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Structure and Some Geometric Properties of Generalized Cesaro Sequence Space

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Abstract. In this paper, we define a modular functional on the generalized Cesaro sequence space $ces[(p_n),(q_n)]$ and investigate structure of this space equipped with the Luxemburg norm. Also we study the geometric properties which are Kadec-Klee (H-property), Rodunt (R) and locally uniformly rodunt (LUR). Finally, we prove that this space posses H-property and this space is not Rotund and therefore not LUR.

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1. Introduction

Throughout the paper, the sets of natural numbers, sets of real numbers and field will be denoted by $\mathbb{N}$, $\mathbb{R}$ and $\mathbb{F}$, respectively. Let $(X, \|\cdot\|)$ be a real Banach space and let $B(X)$ (resp. $S(X)$) be the closed unit ball (resp. unit sphere) of X.

A point $x \in S(X)$ is an $H-point$ of $B(X)$ if for any sequence (x_n) in X such that $\|x_n\| \to 1$ as $n \to \infty$, the weak convergence of (x_n) to x (write $x_n \xrightarrow{w} x$) implies that $\|x_n - x\| \to 0$ as $n \to \infty$. If every point of $S(X)$ is an H-point of $B(X)$; then X is said to have $H-property$ $(Kadec-Klee)$.

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Briefly; X is said to have the property (H), if for any sequence on the unit sphere of X, weak convergence coincides norm convergence.

A point $x \in S(X)$ is an *extreme point* of $B(X)$, if for any $y, z \in S(X)$, the equality $x = \frac{y+z}{2}$ implies $y = z$.

A Banach space X is said to be *rotund*(R) if for every point of $S(X)$ is an extreme point of $B(X)$.

A point $x \in S(X)$ is an *locally uniformly rotund* point of $B(X)$, (*LUR*-point) if for any sequence (x_n) in $B(X)$ such that $\|x_n + x\| \to 2$ as $n \to \infty$, there holds $\|x_n - x\| \to 0$ as $n \to \infty$. If every point of $S(X)$ is a LUR-point of $B(X)$, then X is called *locally uniformly rotund* (LUR). It is known that if X is LUR, then it is (R) and posses property (H). But converse of this last statement is not true in general.

For these geometric notions and their role in Mathematics we refer to the monographs [1], [2], [3], [4] and [5]. Some of these geometric properties were studied for Orlicz spaces in [9], [10], [11] and [12].

If $\{q_n\}$ is positive sequence of real numbers, then for $p = (p_r)$ with $inf p_r > 0$, In [6], Khan and Rahman defined the space $ces\left[(p_n),(q_n)\right]$ by

$$ces\left[((p_n),(q_n)\right] = \left\{ x \in w : \sum_{r=0}^{\infty} \left(\frac{1}{Q_{2^r}} \sum_r q_k |x_k| \right)^{p_r} < \infty \right\},$$

where $Q_{2^r} = q_{2^r} + q_{2^r+1} + ... + q_{2^{r+1}-1}$ and $\sum\limits_r$ denotes summation over the range $2^r \leq k < 2^{r+1}$. Also Khan and Rahman [6] showed that if $q_n = 1$ for all n, then $ces\left[(p_n),(q_n)\right]$ reduces to $ces(p_n)$ studied by Lim [7]. Besides, they showed that; if $p_n = p$ for all n, then $ces\left[(p_n),(q_n)\right]$ reduces to ces_p studied by Lim [8]. In [5], Sanhan and Suantai investigated some geometrical properties of $ces(p_n)$. Moreover Khan and Rahman [6] showed that the space $ces\left[(p_n),(q_n)\right]$ is paranormed by

$$g(x) = \left[\sum_{r=0}^{\infty} \left(\frac{1}{Q_{2^r}} \sum_r q_k |x_k| \right)^{p_r} \right]^{1/M},$$

where $H = sup_r p_r < \infty$ and $M = max(1, H)$. That is, the sequence space $ces\left[(p_n),(q_n)\right]$ has the paranorm $g(x)$.

