omega$. Let the element $F = (f_n(t, \omega))$ and $G = (g_n(t, \omega))$ of the given set denominate equivalent, if there is $n_0$ that for any $t \in T$, almost all $\omega \in \Omega$ and any $n > n_0$: $f_n(t, \omega) = g_n(t, \omega)$.

Let us designate the set of such kind equivalent sequences classes as $G(T, \Omega)$, this is the algebra with on - coordinate operations of addition and multiplication.

Further let

$$\tilde{T} = \{ \tilde{t} = [(t_n)] \in \tilde{R} : \forall (t_n) \in \tilde{t}, \ 0 \leq t_n \leq a, \ n = 1, 2, \cdots \},$$

where $\tilde{R}$ - extended straight line from [6]. Let us designate the algebra of
generalized random processes $\tilde{F}(\tilde{t}, \omega)$ such as

$$\tilde{F}(\tilde{t}, \omega) = [(f_n(t_n, \omega))]$$

as $G(\tilde{T}, \Omega)$, where $\tilde{t} = [(t_n)] \in \tilde{T}$, and $[(f_n(t_n, \omega))] \in G(T, \Omega)$.

We shall affirm, that generalized random process

$$F(t, \omega) = [(f_n(t, \omega))] \in G(T, \Omega)$$

associates the classic random process if for $\forall (f_n) \in F$ at $n \to \infty$, $(f_n(t, \omega)$ converges to the given process in $D'(T)$ and in terms of probability. Let $B(t, \omega), \ t \in T, \ \omega \in \Omega$ be unidimensional standard processes of $F_1$ - Brownian motion [9-11]. Let us designate nonnegative functions from $C^\infty(R)$, which carrier contains in the interval $[0, 1]$, as $\rho(t), \ t \in R$

$$\int_0^1 \rho(t)dt = 1,$$  \hspace{1cm} (1)

and $\rho_n(t) = n \rho(nt), \ n \in N, \ \rho_n^1(t) = \varphi(n)\rho(\varphi(n)t), \ n \in N, \ \varphi : R \to R$ is the monotonic nondecreasing function.

Let us consider the element of algebra $G(T, \Omega)$ as generalized random processes of Brownian motion,

$$\tilde{B}(t, \omega) = [(B_n(t, \omega))] = [(\beta B_n^1(t, \omega) + (1 - \beta)B_n^s(t, \omega))], \ \beta \in R$$

where

$$B_n^s(t, \omega) = (B * \rho_n)(t, \omega) = \int_0^\frac{1}{n} B(t + s, \omega)\rho_n(s)ds,$$

$$B_n^1(t, \omega) = (B * \rho_n^1)(t, \omega) = \int_0^\frac{1}{n} B(t + s, \omega)\rho_n^1(s)ds.$$

(2)
Let us consider partition \(0 = t_0 < t_1 < \cdots < t_m = a\) of interval
\[
T = [0, a], \quad \lambda_m = \max_{1 \leq k \leq m} (t_k - t_{k-1}).
\]

Then for \(\forall \ t \in T \exists p = p(t) \in N, \) that \(t_p < t < t_{p+1}\).

Let us suppose \(S_m f_n(t, \omega) = \sum_{k=1}^{p} [f_n(t_k, \omega) - f_n(t_{k-1}, \omega)]^2 + [f_n(t, \omega) - f_n(t_p, \omega)]^2\), by definition.

1. Second variation of generalized random process of the Brownian motion

**Lemma 1:** Let \(\tilde{f}(t, \omega) = [(f_n(t, \omega))], \ \tilde{g}(t, \omega) = [(g_n(t, \omega))]\) be the elements of \(G(T, \Omega)\) algebra, with

\[
\sup_{t \in T} E[S_m f_n(t, \omega) - f(t, \omega)]^2 \to 0, \ \sup_{t \in T} E[S_m g_n(t, \omega)]^2 \to 0,
\]

when \(m, n \to \infty\).

Then \(\sup_{t \in T} E[S_m (f_n + g_n)(t, \omega) - f(t, \omega)]^2 \to 0, \) when \(m, n \to \infty\).

**Proof:** It is simple to be convinced that

\[
E[S_m (f_n + g_n)(t, \omega) - f(t, \omega)]^2 \leq 3E[S_m f_n(t, \omega) - f(t, \omega)]^2 + 3E[S_m g_n(t, \omega)]^2 \\
+ 12E[\sum_{k=1}^{p} (f_n(t_k, \omega) - f_n(t_{k-1}, \omega))(g_n(t_k, \omega) - g_n(t_{k-1}, \omega)) + (f_n(t, \omega) - f_n(t_p, \omega))(g_n(t, \omega) - g_n(t_p, \omega))]^2 = 3I_1 + 3I_2 + 12I_3.
\]

Under condition

\[
I_1, I_2 \to 0,
\]

uniformly on \(t \in T, \) when \(m, n \to \infty\). Estimating \(I_3, \) we shall using Cauchy-Bunyakovsky inequalities for the finite sums and integrals. As a result we shall obtain

\[
I_3 \leq E[(S_m f_n(t, \omega))(S_m g_n(t, \omega))] \leq (E[S_m f_n(t, \omega)]^2 E[S_m g_n(t, \omega)]^2)^{1/2}.
\]

It is obvious from (3), that \(E[S_m f_n(t, \omega)]^2\) is limited. From here follows that

\[
I_3 \to 0, \text{ when } m, n \to \infty \text{ uniformly on } t \in T.
\]

The proof of the given statement follows from (4)-(6).

