Int. J. Contemp. Math. Sciences, Vol. 3, 2008, no. 5, 223 - 228

Two Linear Transformations Preserving Log-Concavity

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Abstract

In this paper we prove that the linear transformation

$$Z_n = \sum_{i=0}^{n} \begin{bmatrix} n \\ i \end{bmatrix} q^{i(i-n)} x_i, \quad n = 0, 1, 2, \ldots$$

preserves Log-concavity and

$$Z_n = \sum_{i=0}^{n} \begin{bmatrix} n \\ i \end{bmatrix} q^{i(i-n)} x_i y_{n-i}, \quad n = 0, 1, 2, \ldots$$

preserves double Log-concavity.

1 Introduction

Log-concave sequences arise often in combinatorics, algebra, probability and statistics. There has been a considerable amount of research devoted to this topic in recent years. Let $\{x_i\}_{i\geq 0}$ be a sequence of non-negative real numbers. We say that $\{x_i\}$ is *Log-concave* if and only if $x_{i-1}x_{i+1} \leq x_i^2$ for all $i \geq 1$ (relevant results can see [2] and [4]). For our purpose, when a sequence is said to be *Log-concave* we always assume that it has *no internal zeros, i.e.*, there are no three indices $i < j < k$ such that $x_i, x_k \neq 0$ and $x_j = 0$. This is a natural assumption for sequences since most of the Log-concave sequences of interest to us actually meet the condition. Thus a sequences is Log-concave if and only if $x_{i-1}x_{j+1} \leq x_i x_j$ for any $j \geq i \geq 1$ (see [1]). Let $f(x), g(x) \in \mathbb{R}[x]$, we say $f(x) \geq_x g(x)$ if and only if $f(x) - g(x)$ has nonnegative coefficients.

Let $\{f_n(x)\}_{n\geq 0}$ be a sequence of polynomials in x, we say $\{f_n(x)\}_{n\geq 0}$ is *x-log -concave* if for each $n \geq 1$, $f_n^2(x) \geq_x f_{n-1}(x)f_{n+1}(x)$. The concept of the x-log-concavity was first suggested by Stanley (see [3, p.795])

Suppose $\{a(n,k)\}_{0\leq k\leq n}$ be a triangular array of nonnegative numbers. We define two linear transformations:

$$z_n = \sum_{0\leq k\leq n} a(n,k)x_k, \quad n = 0,1,2,\ldots \tag{1}$$

and

$$z_n = \sum_{0\leq k\leq n} a(n,k)x_k y_{n-k}, \quad n = 0,1,2,\ldots \tag{2}$$

If the Log-concavity of $\{x_k\}$ implies the Log-concavity of $\{z_n\}$, we say the transformation (1) preserves log concavity; correspondingly we say the transformation (2) preserves double Log-concavity if the Log-concavity of $\{x_k\}$ and $\{y_k\}$ implies the Log-concavity of $\{z_n\}$.

So far there have been found some important linear transformations preserving log-concavity. For example, it is well known that the linear transformation

$$z_n = \sum_{i=0}^{n} \binom{n}{i} x_i \tag{3}$$

preserves Log-concavity (see [1]). From [6], we know that the linear transformation

$$z_n = \sum_{i=0}^{n} \binom{n}{i} x_i y_{n-i} \tag{4}$$

preserves double Log-concavity.

From [5] we can know that the q-binomial coefficient $\begin{bmatrix} n \\ i \end{bmatrix}$ defined by:

$$\begin{bmatrix} n \\ i \end{bmatrix} = \frac{[\mathbf{n}]!}{[\mathbf{i}]![\mathbf{n\text{-}i}]!}$$

where $[\mathbf{k}]! = [1][2]\cdots[k]$ and $[j] = 1+q+q^2+\cdots+q^{j-1}$. It is well known that it has the following recursion:

$$\begin{bmatrix} n \\ i \end{bmatrix} = \begin{bmatrix} n-1 \\ i \end{bmatrix} + q^{n-i}\begin{bmatrix} n-1 \\ i-1 \end{bmatrix} \tag{5}$$

and

$$\begin{bmatrix} n \\ i \end{bmatrix} = \begin{bmatrix} n-1 \\ i-1 \end{bmatrix} + q^i \begin{bmatrix} n-1 \\ i \end{bmatrix} \tag{6}$$

In this paper we will give a q-analogue of (3) and (4).

