Int. J. Contemp. Math. Sciences, Vol. 3, 2008, no. 7, 311 - 317

Range-Preserving Maps on Operator Algebras

A. Taghavi and E. Dadar

Department of Mathematics
Faculty of Basic Sciences
Mazandaran University, Babolsar, Iran
taghavi@nit.ac.ir

Abstract

Let X, Y and Z are Banach spaces, $B(X,Z)$ denote the algebras of all bounded linear operators from X into Z and $C(X)$ be all of continuous complex valued function on X . We show that if φ be a map from $B(X,Z)$ into $B(Y,Z)$ satisfies $(uof)\varphi(T)(Y) = uT(X)$ for every $T \in B(X,Z)$, $u \in C(X)$ and for some continuous function f from Y into X, then φ is one-to-one. In particular, If $X = Y$, then by choosing $f = I$ we conclude that φ is identity.

Mathematics Subject Classification: Primary 46J10, 47B48

Keywords: Operator algebra, range preserving map

1 Introduction

A map φ from a Banach algebra A into a Banach algebra B is called spectrum-preserving if the spectrum $\sigma(x)$ coincides with the spectrum $\sigma(\varphi(x))$ for every $x \in A$. The linear preserver problems including on spectrum-preserving linear maps mainly concerns with non-commutative Banach algebra and has seen much progress recently.

Kowalski and Slodkowski [4] proved the following theorem: Additively spectrum-preserving functionals on a Banach algebra are linear and multiplicative. Due to the above theorem, linearity and multiplicativity are not hypotheses but conclusions for additively spectrum-preserving maps from Banach algebras into semi-simple commutative Banach algebras. Surjective linear maps between Banach algebras which preserves the spectrum are extensively studied in connection with a longstanding open problem sometimes called Kaplansky's problem oninvertibility preserving linear maps. Jafarian and Sourour in [3] were proved that every surjective linear map on $B(X)$ of all bounded linear operators acting on a Banach space, which preserves the spectrum is a Jordan homomorphism. On the other hand Molnar [5] considered multiplicatively spectrum-preserving surjective maps on Banach algebras and proved that the maps are almost isomorphisms in the sense that isomorphisms multiplicatived by a signum function for the Banach algebra of all complex-valued continuous functions on a first countable compact Housdorff space. Rao and roy [6] generalized the theorem of Molnar for the case of uniform algebras and also proved a similar result for uniformly closed subalgebras of the algebra of complex valued continuous functions which vanish at infinity on locally compact Housdorff spaces. For a compact Housdorff X, the spectrum-preserving mappings on C(X) are studied by Hatiri, Miura and Takagi[2], for mappings φ that satisfy multiplicativity conditions such as

$$(\varphi(f)\varphi(g))(Y) = \ fg(X) \quad or \quad (\bar{\varphi(f)}\varphi(g))(Y) = \bar{f}g(X)$$

for all $f, g \in C(X)$.

In this paper we consider range-preserving mappings on B(X, Z), denote the algebras all of bounded linear operators of Banach space X into Z.

Define

$$P_x = \ \{u \in C(X)\ ;\ u(X) \subset D_1 \cup \{1\},\ u(x) = 1\ \}$$

for every $x \in ch(C(X))$ and

$$Q_x = \{T \in B(X, Z)\ ;\ \|T\| \leq 1, \quad \| Tx \| = 1\}$$

for every $x \in ch(B(X, Z))$.

And also

$$K_f = \{x \in X\ ; \quad f(x) = 1\}$$

and

$$L_T = \{x \in X\ ; \quad \|Tx\| = 1\}$$

for $f \in \mathcal{P} = \cup_{x \in ch(C(X))} P_x$ and $T \in \mathcal{Q} = \cup_{x \in ch(A)} Q_x$.

If $x_0 \in ch(A)$ and F be a closed subset of X with $x_0 \notin F$ and if $\varepsilon > 0$ then there exists a $u \in P_{x_0}$ such that $|u(x)| < \varepsilon$ for $x \in F$. $X \backslash F$ is open and consists x_0 , therefore there exists a neighborhood V_0 in x_0 such that $V_0 \subset F^c$ and since $x_0 \in ch(A)$, from definition for V_0 there is a peak function f that $K_f \subset V_0$. Then we have $|f(x)| < 1$ for $x \in F$. Let

$$u(x) = \begin{cases} \varepsilon f(x) & x \notin K_f \\ f(x) = 1 & x \in K_f \end{cases} \tag{1}$$

and it is easy to see that $u \in P_{x_0}$ and $|u(x)| < \varepsilon$ for $x \in F$.
For every $T \in B(X, Z)$ and $f \in C(X)$, define $Tf(x) = T(x)f(x)$. It is easy to see that Tf is a continuous function on X .

2 Main results

Lemma 2.1: Let $S, T \in \mathcal{Q}$, then $L_T \subset L_S$ if and only if $\|Su(x)\| = 1$ for some $x \in \bar{D}_1$ for every $u \in \mathcal{P}$ with $\|Tu(x)\| = 1$ for some $x \in \bar{D}_1$.