**Lemma 2:** Let \(\tilde{B}(t, \omega) = [B_n(t, \omega)]\) be generalized random process of Brownian motion from the representation (2), then if \(\varphi(n) \to \infty, \ n \to \infty, \ \lambda_m \to 0, \) so that

\[
\lambda_m = o(1/n), \ 1/\varphi(n) = o(\lambda_m),
\]
then for $\forall \beta \in R$

$$\sup_{t \in T} E[S_m(\beta B^l_n + (1 - \beta) B^s_n)(t, \omega) - \beta^2 t]^2 \to 0.$$ 

**Proof:** It is known from [12], that when $n \to \infty$, $\lambda_m \to 0$ so that $\lambda_m = o(1/n)$

$$\sup_{t \in T} E[S_m B^s_n(t, \omega)]^2 \to 0$$

and if $\varphi(n) \to \infty$, $\lambda_m \to 0$ so that $1/\varphi(n) = o(\lambda_m)$, then

$$\sup_{t \in T} E[S_m B^l_n(t, \omega) - t]^2 \to 0. \quad (7)$$

Taking into account that $S_m(\beta f_n(t, \omega)) = \beta^2 S_m f_n(t, \omega)$, from the previous lemma 1 the required is obtained.

Using a technique of proofs of lemmas 1,2 and [8, P.73], [13] it is possible to be convinced of a validity of the following statements.

**Lemma 3:** Let $\tilde{f}(t, \omega) = [(f_n(t, \omega))]$, $\tilde{g}(t, \omega) = [(g_n(t, \omega))]$ be the elements of $G(t, \Omega)$, algebra, and $S_m f_n(t, \omega) - f(t, \omega) \to 0$, $S_m g_n(t, \omega) \to 0$, when $m, n \to \infty$ uniformly on $t \in T$ for almost all $\omega \in \Omega$. Then when $m, n \to \infty$

$$S_m(f_n + g_n)(t, \omega) \to f(t, \omega)$$

uniformly on $t \in T$ for almost all $\omega \in \Omega$.

**Lemma 4:** Let $\tilde{B}(t, \omega) = [(B_n(t, \omega))]$ be generalized random process of Brownian motion from (2), then, if $\varphi(n) \to \infty$, $n \to \infty$, $\lambda_m \to 0$ so that

$$\lambda_m n^{1+q} \to 0, \quad m/\varphi^{1-r}(n) \to 0$$

for any arbitrary small $r, q > 0$, then $\forall \beta \in R$

$$S_m(\beta B^l_n + (1 - \beta) B^s_n)(t, \omega) \to \beta^2 t$$

uniformly on $t \in T$ for almost all $\omega \in \Omega$.

2. **An approximation of stochastic $\alpha$-integrals.**

Let, as previous, $B(t, \omega)$, $t \in T$, $\omega \in \Omega$ be unidimensional standard process of $F_t-$Brownian motion, $f \in C(R)$.

Let designate the stochastic $\alpha$-integrals (see, for example [21], p.191-193) as

$$(\alpha) \int_0^t f(B(s, \omega))dB(s, \omega), \; t \in T, \; \omega \in \Omega, \; \alpha \in [0, 1].$$

At $\alpha = 0$ and $\alpha = 1/2$ we shall write (I) and (S) respectively.
Now we shall consider following generalized random process of Brownian motion
\[ \tilde{B}(t, \omega) = [(B_{n,\alpha}(t, \omega))] = [(\sqrt{1-2\alpha}|B_n^I(t, \omega) + (1 - \sqrt{1-2\alpha})B_n^S(t, \omega))] \tag{8} \]
where \( \alpha \in [0, 1] \), and \( B_n^I(t, \omega), B_n^S(t, \omega) \) are taken from the representation (2).

For functions \( f(x) \in C(R) \) we shall set the element, associated to it, of algebra \( G(R) \) as follows
\[ \tilde{f}(t) = [(f(t))] \quad f_n(t) = (f \ast \rho_n)(t) = \int_{0}^{1/n} f(t+s)\rho_n(s)ds \quad \tag{9} \]

**Theorem 1:** For any function \( f \in C(R) \) and random process of Brownian motion \( B(t, \omega) \) the such elements as \( \tilde{f} \in G(R) \) and \( \tilde{B} \in G(T, \Omega) \), will exist that for any types \( (f_n) \in \tilde{f} \) and \( (B_{n,\alpha}) \in \tilde{B} \) when \( \varphi(n), n \to \infty, \lambda_m \to 0 \)
so that
\[ m/\varphi(n)^{1-r} \to 0 \quad \lambda_m n^{1+q} \to 0, \]
for any arbitrary small \( r, q > 0 \), if
\[ \alpha \in [0, 1], \quad \sum_{k=1}^{m} f_n(\tilde{B}_{n,\alpha}(t_{k-1}, \omega))[B_{n,\alpha}(t_k, \omega) - B_{n,\alpha}(t_{k-1}, \omega)]^2 \to \]
\[ |1 - 2\alpha| \int_{0}^{t} f(B(s, \omega))ds, \]
uniformly on \( t \in T \) for almost all \( \omega \in \Omega \), where \( \tilde{B}_{n,\alpha}(t_{k-1}, \omega) \) is between points \( B_{n,\alpha}(t_k, \omega) \) and \( B_{n,\alpha}(t_{k-1}, \omega) \).

