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Some Paranormed Sequence Spaces and Matrix Transformations

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Abstract

In this paper we characterize the matrix classes with one member as $\bar{m}(u,p)$ or $\bar{m}_o(u,p)$ or $c(u,p)$ or $c_o(u,p)$. Some of these results generalize the existing results. Some are new proved in the general setting.

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1. Introduction and Background

Let w, γ, γ_o, c, c_o and l_∞ be the spaces of all, summable, summable to zero, convergent, null and bounded sequences respectively. The notion of statistical convergence of sequences was introduced by Fast [3], Schoenberg [12] and Buck [1] independently. Later on the idea was exploited from sequence space point of view and linked with summability by Fridy [4], Šalát [11], Kolk [5], Rath and Tripathy [10], Connor [2], Tripathy ([14], [15]) and many others. The basic idea depends on the density of the subsets of $\mathbb{N}$, the set of natural numbers. A subset of E of $\mathbb{N}$ is said to have density $\delta(E)$ if $\delta(E) = \lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \chi_E(k)$ exists, where χ_E is the characteristic function of E.

A sequence (x_k) is said to be statistically convergent to L if for every $\epsilon > 0$, $\delta(\{k \in \mathbb{N} : |x_k - L| \geq \epsilon\}) = 0$. We write $x_k \xrightarrow{st} L$ or $st - \lim x_k = L$.

Tripathy and Sen [17] have generalized the notion on extending it for paranormed sequence spaces. The notion paranormed sequence space was first studied by Nakano [9] and Simons [13]. Later on it was exploited by Maddox [8], Lascarides and Maddox [7], Lascarides [6], Tripathy [16] and many others. Throughout $p = (p_k) \in l_\infty$ denote a non-negative sequence of real numbers. We write $r_k = \frac{1}{p_k}$ for all $k \in \mathbb{N}$.

The following known paranormed sequence spaces will be used.

$$c(u,p) = \{x = (x_k) \in w : |u_k x_k - L|^{p_k} \to 0 \ \ (k \to \infty) \ \text{ for some } \ L\}$$

$$c_o(u,p) = \{x = (x_k) \in w : |u_k x_k|^{p_k} \to 0 \ \ \text{as}(k \to \infty)\}$$

$$l_\infty(u,p) = \{x = (x_k) \in w : \sup_k |u_k x_k|^{p_k} < \infty\}$$

$$\bar{c}(u,p) = \{x = (x_k) \in w : |u_k x_k - L|^{p_k} \xrightarrow{st} 0 \ \text{ for some } \ L\}$$

$$\bar{c}_o(u,p) = \{x = (x_k) \in w : |u_k x_k|^{p_k} \xrightarrow{st} 0\}$$

We write $m(p) = \bar{c}(u,p) \cap l_\infty(u,p)$ and $m_o(u,p) = \bar{c}_o(u,p) \cap l_\infty(u,p)$

The above spaces are paranormed by $g(x_k) = \sup_k |u_k x_k|^{\frac{p_k}{M}}$, where $M = max(1, \sup p_k)$.

2. Preliminaries

The following results will be used for establishing the results of this paper.

Lemma 1. (Tripathy and Sen [17], Theorem 2) The space $m(u,p)$ is closed subspace of $l_\infty(u,p)$.

Lemma 2. (Lascarides [6], Remark, p. 494) Let $p, q \in l_\infty$. Then we have $A = (a_{nk}) \in (c_o(u,p), l_\infty(q))$ if and only if

(2.1) there exists an absolute constant $D > 1$ such that

$$\sup_n \left\{ \sum_k \left| \frac{a_{nk}}{u_k} \right| D^{-r_k} \right\}^{q_n} < \infty.$$

In view of above lemma and using standard techniques we have the following result.