Let w be the space of all (real or complex) sequences and let l_∞ and c respectively be the Banach spaces of bounded and convergent sequences $x = (x_n)$ endowed with the norm,

$$\|x\| = \sup_{k \geq 0} |x_k|.$$

If the functional $\sigma(x)$ on w has the following properties, it is called modular on w:

i) $\sigma(x) = 0 \Longleftrightarrow x = 0$;
ii) $\sigma(\alpha x) = \sigma(x), \forall \alpha \in \mathbb{F}$ with $|\alpha| = 1$, for all $x \in X$;
iii) $\sigma(\alpha x + \beta y) = \sigma(x) + \sigma(y)$ if $\alpha, \beta \geq 0$, $\alpha + \beta = 1$, for all $x, y \in X$,
if the property (iii) is replaced by
iv) $\sigma(\alpha x + \beta y) \leq \alpha\sigma(x) + \beta\sigma(y)$, for all $\alpha, \beta \in R^+$ with $\alpha + \beta = 1$;
then we say σ is a convex modular.

We can introduce the modular $\sigma(x)$ on the sequence space $ces\,[(p_n),(q_n)]$ as follows:

$$\sigma(x) : ces\,[(p_n),(q_n)] \to [0,\infty]\,, \qquad \sigma(x) = \sum_{r=0}^{\infty}\left(\frac{1}{Q_{2^r}}\sum_{r} q_k|x_k|\right)^{p_r}.$$

For all $x \in ces\,[(p_n),(q_n)]$, we define a norm as follows:

$$\|x\| = \inf\left\{\tau > 0 : \sigma\left(\frac{x}{\tau}\right) \leq 1\right\}.$$

The function $||\cdot||$ is called the Luxemburg norm on the sequence space $ces\,[(p_n),(q_n)]$.

Also in this paper we will show that sequence space $ces\,[(p_n),(q_n)]$ is Banach space with respect to Luxemburg norm.

Note that Luxemburg norm on the sequence space $ces\,[(p_n),(q_n)]$ is defined as follows:

$$\|x\| = \inf\left\{\tau > 0 : \sigma\left(\frac{x}{\tau}\right) = \sum_{r=0}^{\infty}\left(\frac{1}{Q_{2^r}}\sum_{r} q_k|x_k|\right)^{p_r}\right\}.$$

From [7], we can easily verify that; the norm defined by Luxemburg norm on the $ces\,[(p_n),(q_n)]$ reduces to

$$\|x\|_{ces(p_n)} = \inf\left\{\tau > 0 : \sigma\left(\frac{x}{\tau}\right) = \sum_{r=1}^{\infty}\left(\frac{1}{k}\sum_{k=1}^{r}|\frac{x_k}{\tau}|\right)^{p_r} \leq 1\right\},$$

if $q_n = 1$ for all $n \in N$.

Furthermore, if $p_n = p$ for all $n \in N$ and $q_n = 1$ for all $n \in N$ then the Luxemburg norm on the sequence space $ces\,[(p_n),(q_n)]$ reduces to norm defined by

$$\|x\|_{ces_p} = \inf\left\{\tau > 0 : \sigma\left(\frac{x}{\tau}\right) = \sum_{r=1}^{\infty}\left(\frac{1}{n}\sum_{k=1}^{r}|\frac{x_k}{\tau}|^p\right)^{1/p} \leq 1\right\},$$

(see; [8]).

Let us give a inequality that we will require throughout this paper: Let $p = (p_k)$ be monotone increasing sequence of positive real numbers, we have

$$|a_k + b_k|^{p_k} \leq C\left[|a_k|^{p_k} + |b_k|^{p_k}\right]$$

where $a_k > 0, b_k > 0$ and $C = 2^{H-1}$, $H = \sup_k p_k$.

One of main purpose of this work, the space $ces\left[(p_n),(q_n)\right]$ equipped with the Luxemburg norm is to show modular space and to investigate some geometric structure of this space.

2. Main Results

In this section we introduce and prove some technical results which will be used in the proof of the main theorems. Let us begin by recalling the following lemmas about the notion of convex modular on the sequence space $ces\left[(p_n),(q_n)\right]$.