**Proof:** Let us use \( \tilde{f} \in G(R) \) and \( \tilde{B} \in G(T, \Omega) \) from the representations (9) and (8) respectively. It is simple to state that for any types \( (f_n) \in \tilde{f} \) and \( (B_{n,\alpha}) \in \tilde{B} \) the following is valid
\[ \left| \sum_{k=1}^{m} f_n(\tilde{B}_{n,\alpha}(t_{k-1}, \omega))[B_{n,\alpha}(t_k, \omega) - B_{n,\alpha}(t_{k-1}, \omega)]^2 - |1 - 2\alpha| \int_{0}^{t} f(B(s, \omega))ds \right| \leq \]
\[ \left| \sum_{k=1}^{m} f_n(\tilde{B}_{n,\alpha}(t_{k-1}, \omega))[B_{n,\alpha}(t_k, \omega) - B_{n,\alpha}(t_{k-1}, \omega)]^2 - \right| 1 - 2\alpha \right| \sum_{k=1}^{m} f(B(t_{k-1}, \omega))[B(t_k, \omega) - B(t_{k-1}, \omega)]^2 | + \]
\[ |1-2\alpha| \sum_{k=1}^{m} f(B(t_{k-1}, \omega))[B(t_k, \omega)-B(t_{k-1}, \omega)]^2 - \int_0^t f(B(s, \omega)) \, ds = I_1 + |1-2\alpha|I_2. \]

Let us consider \( I_2 \). Let \( \varphi_j(t, \omega) \) and \( \psi_j(t, \omega) \) be piece-constant functions at the fixed \( \omega \in \Omega \), such that for any \( t \in T \)

\[ \varphi_j(t, \omega) \leq f(B(t, \omega)) \leq \psi_j(t, \omega), \]

when \( j \to \infty \) uniformly on \( t \in T \) for almost all \( \omega \in \Omega \). Then from the theorem 2.3 ([8], p.66) follows

\[ \sum_{k=1}^{m} \varphi_j(t_{k-1}, \omega)[B(t_k, \omega)-B(t_{k-1}, \omega)]^2 \to \int_0^t \varphi_j(s, \omega) \, ds, \quad (11) \]

\[ \sum_{k=1}^{m} \psi_j(t_{k-1}, \omega)[B(t_k, \omega)-B(t_{k-1}, \omega)]^2 \to \int_0^t \psi_j(s, \omega) \, ds, \quad (12) \]

when \( \lambda_m \to 0 \) uniformly on \( t \in T \) for almost all \( \omega \in \Omega \), for each natural \( j \).

From the theorem of the Lebesgue we shall obtain

\[ \int_0^t \varphi_j(s, \omega) \, ds \to \int_0^t f(B(s, \omega)) \, ds, \quad (13) \]

\[ \int_0^t \psi_j(s, \omega) \, ds \to \int_0^t f(B(s, \omega)) \, ds, \quad (14) \]

when \( j \to \infty \) uniformly on \( t \in T \) for almost all \( \omega \in \Omega \). In the double inequality

\[ \sum_{k=1}^{m} \varphi_j(t_{k-1}, \omega)[B(t_k, \omega)-B(t_{k-1}, \omega)]^2 \leq \sum_{k=1}^{m} f(B(t_{k-1}, \omega))[B(t_k, \omega)-B(t_{k-1}, \omega)]^2 \]

\[ \leq \sum_{k=1}^{m} \psi_j(t_{k-1}, \omega)[B(t_k, \omega)-B(t_{k-1}, \omega)]^2, \]

we shall pass to limit at first at \( \lambda_m \to 0 \), and then at \( j \to \infty \). Then, using relations (11)-(14), we shall obtain

\[ \sup_{t \in T} I_2 \to 0, \quad \text{for almost all } \omega \in \Omega. \quad (15) \]
For $I_1$ the following manipulations (using expression (8)) are valid

$$I_1 \leq (1 - \sqrt{|1 - 2\alpha|^2}) \left| \sum_{k=1}^{m} f_n(\tilde{B}_{n,\alpha}(t_{k-1}, \omega)) [B_n^*(t_k, \omega) - B_n^*(t_{k-1}, \omega)]^2\right| +$$

$$2\sqrt{|1 - 2\alpha|(1 - \sqrt{|1 - 2\alpha|})} \left| \sum_{k=1}^{m} f_n(\tilde{B}_{n,\alpha}(t_{k-1}, \omega)) [B_n^*(t_k, \omega) - B_n^*(t_{k-1}, \omega)][B_n^I(t_k, \omega)] -$$

$$\sum_{k=1}^{m} f(B(t_{k-1}, \omega))[B(t_k, \omega) - B(t_{k-1}, \omega)]^2\right| =$$

$$(1 - \sqrt{|1 - 2\alpha|^2})I_{11} + 2\sqrt{|1 - 2\alpha|(1 - \sqrt{|1 - 2\alpha|})}I_{12} + |1 - 2\alpha|I_{13}.$$

From the lemma 4 and limitation of $f_n(\tilde{B}_{n,\alpha}(t_{k-1}, \omega))$ in the aggregate with the fixed $\omega \in \Omega$ we obtain that

$$\text{Sup}_{t \in T} I_{11} \to 0,$$  \hspace{1cm} (16)

at $m, n \to \infty$, $\lambda_m \to 0$, so that $\lambda_m n^{1+q} \to 0$ for arbitrary small $q > 0$ for almost all $\omega \in \Omega$.