2 The main results

Suppose $\{a(n,k)\}_{0\le k\le n}$ be a triangular array of nonnegative numbers. Let

$$\mathcal{A}_r(n,x) = \sum_{r\le k\le n} a(n,k)x^k$$

and

$$\mathcal{A}_r^*(n,x) = \sum_{r\le k\le n} a(n,n-k)x^k.$$

In [6], Wang and Yeh gave a sufficient condition for a linear transformation preserves Log-concavity and double Log-concavity.

Theorem 2.1 *[6] Let $\{a(n,k)\}, \mathcal{A}_r(n,x)$ and $\mathcal{A}_r^*(n,x)$ be defined as above.*

(i) If $\mathcal{A}_r(n,x)$ is x-log-concave in n, then the linear transformation (1) preserves Log-concavity;

(ii) If $\mathcal{A}_r(n,x)$ and $\mathcal{A}_r^*(n,x)$ both are x-log-concave in n, then the linear transformation (2) preserves double Log-concavity.

For $0 \le k \le n$, let $B_{(n,k)}(x) = \sum_{i=k}^{n} \begin{bmatrix} n \\ i \end{bmatrix} x^i q^{i(i-n)}$. Then we have the following lemma.

Lemma 2.2 *Suppose $0 \le k \le n$ and $q \ge 1$, then we have*

$$B_{(n,k)}^2(x) \ge_x B_{(n-1,k)}(x)B_{(n+1,k)}(x).$$

Proof. Firstly, by induction on k, we prove the following inequality

$$B_{(n,k)}(x)B_{(n-1,k)}(x/q) \ge_x B_{(n-1,k)}(x)B_{(n,k)}(x/q). \tag{7}$$

Since

$$\begin{aligned}
& B_{(n,n-1)}(x)B_{(n-1,n-1)}(x/q) - B_{(n-1,n-1)}(x)B_{(n,n-1)}(x/q) \\
= \ & \frac{x^{n-1}}{q^{n-1}}\left(\begin{bmatrix} n \\ n-1 \end{bmatrix} x^{n-1}q^{-(n-1)} + x^n\right) - x^{n-1}\left(\begin{bmatrix} n \\ n-1 \end{bmatrix} x^{n-1}q^{-2(n-1)} + x^n q^{-n}\right) \\
= \ & \frac{q-1}{q^n}x^{2n-1},
\end{aligned}$$

then (7) holds in the case $k = n-1$. By definition we have that

$$B_{(n,k)}(x) \;=\; B_{(n,k+1)}(x) + \begin{bmatrix} n \\ k \end{bmatrix} x^k q^{k(n-k)}.$$

Then

$$\begin{aligned}
& B_{(n,k-1)}(x) y_{(n-1,k-1)}(x/q) - y_{(n-1,k-1)}(x) y_{(n,k-1)}(x/q) \\
= \; & \det \begin{bmatrix} B_{(n,k)}(x) + \begin{bmatrix} n \\ k-1 \end{bmatrix} x^{k-1} q^{(k-1)(k-n-1)} & B_{(n,k)}(x/q) + \begin{bmatrix} n \\ k-1 \end{bmatrix} x^{k-1} q^{(k-1)(k-n)} \\ B_{(n-1,k)}(x) + \begin{bmatrix} n-1 \\ k-1 \end{bmatrix} x^{k-1} q^{(k-1)(k-n)} & B_{(n-1,k)}(x/q) + \begin{bmatrix} n-1 \\ k-1 \end{bmatrix} x^{k-1} q^{(k-1)(k-n+1)} \end{bmatrix} \\
= \; & B_{(n,k)}(x) y_{(n-1,k)}(x/q) - B_{(n,k)}(x/q) B_{(n-1,k)}(x) \\
& + \begin{bmatrix} n \\ k-1 \end{bmatrix} x^{k-1} q^{(k-1)(k-n-1)} B_{(n-1,k)}(x/q) - \begin{bmatrix} n-1 \\ k-1 \end{bmatrix} x^{k-1} q^{(k-1)(k-n)} B_{(n,k)}(x/q) \\
& + \begin{bmatrix} n-1 \\ k-1 \end{bmatrix} x^{k-1} q^{(k-1)(k-n+1)} B_{(n,k)}(x) - \begin{bmatrix} n \\ k-1 \end{bmatrix} x^{k-1} q^{(k-1)(k-n)} B_{(n-1,k)}(x),
\end{aligned}$$