Lemma 1. *The functional σ is a convex modular on $ces\left[(p_n),(q_n)\right]$.*

Proof. Let $x, y \in ces\left[(p_n),(q_n)\right]$. It is obvious that;
(i) $\sigma(x) = 0 \Leftrightarrow x = 0$ and
(ii) $\sigma(\lambda x) = \sigma(x)$ for all scalar λ with $|\lambda| = 1$.

$$\begin{aligned} \sigma(\lambda x) &= \sum_{r=0}^{\infty}\left(\frac{1}{Q_{2^r}}\sum_{r} q_k|\lambda x_k|\right)^{p_r} \\ &= \sum_{r=0}^{\infty}|\lambda|^{p_r}\left(\frac{1}{Q_{2^r}}\sum_{r} q_k|x_k|\right)^{p_r} \\ &= \sum_{r=0}^{\infty}\left(\frac{1}{Q_{2^r}}\sum_{r} q_k|x_k|\right)^{p_r} \\ &= \sigma(x) \end{aligned}$$

(iii) For $\lambda, \beta \geq 0$ with $\lambda + \beta = 1$, by the convexity $t \rightarrow |t|^{p_k}$, for every $k \in N$, we have

$$\begin{aligned} \sigma(\lambda x + \beta y) &= \sum_{r=0}^{\infty}\left(\frac{1}{Q_{2^r}}\sum_{r} q_k|\lambda x_k + \beta y_k|\right)^{p_r} \\ &\leq \sum_{r=0}^{\infty}\left(\frac{1}{Q_{2^r}}\left[\lambda\sum_{r} q_k|x_k| + \beta\sum_{r} q_k|y_k|\right]\right)^{p_r} \\ &\leq \sum_{r=0}^{\infty}\left[\lambda\left(\frac{1}{Q_{2^r}}\sum_{r} q_k|x_k|\right) + \beta\left(\frac{1}{Q_{2^r}}\sum_{r} q_k|y_k|\right)\right]^{p_r} \\ &\leq \lambda^{p_r}\sum_{r=0}^{\infty}\left(\frac{1}{Q_{2^r}}\sum_{r} q_k|x_k|\right)^{p_r} + \beta^{p_r}\sum_{r=0}^{\infty}\left(\frac{1}{Q_{2^r}}\sum_{r} q_k|y_k|\right)^{p_r} \\ &\leq \lambda\sigma(x) + \beta\sigma(y). \end{aligned}$$

∎

Lemma 2. *For $x \in ces\,[(p_n),(q_n)]$, the modular σ on $ces\,[(p_n),(q_n)]$ satisfies the following properties:*

(i) if $0 < a < 1$, then $a^H\sigma(\frac{x}{a}) \leq \sigma(x)$ and $\sigma(ax) \leq a\sigma(x)$,

(ii) if $a > 1$, then $\sigma(x) \leq a^H\sigma(\frac{x}{a})$,

(iii) if $a \geq 1$, then $\sigma(x) \leq a\sigma(x) \leq \sigma(ax)$.

Proof. (i) Let $0 < a < 1$. Then we have

$$\begin{aligned}
\sigma(x) &= \sum_{r=0}^{\infty}\left(\frac{1}{Q_{2^r}}\sum_r q_k|x_k|\right)^{p_r} \\
&= \sum_{r=0}^{\infty}\left(\frac{1}{Q_{2^r}}a\sum_r q_k|\frac{x_k}{a}|\right)^{p_r} \\
&= \sum_{r=0}^{\infty}a^{p_r}\left(\frac{1}{Q_{2^r}}\sum_r q_k|\frac{x_k}{a}|\right)^{p_r} \\
&\geq a^H\sigma(\frac{x}{a}).
\end{aligned}$$

(ii) Let $a > 1$. Then we have

$$\begin{aligned}
\sigma(x) &= \sum_{r=0}^{\infty}\left(\frac{1}{Q_{2^r}}\sum_r q_k|x_k|\right)^{p_r} \\
&= \sum_{r=0}^{\infty}\left(\frac{1}{Q_{2^r}}a\sum_r q_k|\frac{x_k}{a}|\right)^{p_r} \\
&= \sum_{r=0}^{\infty}a^{p_r}\left(\frac{1}{Q_{2^r}}\sum_r q_k|\frac{x_k}{a}|\right)^{p_r} \\
&\leq a^H\sum_{r=0}^{\infty}\left(\frac{1}{Q_{2^r}}\sum_r q_k|\frac{x_k}{a}|\right)^{p_r} \\
&\leq a^H\sigma(\frac{x}{a}).
\end{aligned}$$