Proof: Let $L_T \subset L_S$ and for $u \in \mathcal{P}$ and $x \in \bar{D}_1$ have $\|Tu(x)\| = 1$ then $|u(x)|\|Tx\| = 1$. Since $u(X) \subset D_1 \cup \{1\}$ and $\|T\| \leq 1$ then $|u(x)| = \|Tx\| = 1$ and hence $x \in L_T \subset L_S$. It means $\|S(x)\| = 1$ and therefore $\|Su(x)\| = |u(x)|\|Sx\| = 1$.
Suppose $L_T \not\subset L_S$ and consider $x_0 \in L_T \backslash L_S$. Since L_S is a closed set and $x_0 \notin L_S$, there exists $u \in P_{x_0}$ such that $|u(x)| < 1$ for $x \in L_S$. On the other hand we have $\|Tx_0u(x_0)\| = 1$ while $\|Sx_0u(x_0)\| = \|S(x_0)\| \neq 1$ and the proof is completed. □

Lemma 2.2: Suppose $T \in B(X, Z)$, $x_0 \in ch(A)$ such that $Tx_0 = y$ and $y \neq 0$ then there exists $u \in P_{x_0}$ such that $\frac{1}{\|y\|}\ Tu(\bar{D}_1) \subset D_1 \cup \{y\}$ and $\frac{1}{\|y\|}\ \|Tu(x_0)\| = 1$.

Proof: Define

$$F_0 = \{\ x \in \bar{D}_1\ ;\ \ \|Tx - y\| \geq \|y\|/2\}$$

$$F_n = \{\ x \in \bar{D}_1\ ;\ \ \|y\|/2^{n+1} \leq \|Tx - y\| \leq \|y\|/2^n\} \qquad n = 1, 2, ...$$

It is clear that F_n for $n = 0, 1, 2, ...$ are closed subset of X with $x_0 \notin F_n$. Hence there exists u_0, u_1, u_2, ... in P_{x_0} that

$$|u_0(x)| < \frac{\|y\|}{\|T\|} \qquad \forall x \in F_0$$

$$|u_n(x)| < \frac{1}{2^n + 1} \qquad \forall x \in F_n$$

Now put

$$u = u_0 \sum_{k=1}^{\infty} \frac{u_k}{2^k}$$

The above series is majorized by the convergent series $\Sigma \frac{1}{2^k}$, so u is well define and $u \in P_{x_0}$. Put $S = \frac{1}{\|y\|}\ Tu$. Let $x \in \bar{D_1}$. If $x \in F_0$ then we have

$$\|Sx\| = \frac{1}{\|y\|}\ \|Tx\|\ |u_0(x)| \sum_{k=1}^{\infty} \frac{|u_k(x)|}{2^k} < \frac{1}{\|y\|}\ \|T\|\ \frac{\|y\|}{\|T\|} \sum_{k=1}^{\infty} \frac{1}{2^k} = 1$$

If $x \in F_n$ for some $n \in \{\ 1, 2, ...\}$, then

$$\|Sx\| = \frac{1}{\|y\|}\ \|Tx\|\ |u_0(x)|(\frac{|u_n(x)|}{2^n} + \sum_{k \neq n} \frac{|u_k(x)|}{2^k})$$

$$\leq \frac{1}{\|y\|}\ (\ \|Tx - y\| + \|y\|)\ (\frac{|u_n(x)}{2^n} + \sum_{k \neq n} \frac{1}{2^k})$$

$$< \frac{1}{\|y\|}\ (\frac{\|y\|}{2^n} + \|y\|)\ (\frac{1}{2^n} \frac{1}{2^n + 1} + 1 - \frac{1}{2^n}) = 1$$

If $x \in \bar{D_1} \backslash \cup_{n=1}^{\infty} F_n$ then $Tx = y$ and so $Sx = \frac{1}{\|y\|}\ Tx\ u(x) = \frac{y}{\|y\|} u(x)$ then $\|Sx\| \leq 1$ and $S(\bar{D_1}) \subset D_1 \cup \{y\}$ and also $\|S(x_0)\| = 1$ and the proof is completed. □

Lemma 2.3: For every $T, S \in B(X, Z)$, $T = S$ if and only if $(Tu)(X) = (Su)(X)$ for all $u \in \mathcal{P}$.