The following estimation for $I_{12}$, obtained by using the inequality of Cauchy-Bunyakovsky for finite sums, and in view of limitation of $f_n(\tilde{B}_{n,\alpha}(t_{k-1}, \omega))$ in the aggregate with the fixed $\omega \in \Omega$, is valid

$$I_{12} \leq C \left( \sum_{k=1}^{m} [B_n^*(t_k, \omega) - B_n^*(t_{k-1}, \omega)]^2 \right)^{1/2} \left( \sum_{k=1}^{m} [B_n^I(t_k, \omega) - B_n^I(t_{k-1}, \omega)]^2 \right)^{1/2}.$$

Passing here to the limit at $\varphi(n)$, $n, m \to \infty$, $\lambda_m \to 0$, so that

$$m/\varphi(n)^{1-r} \to 0, \quad \lambda_m n^{1+q} \to 0,$$

at the any arbitrary small $r, q > 0$ under the lemma 4 we shall obtain

$$\text{Sup}_{t \in T} I_{12} \to 0,$$  \hspace{1cm} (17)

for all almost all $\omega \in \Omega$. Let us consider $I_{13}$. It is obvious that

$$I_{13} \leq \left| \sum_{k=1}^{m} f_n(\tilde{B}_{n,\alpha}(t_{k-1}, \omega)) [B_n^I(t_k, \omega) - B_n^I(t_{k-1}, \omega)]^2 \right| -$$

$$\sum_{k=1}^{m} f(\tilde{B}_{n,\alpha}(t_{k-1}, \omega)) [B_n^I(t_k, \omega) - B_n^I(t_{k-1}, \omega)]^2\right| +$$
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\[
\left| \sum_{k=1}^{m} f(\tilde{B}_{n,\alpha}(t_{k-1}, \omega)) [B_{n}^{I}(t_{k}, \omega) - B_{n}^{I}(t_{k-1}, \omega)]^{2} \right| - \\
\sum_{k=1}^{m} f(B(t_{k-1}, \omega)) [B_{n}^{I}(t_{k}, \omega) - B_{n}^{I}(t_{k-1}, \omega)]^{2} + \\
\left| \sum_{k=1}^{m} f(B(t_{k-1}, \omega)) [B_{n}^{I}(t_{k}, \omega) - B_{n}^{I}(t_{k-1}, \omega)]^{2} \right| - \\
\sum_{k=1}^{m} f(B(t_{k-1}, \omega)) [B_{n}^{I}(t_{k}, \omega) - B_{n}^{I}(t_{k-1}, \omega)]^{2}
\]

\[= I_{13}^{1} + I_{13}^{2} + I_{13}^{3}.\]

From the uniform continuity of $f(B(t, \omega))$ and boundedness of the sum

\[
\sum_{k=1}^{m} [B_{n}^{I}(t_{k}, \omega) - B_{n}^{I}(t_{k-1}, \omega)]^{2}
\]
on $T$ for almost all $\omega \in \Omega$ follows

\[\text{Sup}_{t \in T} I_{13}^{2} \to 0, \quad (18)\]
at $\varphi(n), m \to \infty, \lambda_{m} \to 0$, so that $m/\varphi(n)^{1-r} \to 0$ for any arbitrary small $r > 0$. Taking into account that $f_{n}(x) \to f(x)$ uniformly on any compact set from $R$ at $n \to \infty$, we shall obtain

\[\text{Sup}_{t \in T} I_{13}^{1} \to 0. \quad (19)\]

Let $\varphi_{j}(t, \omega)$ and $\psi_{j}(t, \omega)$ are form the expression (10), then double inequality is obvious

\[
\sum_{k=1}^{m} \varphi_{j}(t_{k-1}, \omega) [B_{n}^{I}(t_{k}, \omega) - B_{n}^{I}(t_{k-1}, \omega)]^{2} - \sum_{k=1}^{m} \psi_{j}(t_{k-1}, \omega) [B(t_{k}, \omega) - B(t_{k-1}, \omega)]^{2} \leq \\
I_{13}^{3} \leq \sum_{k=1}^{m} \psi_{j}(t_{k-1}, \omega) [B_{n}^{I}(t_{k}, \omega) - B_{n}^{I}(t_{k-1}, \omega)]^{2} - \sum_{k=1}^{m} \varphi_{j}(t_{k-1}, \omega) [B(t_{k}, \omega) - B(t_{k-1}, \omega)]^{2}.\]

Passing to limit at first at $\varphi(n), m \to \infty, \lambda_{m} \to 0$, so that $m/\varphi(n)^{1-r} \to 0$ at any arbitrary small $r > 0$ and then at $j \to \infty$ taking into account the theorem 2.3 [8, p.66] and lemma 4 from (13), (14) we shall obtain

\[\text{Sup}_{t \in T} I_{13}^{3} \to 0, \quad (20)\]
at the almost all $\omega \in \Omega$. The proof of the theorem 1 follows from the relations (15)-(20).

**Theorem 2:** For any function $f \in C^1(R)$ and random process of Brownian motion $B(t, \omega)$ the such elements as $\tilde{f} \in G(R)$ and $\tilde{B} \in G(T, \Omega)$, will exist, that for any types $(f_n) \in \tilde{f}$ and $(B_{n,\alpha}) \in \tilde{B}$ at $\varphi(n)$, $n, m \to \infty$, $\lambda_m \to 0$ so that

$$m/\varphi(n)^{1-r} \to 0 \quad \lambda_m n^{1+q} \to 0, \quad (21)$$

for any arbitrary small $r, q > 0$, and $\alpha \in [0, 1/2]$

$$\sum_{k=1}^{m} f_n(B_{n,\alpha}(t_{k-1}, \omega))[B_{n,\alpha}(t_k, \omega) - B_{n,\alpha}(t_{k-1}, \omega)] \to (\alpha) \int_0^t f(B(s, \omega))dB(s, \omega),$$

uniformly on $t \in T$ for almost all $\omega \in \Omega$.