Lemma 3. Let $(p_k) \in l_\infty$. Then $A = (a_{nk}) \in (l_\infty, l_\infty(u,p))$ if and only if

$$\sup_n \Big\{ \sum_k |u_k a_{nk}| \Big\}^{q_n} < \infty. \tag{2.2}$$

Lemma 4. (Lascarides [6], Theorem 9) Let $p \in l_\infty$. Then $A = (a_{nk}) \in (c(u,p), c)$ if and only if

(2.3) there exists an absolute constant $D > 1$ such that

$$\sup_n \sum_k \left| \frac{a_{nk}}{u_k} \right| D^{-r_k} < \infty,$$

$$\lim_{n\to\infty} \frac{a_{nk}}{u_k} = \alpha_k \text{ exists for every fixed } k. \tag{2.4}$$

$$\lim_{n\to\infty} \sum_k \frac{a_{nk}}{u_k} = \alpha \text{ exists.} \tag{2.5}$$

Lemma 5. Let $0 < \inf p_k \le \sup p_k = h < \infty$. Then for any linear subspace X of $l_\infty(u,p)$, the following are equivalent:

(2.6) X is complete with respect to g.

(2.7) If $\sum\limits_k a_{nk}$ converges uniformly to a_n for each $n \in \mathbb{N}$ and for each $k \in \mathbb{N}$,

$$a^k = (a_{nk})_{n\in\mathbb{N}} \in X, \text{ then } a = (a_n) \in X.$$

Proof. (2.6)⇒(2.7). Suppose $\sum\limits_k a_{nk}$ converges uniformly to a_n for each $n \in \mathbb{N}$ and $a^k = (a_{nk})_{n\in\mathbb{N}} \in X$, for each $k \in \mathbb{N}$. Since X is linear, so $s^j = \sum\limits_{k=1}^{j} a^k \in X$, $j \in \mathbb{N}$. We have $g(s^j - a) = \sup\limits_n \Big| \sum\limits_{k>j} \frac{a_{nk}}{u_k} \Big|^{\frac{p_k}{M}}$.

Since the convergence of $\sum\limits_k a_{nk}$ is uniform, so given $1 > \epsilon > 0$, there exists j_o such that $g(s^j - a) < \epsilon^{\frac{h}{M}}$ for all $j > j_o$. Thus we have $a \in X$, since X is complete.

(2.7)⇒(2.6). Let (x^m), where $x^m = (x_k^m)_{k\in\mathbb{N}}$, be a Cauchy sequence in X. Then (x^m) converges (say to x) in $l_\infty(u,p)$, since $l_\infty(u,p)$ is complete. Write $a_{km} = x_k^m - x_k^{m-1}$ $(x_k^o = 0)$. Then $\sum\limits_m a_{km}$ converges uniformly to x_k and $(a_{km})_{k\in\mathbb{N}} = a^m \in X$.

Note 1. Taking $p_n = 1$ for all $n \in \mathbb{N}$, one will get Lemma 4 of Rath and Tripathy [10] as particular case.

Lemma 6. Let $(p_k) \in l_\infty$, then $A = (a_{nk}) \in (\gamma, l_\infty(u,p))$ if and only if

(2.8)
$$T = \sup_n \Big\{ |u_k \sum_k \triangle a_{nk}| \Big\}^{p_n} < \infty, \text{ where } \triangle a_{nk} = a_{nk} - a_{n,k+1}, \text{ for all } k \in \mathbb{N}.$$

and

(2.9). $$(a_{n1}) \in l_\infty(u,p)$$

Proof. Let $s = (s_k) \in \gamma$ and $S_n = \sum_{k=1}^{n} s_k \to S$ as $n \to \infty$. Then by Abel's summation formula we have

(2.10) $$A_n s = \sum_{k=1}^{\infty} a_{nk} s_k = S a_{n1} + \sum_{k=1}^{\infty} \triangle a_{nk}(S_k - S)$$

The rest of the proof is a routine work in view of Lemma 2 and using standard techniques.

The proof of the following result is a routine work in view of Lemma 6.

Lemma 7. Let $(p_k) \in l_\infty$, then $A = (a_{nk}) \in (\gamma_o, l_\infty(u,p))$ if and only if (2.8) holds.

3. Main Result

In this section we establish the results of this paper.