Proof: Suppose $T \neq S$ on X, then $T \neq S$ on ch(A), that is there exists $x_0 \in ch(A)$such that $Tx_0 \neq Sx_0$. Without loss of generality, we can assume that $\|Tx_0\| \leq \|Sx_0\|$ and $x_0 \in \bar{D_1}$. If $Tx_0 \neq 0$, then the Lemma 2.2 gives a $u \in P_{x_0} \subseteq \mathcal{P}$ such that $\frac{1}{\|y\|} Tu(\bar{D_1}) \subset D_1 \cup \{y\}$ and $\frac{1}{\|y\|}\ \|Tu(x_0)\| = 1$

while $Su(x_0) = Sx_0$ and Sx_0 cannot lie in $D_1 \cup \{y\}$, where $y = Tx_0$. Hence $Su(X) \neq Tu(X)$.
Now if $Tx_0 = 0$ then $Sx_0 \neq 0$. Let $r = \|Sx_0\|$ and

$$F = \{x \in \bar{D_1}\ ;\ \ \|Tx\| \geq r\ \}$$

Since F is a closed subset of X and $x_0 \notin F$, there exists $u \in P_{x_0}$such that $|u(x)| < \frac{r}{\|T\|+1}$ for all $x \in F$. Consequently

$$\|Tu(x)\| = \|Tx\|\ |u(x)| \begin{cases} \leq \|T\|\frac{r}{\|T\|+1} < r & x \in F \\ < r\|u\| = r & x \in X \backslash F \end{cases} \tag{2}$$

Then for every $x \in X$, have $\|Tu(x)\| < r = \|Su(x_0)\|$. Hence $(Tu)(X) \neq (Su)(X)$. □

Theorem 2.4. Let X, Y and Z are Banach spaces and $\varphi : B(X,Z) \to B(Y,Z)$ be a map with

$$(uof)\varphi(T)(Y) = uT(X) \qquad T \in B(X,Z),\ u \in C(X)$$

for some continuous function f from Y into X, then φ is one-to-one.

Proof: Suppose $\varphi(T) = \varphi(S)$ for $S,\ T \in B(X,Z)$. We have

$$uT(X) = (uof)\varphi(T)(Y) = (uof)\varphi(S)(Y) = uS(X),$$

for all $u \in \mathcal{P}$ and for some f for some continuous function f from Y into X that from the lemma 2.3 imply $T = S$. Hence φ is one-to-one. □

Remark If in the above theorem $X = Y$, then by choosing $f = I$ Lemma 2.3 implies that φ is identity.

Theorem 2.5. Let X be a Banach space and $\varphi : B(X) \to B(X)$ be a map with

$$u(\varphi(T)\ \varphi(S))(X) = uT\ S(X) \qquad T,S \in B(X), \quad u \in C(X),$$

then $\varphi(T) = \varphi(I)T$ for every $T \in B(X)$.

Proof: Let $\tilde{\varphi}$ be a map from $B(X)$ into itself defining by

$$\tilde{\varphi}(T) = \varphi(I)\ \varphi(T)$$

clearly $\tilde{\varphi}$ is a map from $B(X)$ into $B(X)$ with

$$u\tilde{\varphi}(T)(X) = u\varphi(I)\ \varphi(T)(X) = uI\ T(X) = uT(X)$$

for all $u \in \mathcal{P}$ and $T \in B(X)$ According to above remark we find $\tilde{\varphi}$ is identity.

Hence $\varphi(I)\varphi(T) = T$. It implies that $\varphi(I)^2 = I$. Therefore, $\varphi(T) = \varphi(I)T$ for every $T \in B(X)$ and the proof is completed. □

Theorem 2.6. Let X be a Banach space and $\varphi : B(X) \to B(X)$ be a map with

$$(\varphi(T)\ \varphi(S))(X) = T\ S(X) \qquad T, S \in B(X),$$

then φ is a range preserving map.

Proof: We have

$$\varphi(I)^2(X) = \varphi(I)\varphi(I)(X) = I^2(X) = X.$$

Hence,

$$X = \varphi(I)^2(X) \subset \varphi(I)(X) \subset X.$$

$$\varphi(I)(X) = X.$$

This means that $\varphi(I)$ is onto. By the hypothesis $\varphi(S)(X) = \varphi(S)\varphi(I)(X) = SI(X) = S(X)$. □

References

1. O.Hatori,T.Miura and H.Takagi, *Characterizations of isometric isomorphisms between uniform algebras via nonlinear range-preserving properties,* Proc.Amer.Math.Soc.134(2006) 2923-2930.

2. O.Hatori,T.Miura and H.Takagi, *unital and multiplicatively spectrum-preserving surjections between semi-simple commutative Banach algebras are linear and multiplicative,* Proc.Amer.Math.Soc. (2007)

3. A.A. Jafarian and A.R. Sourour, Spectrum-preserving linear maps, J. Funct. Anal. 66 (1986), 255-261. MR 86m:47011.

4. S.Kowalski and Z.Slodkowski, *A characterization of multiplicative linear functionals in Banach algebras,* Studia Math. 67(1980) 215-223.

5. L.Molnar *Some characterization of the automorphisms of B(H) and C(X),* Proc.Amer.Math.Soc. 130(2002) 111-120.

6. N.V.Rao and A.K.Roy, *Multiplicatively spectrum-preserving maps of function algebras,* Proc.Amer.Math.Soc.133(2005) 1135-1142.

Received: September 21, 2007