**Proof:** Let us use $\tilde{f} \in G(R)$ and $\tilde{B} \in G(T, \Omega)$ from the representations (9) and (8) respectively. Let us use the function $\varphi_n(x) = \int_d^x f_n(t)dt$. Then under the formula of Taylor

$$\varphi_n(B_{n,\alpha}(t, \omega)) - \varphi_n(B_{n,\alpha}(0, \omega)) = \sum_{k=1}^{m} (\varphi_n(B_{n,\alpha}(t_k, \omega)) - \varphi_n(B_{n,\alpha}(t_{k-1}, \omega))) =$$

$$\sum_{k=1}^{m} \varphi_n'(B_{n,\alpha}(t_{k-1}, \omega))[B_{n,\alpha}(t_k, \omega) - B_{n,\alpha}(t_{k-1}, \omega)] +$$

$$\frac{1}{2} \sum_{k=1}^{m} \varphi_n''(\tilde{B}_{n,\alpha}(t_{k-1}, \omega))[B_{n,\alpha}(t_k, \omega) - B_{n,\alpha}(t_{k-1}, \omega)]^2, \quad (22)$$

where $\tilde{B}_{n,\alpha}(t_{k-1}, \omega)$ is the point laying on the straight-line, connecting the points $B_{n,\alpha}(t_k, \omega)$ and $B_{n,\alpha}(t_{k-1}, \omega)$.

Taking into account (22) and that $\varphi_n'(x) = f_n(x)$, $\varphi_n''(x) = f_n'(x)$ we obtain

$$\int_d^{B_{n,\alpha}(t, \omega)} f_n(x)dx - \int_d^{B_{n,\alpha}(0, \omega)} f_n(x)dx$$

$$= \sum_{k=1}^{m} f_n(B_{n,\alpha}(t_{k-1}, \omega))[B_{n,\alpha}(t_k, \omega) - B_{n,\alpha}(t_{k-1}, \omega)] +$$

$$\frac{1}{2} \sum_{k=1}^{m} f_n'(\tilde{B}_{n,\alpha}(t_{k-1}, \omega))[B_{n,\alpha}(t_k, \omega) - B_{n,\alpha}(t_{k-1}, \omega)]^2. \quad (23)$$
Similarly, it is possible to obtain

\[
\int_d^{B(t, \omega)} f(x)\,dx - \int_d^{B(0, \omega)} f(x)\,dx = \sum_{k=1}^{m} f(B(t_{k-1}, \omega))[B(t_k, \omega) - B(t_{k-1}, \omega)] + \\
\frac{1}{2} \sum_{k=1}^{m} f'(\tilde{B}(t_{k-1}, \omega))[B(t_k, \omega) - B(t_{k-1}, \omega)]^2,
\]

where \( \tilde{B}(t_{k-1}, \omega) \) lays between \( B(t_k, \omega) \) and \( B(t_{k-1}, \omega) \).

The relation between \( \alpha \)-integral and integral of Ito is known (see [21], p.208)

\[
(\alpha) \int_0^t f(B(s, \omega))\,dB(s, \omega) = (I) \int_0^t f(B(s, \omega))\,dB(s, \omega) + (\alpha) \int_0^t f'(B(s, \omega))\,ds.
\]

Using (23)-(25), let us estimate the following residual

\[
|\sum_{k=1}^{m} f_n(B_{n, \alpha}(t_{k-1}, \omega))[B_{n, \alpha}(t_k, \omega) - B_{n, \alpha}(t_{k-1}, \omega)]| - (\alpha) \int_0^t f(B(s, \omega))\,dB(s, \omega)| \leq \\
| \int_d^{B_{n, \alpha}(t, \omega)} f_n(x)\,dx - \int_d^{B(t, \omega)} f(x)\,dx | + | \int_d^{B(0, \omega)} f(x)\,dx - \int_d^{B_{n, \alpha}(0, \omega)} f_n(x)\,dx | + \\
\frac{1}{2} \sum_{k=1}^{m} f'(\tilde{B}(t_{k-1}, \omega))[B(t_k, \omega) - B(t_{k-1}, \omega)]^2 - (\alpha) \int_0^t f'(B(s, \omega))\,ds
\]

\[
- \frac{1}{2} \sum_{k=1}^{m} f_n'(\tilde{B}_{n, \alpha}(t_{k-1}, \omega))[B_{n, \alpha}(t_k, \omega) - B_{n, \alpha}(t_{k-1}, \omega)]^2| + \\
| \sum_{k=1}^{m} f(B(t_{k-1}, \omega))[B(t_k, \omega) - B(t_{k-1}, \omega)] - (I) \int_0^t f(B(s, \omega))\,dB(s, \omega) | \\
= I_1 + I_2 + I_3 + I_4.
\]

By the definition of integral by Ito

\[
\text{Sup}_{t \in T} I_4 \to 0,
\]

at \( \lambda_m \to 0 \) for almost all \( \omega \in \Omega \). For \( I_1 \) following estimations are valid

\[
I_1 \leq | \int_d^{B(t, \omega)} (f_n(x) - f(x))\,dx | + | \int_{B(t, \omega)}^{B_{n, \alpha}(t, \omega)} f_n(x)\,dx |.
\]
As $f_n(x) \to f(x)$ uniformly on the any compact set form $R$ at $n \to \infty$ and $B_{n,\alpha}(t, \omega) \to B(t, \omega)$ when $n \to \infty$ uniformly on $t \in T$ for almost all $\omega \in \Omega$,

$$\operatorname{Sup}_{t \in T} I_1 \to 0,$$

(27)

at $\varphi(n), n \to \infty$ for almost all $\omega \in \Omega$. Similarly

$$\operatorname{Sup}_{t \in T} I_2 \to 0,$$

(28)

at $\varphi(n), n \to \infty$ for almost all $\omega \in \Omega$.