(iii) It is clear that (iii) is satisfied by the convexity of σ. ∎

Lemma 3. *For any $x \in ces\,[(p_n),(q_n)]$,*

(i) if $\|x\| < 1$, then $\sigma(x) \leq \|x\|$,

(ii) if $\|x\| > 1$, then $\sigma(x) \geq \|x\|$,

(iii) $\|x\| = 1$ if and only if $\sigma(x) = 1$,

(iv) if $\|x\| < 1$, then $\sigma(x) < 1$,

(v) if $\|x\| > 1$, then $\sigma(x) > 1$.

Proof. (i) Let $\varepsilon > 0$ be such that $0 < \varepsilon < 1 - ||x||$. Then we obtain that $\varepsilon + ||x|| < 1$. By definition of the norm; there exists $\tau > 0$ such that $\varepsilon + ||x|| > \tau$ and $\sigma(x\tau) \leq 1$. From Lemma 2 (i) and (iii), we have

$$\begin{aligned} \sigma(x) &\leq \sigma\left((\varepsilon + \|x\|)x\tau\right) = \sigma\left((\varepsilon + \|x\|)\frac{x}{\tau}\right) \\ &\leq (\varepsilon + \|x\|)\,\sigma(x\tau) \leq \varepsilon + \|x\|\,. \end{aligned}$$

Hence we obtain that; $\sigma(x) \leq \|x\|$, so (i) is satisfied.

(ii) If $\|x\| > 1$ and $\varepsilon > 0$ then $0 > 1 - ||x||$ and $0 > \frac{1-||x||}{||x||}$. Hence we get that $\frac{||x||-1}{||x||} > 0$. Let $\varepsilon > 0$ be such that $0 < \varepsilon < \frac{||x||-1}{||x||}$. Since $\frac{||x||-1}{||x||} > 0$ and $||x||(\varepsilon - 1) < -1$, we can write that -$\frac{1}{||x||(\varepsilon-1)} < 1 < \frac{1}{||x||(\varepsilon-1)}$.

By the definition of norm and Lemma 2 (i) , we have $1 < \sigma(\frac{x}{||x||(1-\varepsilon)}) \leq \frac{1}{||x||(1-\varepsilon)}\sigma(x)$.

So $||x||(1-\varepsilon) < \sigma(x)$ for all $\varepsilon \in \left(0, \frac{||x||-1}{||x||}\right)$. Put A= $\left\{(1-\varepsilon)||x|| : 0 < \varepsilon < \frac{||x||-1}{||x||}\right\}$ then $||x|| = supA$. Since $\sigma(x)$ is an upper bound of A, we have $\|x\| \leq \sigma(x)$.

(iii) Assume that $||x|| = 1$. Let $\varepsilon > 0$, then there exists $\delta > 0$ such that $1+\varepsilon > \delta > ||x||$ and $\sigma(x\delta) \leq 1$. By Lemma 2 (ii) we have $\sigma(x) \leq \delta^H\ \sigma(x\delta) \leq \delta^H < (1+\varepsilon)^H$, so $(\sigma(x))^{1/H} \leq 1 + \varepsilon$ for all $\varepsilon > 0$ which implies that $\sigma(x) \leq 1$.

If $\sigma(x) < 1$, let $a \in (0,1)$ such that $\sigma(x) < a^H < 1$. From Lemma 2 (i) we have $\sigma(xa) \leq \frac{1}{a^H}\sigma(x) \leq 1$, hence $||x|| \leq a < 1$, which is a contradiction. Thus we have $\sigma(x) = 1$.

Conversely, assume that $\sigma(x) = 1$, by the definition of norm, we get that $\|x\| \leq 1$. If $\|x\| \leq 1$, then by (i), we have that $\sigma(x) \leq ||x||$, which contradicts to our assumption, so we obtain that $||x|| = 1$.