for $k \le i \le n$, the coefficient of x^{i+k-1} in the last two terms is

$$q^{(k-1)(k-n+1)+i(i-n)} \frac{[n]![n-1]![i-k]!(q^{i+1-k}-1)}{[k-1]![i]![n-k+1]![n-i]!}.$$

Thus (7) holds by induction.

By (5), we have

$$\begin{aligned}
B_{(n,k)}(x) \;&=\; \sum_{i \ge k} \begin{bmatrix} n \\ i \end{bmatrix} x^i q^{i(i-n)} \\
&=\; \sum_{i \ge k} \begin{bmatrix} n-1 \\ i \end{bmatrix} x^i q^{i(i-n)} + \sum_{i \ge k} \begin{bmatrix} n-1 \\ i-1 \end{bmatrix} x^i q^{i(i-n)+(n-i)} \\
&=\; B_{(n-1,k)}(x/q) + x B_{(n-1,k)}(x) + \begin{bmatrix} n-1 \\ k-1 \end{bmatrix} x^{k-1} q^{(k-1)(k-n)}.
\end{aligned}$$

Then

$$\begin{aligned}
& B^2_{(n+1,k)}(x) - B_{(n+2,k)}(x) B_{(n,k)}(x) \\
= \; & B_{(n+1,k)}(x) \left(B_{(n,k)}(x/q) + x B_{(n,k)}(x) + \begin{bmatrix} n \\ k-1 \end{bmatrix} x^{k-1} q^{(k-1)(k-1-n)} \right) \\
& - B_{(n,k)}(x) \left(B_{(n+1,k)}(x/q) + x B_{(n+1,k)}(x) + \begin{bmatrix} n+1 \\ k-1 \end{bmatrix} x^{k-1} q^{(k-1)(k-n-2)} \right) \\
= \; & B_{(n+1,k)}(x) B_{(n,k)}(x/q) - B_{(n,k)}(x) B_{(n+1,k)}(x/q) \\
& + \begin{bmatrix} n \\ k-1 \end{bmatrix} x^{k-1} q^{(k-1)(k-n-1)} B_{(n+1,k)}(x) - \begin{bmatrix} n+1 \\ k-1 \end{bmatrix} x^{k-1} q^{(k-1)(k-n-2)} B_{(n,k)}(x).
\end{aligned}$$

For $k \leq i \leq n$, the coefficient of x^{i+k} is $q^{(k-1)(k-n-1)+i(i-n-1)}\frac{[n]![n+1]![i+1-k]!}{[k-1]![i]![n-k+2]![n-i+1]!}$ in the last line. Combining with (7) we complete the proof. □

By Theorem 2.1 and Lemma 2.2, we immediately obtain the following result.

Theorem 2.3 *The linear transformation*

$$Z_n = \sum_{i=0}^{n} \begin{bmatrix} n \\ i \end{bmatrix} q^{i(i-n)} x_i, \ n = 0, 1, 2, \ldots$$

preserves Log-concavity.

For $0 \leq k \leq n$, let $B^*_{(n,k)}(x) = \sum_{i=k}^{n} \begin{bmatrix} n \\ n-i \end{bmatrix} x^i q^{-i(n-i)}$. It is easy to find that $B_{(n,k)}(x) = B^*_{(n,k)}(x)$. Then by Theorem 2.1 and Lemma 2.2, we can obtain another result as follows.

Theorem 2.4 *The linear transformation*

$$Z_n = \sum_{i=0}^{n} \begin{bmatrix} n \\ i \end{bmatrix} q^{i(i-n)} x_i y_{n-i}, \ n = 0, 1, 2, \ldots$$

preserves double Log-concavity.

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Received: August 17, 2007