At $\alpha \in [0, 1/2]$ for $I_3$ the following inequalities are valid

$$I_3 \leq \left| \frac{1}{2} \sum_{k=1}^{m} f'(\tilde{B}(t_{k-1}, \omega))[B(t_k, \omega) - B(t_{k-1}, \omega)]^2 - \frac{1}{2} \int_0^t f'(B(s, \omega))ds \right| +$$

$$\frac{1}{2} \sum_{k=1}^{m} f_n'(\tilde{B}_{n,\alpha}(t_{k-1}, \omega))[B_{n,\alpha}(t_k, \omega) - B_{n,\alpha}(t_{k-1}, \omega)]^2 - \frac{1}{2}(1-2\alpha) \int_0^t f'(B(s, \omega))ds$$

$$= I_{31} + I_{32}.$$

Under the theorem 1, as $\alpha \in [0; 1/2]$, the following expression is valid

$$\operatorname{Sup}_{t \in T} I_{32} \to 0,$$

(29)

at $\varphi(n), n, m \to \infty, \lambda_m \to 0$, so that (21) is valid for almost all $\omega \in \Omega$.

Similarly, as in the proof of the theorem 1, it is possible to obtain

$$\operatorname{Sup}_{t \in T} I_{31} \to 0,$$

(30)

at $m \to \infty, \lambda_m \to 0$, for almost all $\omega \in \Omega$.

The proof of the theorem 2 follows from the relations (26)-(30).

**Theorem 3:** For any function $f \in C^1(R)$ and random process of Brownian motion $B(t, \omega)$ the such elements as $\tilde{f} \in G(R)$ and $\tilde{B} \in G(T, \Omega)$, will exist, that for any types $(f_n) \in \tilde{f}$ and $(B_{n,\alpha}) \in \tilde{B}$ at $\varphi(n), n, m \to \infty, \lambda_m \to 0$ so that

$$m/\varphi(n)^{1-r} \to 0 \quad \lambda_m n^{1+q} \to 0,$$

(31)

for any arbitrary small $r, q > 0$, and $\alpha \in [1/2, 1]$

$$\sum_{k=1}^{m} f_n(B_{n,\alpha}(t_k, \omega))[B_{n,\alpha}(t_k, \omega) - B_{n,\alpha}(t_{k-1}, \omega)] \to (\alpha) \int_0^t f(B(s, \omega))dB(s, \omega),$$
uniformly on \( t \in T \) for almost all \( \omega \in \Omega \).

**Proof:** Let us use the elements \( \tilde{f} \in G(\mathbb{R}) \) and \( \tilde{B} \in G(T, \Omega) \) from the representations (9) and (8) respectively. Let us assume, under the definition \( \varphi_n(x) = \int_0^x f_n(t)dt \). Then under the formula of Taylor

\[
\varphi_n(B_{n,\alpha}(t, \omega)) - \varphi_n(B_{n,\alpha}(0, \omega)) = \sum_{k=1}^m (\varphi_n(B_{n,\alpha}(t_k, \omega)) - \varphi_n(B_{n,\alpha}(t_k-1, \omega))) =
\]

\[
\sum_{k=1}^m \varphi_n'(B_{n,\alpha}(t_k, \omega))[B_{n,\alpha}(t_k, \omega) - B_{n,\alpha}(t_k-1, \omega)] -
\]

\[
\frac{1}{2} \sum_{k=1}^m \varphi_n''(\tilde{B}_{n,\alpha}(t_k-1, \omega))[B_{n,\alpha}(t_k, \omega) - B_{n,\alpha}(t_k-1, \omega)]^2,
\]

where \( \tilde{B}_{n,\alpha}(t_k-1, \omega) \) lays between \( B_{n,\alpha}(t_k, \omega) \) and \( B_{n,\alpha}(t_k-1, \omega) \).

Taking into account that \( \varphi_n'(x) = f_n(x) \), \( \varphi_n''(x) = f_n'(x) \) we obtain

\[
\int_d^{B_{n,\alpha}(t, \omega)} f_n(x)dx - \int_d^{B_{n,\alpha}(0, \omega)} f_n(x)dx = \sum_{k=1}^m f_n(B_{n,\alpha}(t_k, \omega))[B_{n,\alpha}(t_k, \omega) - B_{n,\alpha}(t_k-1, \omega)] -
\]

\[
\frac{1}{2} \sum_{k=1}^m f_n'(\tilde{B}_{n,\alpha}(t_k-1, \omega))[B_{n,\alpha}(t_k, \omega) - B_{n,\alpha}(t_k-1, \omega)]^2. \tag{32}
\]

Similarly, as in theorem 2, it is possible to obtain

\[
\int_d^{B(t, \omega)} f(x)dx - \int_d^{B(0, \omega)} f(x)dx = \sum_{k=1}^m f(B(t_k-1, \omega))[B(t_k, \omega) - B(t_k-1, \omega)] +
\]

\[
\frac{1}{2} \sum_{k=1}^m f'(\tilde{B}(t_k-1, \omega))[B(t_k, \omega) - B(t_k-1, \omega)]^2, \tag{33}
\]

where \( \tilde{B}(t_k-1, \omega) \) lays between \( B(t_k, \omega) \) and \( B(t_k-1, \omega) \).