(iv) If $\|x\| < 1$, we have by (i) that $\sigma(x) < \|x\| < 1$. $\sigma(x) < 1$, it follows by (ii) and (iii) that $||x|| < 1$, so (iv) is obtained.

(v) it follows from (iii) and (iv). ∎

Lemma 4. *For $x \in ces\left[(p_n),(q_n)\right]$ we have*
(i) if $0 < a < 1$ and $||x|| > a$, then $\sigma(x) > a^H$,
(ii) if $a \geq 1$ and $||x|| < a$, then $\sigma(x) < a^H$.

Proof. (i) We suppose that $0 < a < 1$ and $||x|| > a$. Then $\left\|\frac{x}{a}\right\| > 1$. By Lemma 3 (ii), we have $\sigma\left(\frac{x}{a}\right) > \left\|\frac{x}{a}\right\| > 1$. Hence by Lemma 2 (i), we get that $\sigma\left(\frac{x}{a}\right) \geq a^H\sigma\left(\frac{x}{a}\right) > a^H$.

(ii) Assume that $a > 1$ and $\|x\| < a$. Then $\left\|\frac{x}{a}\right\| < 1$. By the Lemma 3 (i) $\sigma\left(\frac{x}{a}\right) \leq \left\|\frac{x}{a}\right\| < 1$. If a=1, then we have $\sigma(x) < 1$, by Lemma 2 (ii), we obtain that $\sigma(x) < a^H\sigma\left(\frac{x}{a}\right) < a^H$. ∎

Lemma 5. *Let (x_n) be sequence in $ces\left[(p_n),(q_n)\right]$,* *(i) if* $\lim_{n\to\infty} ||x_n|| = 1$, *then* $\lim_{n\to\infty} \sigma(x_n) = 1$,

(ii) if $\lim_{n\to\infty} \sigma(x_n) = 0$, *then* $\lim_{n\to\infty} ||x_n|| = 0$.

Proof. (i) Suppose that $\lim_{n\to\infty} ||x_n|| = 1$. Let $\varepsilon \in (0,1)$. Then there exists n_0 such that $1-\varepsilon < ||x_n|| < 1+\varepsilon$ for all $n \geq n_0$. By Lemma 4, for all $n \geq n_0$, $(1-\varepsilon)^H < ||x_n|| < (1+\varepsilon)^H$ implies that $\lim_{\infty} \sigma(x_n) = 1$.

(ii) Suppose that $||x_n|| \nrightarrow 0$. Then there is an $\varepsilon \in (0,1)$ and a subsequence (x_{n_k}) of (x_n) such that $||x_{n_k}|| > \varepsilon$ for all $k \in N$. By Lemma 4 (i), we obtain that $\sigma(x_{n_k}) > \varepsilon^H$ for all $k \in N$. This implies that $\sigma(x_{_{nk}}) \nrightarrow 0$ as $n \to \infty$. Hence $\sigma(x_n) \nrightarrow 0$ ∎

We now show that the *ces* $[(p_n), (q_n)]$ is a Banach space with respect to Luxemburg norm.

Theorem 1. *The space ces* $[(p_n), (q_n)]$ *is a Banach space with respect to Luxemburg norm defined by*

$$||x|| = inf \{\rho > 0 : \sigma(x\rho) \leq 1\}.$$

Proof. We will show that every Cauchy sequence in *ces* $[(p_n), (q_n)]$ is convergent according to the Luxemburg norm. Let (x_k^n) be a Cauchy sequence in *ces* $[(p_n), (q_n)]$ and $\varepsilon \in (0,1)$. Thus there exists n_0 such that $||x^n - x^m|| < \varepsilon^M$, for all $m, n \geq n_0$. By the Lemma 3 (i) we obtain

$$\sigma(x^n - x^m) < ||x^n - x^m|| < \varepsilon^M, \tag{2.1}$$

for all $n, m \geq n_0$. That is,

$$\sum_{r=0}^{\infty} \left(\frac{1}{Q_{2^r}} \sum_{r} q_k |x_k^n - x_k^m| \right)^{p_r} < \varepsilon^M,$$