Theorem 1. Let $0 < \inf p_k \le \sup p_k < \infty$. Then $A = (a_{nk}) \in (\gamma, m(u,p))$ if and only if (2.8) holds and

(3.1) $$(a_{nk})_{n \in \mathbb{N}} \in m(u,p), \quad \text{for every} \quad k \in \mathbb{N}.$$

Proof. The necessity of (2.8) follows from the inclusion $(\gamma, m(u,p)) \subset (\gamma, l_\infty(u,p))$ and Lemma 5 and that of (3.1) on considering the sequence $e_k = (0, 0, ...0, 1, 0, ...)$ in γ, where the only 1 appears at the k-th place.

Sufficiency. Let $s = (s_k) \in \gamma$. We have by (3.1) that $(\triangle a_{nk})_{n \in \mathbb{N}} \in m(u,p)$ for all $k = 1, 2, 3, ...$

Hence we have $Sa_{nk} + \sum_{k \leq j_o} \triangle a_{nk}(S_k - S) \in m(u,p)$, by the linearity.

Next we have

$$\Big|\sum_{k>j_o} \frac{\triangle a_{nk}}{u_k}(S_k - S)\Big| \leq T^{\frac{1}{h}} \max_{k>j_o} |S_k - S|$$
$$\to 0, \text{ uniformly in } n \text{ as } j_o \to \infty$$

Hence by Lemma 1, Lemma 5 and (2.8) we have $As \in m(u,p)$.

This completes the proof of the Theorem.

The proof of the following result is obvious in view of the above result.

Corollary 1. Let $0 < \inf p_k < \sup p_k < \infty$. Then $A = (a_{nk}) \in (\gamma, m_o(u,p))$ if and only if (2.8) holds and $(a_{nk})_{n\in\mathbb{N}} \in m_o(u,p)$, for every $k \in \mathbb{N}$.

Following the techniques of Tripathy [15] and the arguments of Theorem 1, we have the following result.

Theorem 2. Let $0 < \inf p_k \leq \sup p_k < \infty$. Then $A = (a_{nk}) \in (\gamma, m(u,p); P)$ if and only if (2.8) holds and $(a_{nk} - 1)_{n\in\mathbb{N}} \in m_o(u,p)$ for all $k = 1, 2, 3, ...$ In this transformation, the limit is preserved.

Theorem 3. Let $0 < \inf p_k \leq \sup p_k < \infty$. Then $A = (a_{nk}) \in (\gamma_o, m(u,p))$ if and only if (2.8) holds and

(3.2) $$(\triangle a_{nk})_{n\in\mathbb{N}} \in m(u,p), \text{ for each fixed } k \in \mathbb{N}.$$

Proof. The necessity of (2.8) follows from the inclusion $(\gamma_o, m(u,p)) \subset (\gamma_o, l_\infty(u,p))$ and Lemma 7. The necessity of (3.2) follows on considering the series (s_k) whose k-th term is 1 and $(k-1)$-th term is -1 and rest are zero. Putting $S = 0$ in (2.10) we have

$$A_n s = \sum_{k=1}^{\infty} u_k \triangle a_{nk} S_k, \text{ for all } n = 1, 2, 3....$$

Following the techniques of Theorem 1, it can be shown that $As \in m(u,p)$. This completes the proof of the Theorem.

The following result is an easy consequence of the above Theorem.

Corollary 2. Let $0 < \inf p_k \leq \sup p_k < \infty$. Then $A = (a_{nk}) \in (\gamma_o, m_o(u,p))$ if and only if (2.8) holds and

(3.3) $$(\triangle a_{nk})_{n\in\mathbb{N}} \in m_o(u,p), \quad \text{for all } k = 2, 3....$$

Note 2. Taking $p_n = 1$ for all $n \in \mathbb{N}$ in Theorem 1, Corollary 1, Theorem 2, Theorem 3 and Corollary 2, we have the characterization of the matrix classes (γ, m), (γ, m_o), $(\gamma, m; P)$, (γ_o, m), and (γ_o, m_o) i.e. the result of Tripathy [15] as particular cases.

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