Let us estimate the residual, using (32), (33), (25)

\[
| \sum_{k=1}^m f_n(B_{n,\alpha}(t_k, \omega))[B_{n,\alpha}(t_k, \omega) - B_{n,\alpha}(t_k-1, \omega)] - (\alpha) \int_0^t f(B(s, \omega))dB(s, \omega)| \leq
\]
\[\left| \int_{\Omega} B_{t,\omega}(t,\omega) f_n(x) \, dx - \int_{\Omega} B(t,\omega) f(x) \, dx \right| + \left| \int_{\Omega} B_{t,\omega}(0,\omega) f_n(x) \, dx \right| + \left| \int_{\Omega} B_{t,\omega}(0,\omega) f(x) \, dx \right| + \left| \int_{\Omega} B_{t,\omega}(t,\omega) f_n(x) \, dx \right|\]

\[\left| \frac{1}{2} \sum_{k=1}^{m} f_n'(\tilde{B}(t_{k-1},\omega))[B(t_k,\omega) - B(t_{k-1},\omega)]^2 \right| - (\alpha) \int_{0}^{t} f'(B(s,\omega)) \, ds + \]

\[+ \left| \frac{1}{2} \sum_{k=1}^{m} f_n'(\tilde{B}_{n,\alpha}(t_{k-1},\omega))[B_{n,\alpha}(t_k,\omega) - B_{n,\alpha}(t_{k-1},\omega)]^2 \right| + \]

\[\left| \sum_{k=1}^{m} f(B(t_{k-1},\omega))[B(t_k,\omega) - B(t_{k-1},\omega)] - (I) \int_{0}^{t} f(B(s,\omega)) \, dB(s,\omega) \right| = I_1 + I_2 + I_3 + I_4.\]

By the definition of integral by Ito

\[\sup_{t \in T} I_4 \to 0, \quad (34)\]

at \(\lambda_m \to 0\) for almost all \(\omega \in \Omega\).

As in the proof of the theorem 2 it is possible to obtain

\[\sup_{t \in T} I_1 \to 0, \quad \sup_{t \in T} I_2 \to 0, \quad (35)\]

at \(\varphi(n), n \to \infty\) for almost all \(\omega \in \Omega\).

At \(\alpha \in [1/2; 1]\) for \(I_3\) the following inequalities are valid

\[I_3 \leq \left| \frac{1}{2} \sum_{k=1}^{m} f'(\tilde{B}(t_{k-1},\omega))[B(t_k,\omega) - B(t_{k-1},\omega)]^2 \right| - \frac{1}{2} \int_{0}^{t} f'(B(s,\omega)) \, ds + \]

\[\left| \frac{1}{2} \sum_{k=1}^{m} f_n'(\tilde{B}_{n,\alpha}(t_{k-1},\omega))[B_{n,\alpha}(t_k,\omega) - B_{n,\alpha}(t_{k-1},\omega)]^2 \right| - \frac{1}{2} (2\alpha - 1) \int_{0}^{t} f'(B(s,\omega)) \, ds \]

\[= I_{31} + I_{32}.\]

Under the theorem 1 follows

\[\sup_{t \in T} I_{32} \to 0, \quad (36)\]

at \(\varphi(n), n, m \to \infty, \lambda_m \to 0\), so that (21) is valid for almost all \(\omega \in \Omega\).

Similarly, as in the proof of the theorem 1, it is possible to obtain

\[\sup_{t \in T} I_{31} \to 0, \quad (37)\]
Approximation of stochastic $\alpha$-integrals

at $m \to \infty$, $\lambda_m \to 0$, for almost all $\omega \in \Omega$.

The proof of the theorem 3 follows from the relations (34)-(37).

**Comment 1:** It is follows from the theorem 3 that concept of the stochastic $\alpha-$integral at $\alpha \in [1/2; 1]$ is unnatural in the algebra of generalized random processes, because it reduces to the finite-residual equations with an advancing.

3. The associated formula of Ito for the $\alpha-$integrals.

Let us find the aspect of the associated formula of Ito in the algebra $G(T, \Omega)$. For this purpose let us consider the expression

$$\tilde{f}(\tilde{B}(\tilde{t}, \omega)) - \tilde{f}(\tilde{B}(\tilde{0}, \omega)), \tag{38}$$

where $\tilde{B}(t, \omega)$—generalized random process of Brownian motion from the representation (8), $\tilde{f}$ is generalized function from $G(R)$, from the representation (9), $\tilde{t} = [(t, \cdots, t, \cdots)]$ and $\tilde{0} = [(0, \cdots, 0, \cdots)]$ are the standard elements from $\tilde{R}$.