for $m, n \geq n_0$.For fixed k we get that

$$|x_k^n - x_k^m| < \varepsilon.$$

Hence we obtain that the sequence (x_k^n) is a Cauchy sequence in $\mathbb{R}$. Since the real number $\mathbb{R}$ is complete, $x_k^m \to x_k$ as $m \to \infty$. Therefore, for fixed k

$$|x_k^n - x_k| < \varepsilon \qquad\qquad \text{for } n \geq n_0.$$

Now we will show that the sequence (x_k) is the element of *ces* $[(p_n), (q_n)]$. From (2.1) inequality, we can write

$$\sum_{r=0}^{\infty}\left(\frac{1}{Q_{2^r}}\sum_{r}q_k|x^n-x^m|\right)^{p_r}<\varepsilon \qquad \text{for all } \; m,n\geq n_0.$$

For every $k\in\mathbb{N}$, we have $x_k^m\to x_k$, so we obtain that

$$\sigma(x^n-x^m)\to\sigma(x^n-x) \qquad \text{as } m\to\infty.$$

So, we can write

$$\sum_{r=0}^{\infty}\left(\frac{1}{Q_{2^r}}\sum_{r}q_k|x_k^n-x_k^m|\right)^{p_r}\to\sum_{r=0}^{\infty}\left(\frac{1}{Q_{2^r}}\sum_{r}q_k|x_k^n-x_k|\right)^{p_r}$$

as $m\to\infty$. Hence we have

$$\sigma(x^n-x)<\varepsilon \qquad \text{for all } \; n\geq n_0.$$

That is,

$$\sigma(x^n-x)<\varepsilon\Rightarrow x_n\to x.$$

Secondly, from the linearity of the sequence space $ces\left[(p_n),(q_n)\right]$, we can write as follows:

$$x=(x-x^n)+x^n.$$

By the following calculations, we obtain that,

$$\begin{aligned}\sum_{r=0}^{\infty}\left(\frac{1}{Q_{2^r}}\sum_{r}q_k|x_k|\right)^{p_r} &= \sum_{r=0}^{\infty}\left(\frac{1}{Q_{2^r}}\sum_{r}q_k|x_k-x_k^n|+\frac{1}{Q_{2^r}}\sum_{r}q_k|x_k^n|\right)^{p_r}\\ &\leq \sum_{r=0}^{\infty}\left(\frac{1}{Q_{2^r}}\sum_{r}q_k|x_k^n-x_k|\right)^{p_r}+\sum_{r=0}^{\infty}\left(\frac{1}{Q_{2^r}}\sum_{r}q_k|x_k^n|\right)^{p_r}\\ &\leq \varepsilon.\end{aligned}$$

That is, the sequence (x^n-x) convergences to sequence x in $ces\left[(p_n),(q_n)\right]$. Hence the sequence space $ces\left[(p_n),(q_n)\right]$ is Banach space with respect to Luxemburg norm. This completes the proof of theorem. ∎

Lemma 6. *Let* $x\in ces\left[(p_n),(q_n)\right]$ *and* $(x_n)\subseteq ces\left[(p_n),(q_n)\right]$. *If* $\sigma(x_n)\to\sigma(x)$ *as* $n\to\infty$ *and* $x_n(i)\to x(i)$ *as* $n\to\infty$ *for all* $i\in N$, *then* $x_n\to x$ *as* $n\to\infty$.

Proof. Let $\varepsilon>0$. Since $\sigma(x)=\sum\limits_{r=0}^{\infty}\left(\frac{1}{Q_{2^r}}\sum\limits_{r}q_k|x_k|\right)^{p_r}<\infty$, there exists $k\in N$ such that

$$\sum_{r=r_0+1}^{\infty}\left(\frac{1}{Q_{2^r}}\sum_{r} q_k|x_k|\right)^{p_r} < \frac{\varepsilon}{3}\frac{1}{2^M}, \tag{2.2}$$

where $M = max\{1,\, 2^{H-1}\}$, $H = \sup_r p_r$.