Let $(f_n)$ and $(B_{n,\alpha})$ are some types of $\tilde{f}(t)$ and $\tilde{B}(t, \omega)$ respectively. Then at a level of types it is possible to introduce the expression (38) as

$$f_n(B_{n,\alpha}(t, \omega)) - f_n(B_{n,\alpha}(0, \omega)) = \sum_{k=1}^{m} [f_n(B_{n,\alpha}(t_k, \omega)) - f_n(B_{n,\alpha}(t_{k-1}, \omega))], \tag{39}$$

where $0 = t_0 < t_1 < \cdots < t_m = t$ is the partition of the interval $[0; t]$, $\lambda_m = \max_{1 \leq k \leq m} |t_k - t_{k-1}|$. Applying in the right part of an identity the formula of Taylor, we shall obtain

$$f_n(B_{n,\alpha}(t, \omega)) - f_n(B_{n,\alpha}(0, \omega)) = \sum_{k=1}^{m} f'_n(B_{n,\alpha}(t_{k-1}, \omega))[B_{n,\alpha}(t_k, \omega) - B_{n,\alpha}(t_{k-1}, \omega)] +$$

$$\frac{1}{2} \sum_{k=1}^{m} f''_n(\tilde{B}_{n,\alpha}(t_{k-1}, \omega))[B_{n,\alpha}(t_k, \omega) - B_{n,\alpha}(t_{k-1}, \omega)]^2, \tag{40}$$

where $\tilde{B}_{n,\alpha}(t_{k-1}, \omega)$—point laying on the straight-line, connecting points $B_{n,\alpha}(t_k, \omega)$ and $B_{n,\alpha}(t_{k-1}, \omega)$.

Let us pass in the expression (40) to the limit at $n \to \infty$, $m \to \infty$, $\varphi(n) \to \infty$, $\lambda_m \to \infty$, so that $m/\varphi(n)^{1-r} \to 0$, $\lambda_m n^{1+q} \to 0$ for any arbitrary small $r$ and $q > 0$. It is follows under the theorem 2 that at the made above assumptions and proceeding from fact that at $n \to \infty$

$$f_n(t) \to f(t), \quad B_{n,\alpha}(t, \omega) \to B(t, \omega),$$
uniformly on \( t \) at the any compact set for almost all \( \omega \in \Omega \), for \( \alpha \in [0; 1/2] \)

\[
\sum_{k=1}^{m} f'_n(B_n,\alpha(t_{k-1},\omega))[B_n,\alpha(t_k,\omega) - B_n,\alpha(t_{k-1},\omega)] \to (\alpha) \int_0^t f'(B(s,\omega))dB(s,\omega),
\]

(41)

uniformly on \( t \in T \) for almost all \( \omega \in \Omega \) and under the theorem 1 follows that

\[
\frac{1}{2} \sum_{k=1}^{m} f''_n(\tilde{B}_n,\alpha(t_{k-1},\omega))[B_n,\alpha(t_k,\omega) - B_n,\alpha(t_{k-1},\omega)]^2 \to \left( \frac{1}{2} - \alpha \right) \int_0^t f''(B(s,\omega))ds,
\]

(42)

uniformly on \( t \in T \) for almost all \( \omega \in \Omega \).

If \( \alpha \in [1/2; 1] \) then after applying of formula by Taylor in the right part of equality (39) we shall obtain

\[
f_n(B_n,\alpha(t,\omega)) - f_n(B_n,\alpha(0,\omega)) = \sum_{k=1}^{m} f'_n(B_n,\alpha(t_k,\omega))[B_n,\alpha(t_k,\omega) - B_n,\alpha(t_{k-1},\omega)] - \frac{1}{2} \sum_{k=1}^{m} f''_n(\tilde{B}_n,\alpha(t_{k-1},\omega))[B_n,\alpha(t_k,\omega) - B_n,\alpha(t_{k-1},\omega)]^2,
\]

(43)

where \( \tilde{B}_n,\alpha(t_{k-1},\omega) \) is the point laying on the straight-line, connecting points \( B_n,\alpha(t_k,\omega) \) and \( B_n,\alpha(t_{k-1},\omega) \).

Passing in (43) to the limit at \( n \to \infty, m \to \infty, \varphi(n) \to \infty, \lambda_m \to 0 \), so that \( m/\varphi(n)^{1-r} \to 0, \lambda_m n^{1+q} \to 0 \) for any arbitrary small \( r \) and \( q > 0 \), taking into account above made assumptions and under the theorem 3, as \( \alpha \in [1/2; 1] \) we obtain

\[
\sum_{k=1}^{m} f'_n(B_n,\alpha(t_k,\omega))[B_n,\alpha(t_k,\omega) - B_n,\alpha(t_{k-1},\omega)] \to (\alpha) \int_0^t f'(B(s,\omega))dB(s,\omega),
\]

(44)

uniformly on \( t \in T \) for almost all \( \omega \in \Omega \) and under the theorem 1 follows that

\[
\frac{1}{2} \sum_{k=1}^{m} f''_n(\tilde{B}_n,\alpha(t_{k-1},\omega))[B_n,\alpha(t_k,\omega) - B_n,\alpha(t_{k-1},\omega)]^2 \to (\alpha - \frac{1}{2}) \int_0^t f''(B(s,\omega))ds,
\]

(45)

uniformly on \( t \in T \) for almost all \( \omega \in \Omega \).
It is follows from the expressions (40)-(45) that at $\alpha \in [0; 1]$

$$f(B(t, \omega))-f(B(0, \omega)) = (\alpha) \int_0^t f'(B(s, \omega))dB(s, \omega) + \left(\frac{1}{2}-\alpha\right) \int_0^t f''(B(s, \omega))ds,$$

for almost all $\omega \in \Omega$, $t \in T$, $f \in C^2(R)$, and this is the formula of Ito for $\alpha-$integrals (see, for example [21], p.206), that is the theorem is proved.

**Theorem 4:** One of the formulas by Ito for $\alpha-$integrals, associated in the algebra of generalized random processes $G(T, \Omega)$, is match with the classic one.

**References**


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