Since $\sigma(x_n) - \sum_{r=0}^{r_0}\left(\frac{1}{Q_{2^r}}\sum_{r} q_k|x_n(i)|\right)^{p_r} \to \sigma(x) - \sum_{r=0}^{r_0}\left(\frac{1}{Q_{2^r}}\sum_{r} q_k|x(i)|\right)^{p_r}$ as $n\to\infty$ and $x_n(i)\to x(i)$ as $n\to\infty$ for all $i\in N$, there exists $n_0\in N$ such that

(2.3)

$$\sigma(x_n) - \sum_{r=0}^{r_0}\left(\frac{1}{Q_{2^r}}\sum_{r} q_k|x_n(i)|\right)^{p_r} \to \sigma(x) - \sum_{r=0}^{r_0}\left(\frac{1}{Q_{2^r}}\sum_{r} q_k|x(i)|\right)^{p_r} + \frac{\varepsilon}{32^M}$$

for all $n\geq n_0$ and since $x_n(i)\to x(i)$ as $n\to\infty$ for all $i\in N$, and we know also that $\sigma(x_n)\to\sigma(x)$ as $n\to\infty$. Hence for all $n\geq n_0$ we have $|x_n(i)-x(i)|<\varepsilon$. As a result for all $n\geq n_0$,we have

$$\sum_{r=0}^{\infty}\left(\frac{1}{Q_{2^r}}\sum_{r} q_k|x_n(i)-x(i)|\right)^{p_r} < \frac{\varepsilon}{3}. \tag{2.4}$$

Then, from (2.1), (2.2) and (2.3) it follows that for $n\geq n_0$,

$$\begin{aligned}
\sigma(x_n - x) &= \sum_{r=0}^{\infty}\left(\frac{1}{Q_{2^r}}\sum_{r} q_k|x_n(i)-x(i)|\right)^{p_r}\\
&= \sum_{r=0}^{r_0}\left(\frac{1}{Q_{2^r}}\sum_{r} q_k|x_n(i)-x(i)|\right)^{p_r} + \sum_{r=r_0+1}^{\infty}\left(\frac{1}{Q_{2^r}}\sum_{r} q_k|x_n(i)-x(i)|\right)^{p_r}\\
&< \frac{\varepsilon}{3} + 2^M\left[\sum_{r=r_0+1}^{\infty}\left(\frac{1}{Q_{2^r}}\sum_{r} q_k|x_n(i)|\right)^{p_r} + \sum_{r=r_0+1}^{\infty}\left(\frac{1}{Q_{2^r}}\sum_{r} q_k|x(i)|\right)^{p_r}\right]\\
&= \frac{\varepsilon}{3} + 2^M\left[\sigma(x_n) - \sum_{r=0}^{r_0}\left(\frac{1}{Q_{2^r}}\sum_{r} q_k|x_n(i)|\right)^{p_r} + \sum_{r=r_0+1}^{\infty}\left(\frac{1}{Q_{2^r}}\sum_{r} q_k|x(i)|\right)^{p_r}\right]\\
&< \frac{\varepsilon}{3} + 2^M\left[\sigma(x) - \sum_{r=0}^{r_0}\left(\frac{1}{Q_{2^r}}\sum_{r} q_k|x(i)|\right)^{p_r} + \frac{\varepsilon}{3}\frac{1}{2^M}\sum_{r=r_0+1}^{\infty}\left(\frac{1}{Q_{2^r}}\sum_{r} q_k|x(i)|\right)^{p_r}\right]\\
&= \frac{\varepsilon}{3} + 2^M\left[\sum_{r=r_0+1}^{\infty}\left(\frac{1}{Q_{2^r}}\sum_{r} q_k|x(i)|\right)^{p_r} + \frac{\varepsilon}{3}\frac{1}{2^M}\sum_{r=r_0+1}^{\infty}\left(\frac{1}{Q_{2^r}}\sum_{r} q_k|x(i)|\right)^{p_r}\right]\\
&= \frac{\varepsilon}{3} + 2^M\left[2\sum_{r=r_0+1}^{\infty}\left(\frac{1}{Q_{2^r}}\sum_{r} q_k|x(i)|\right)^{p_r} + \frac{\varepsilon}{3}\frac{1}{2^M}\right]\\
&< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon
\end{aligned}$$

This shows that $\sigma(x_n - x) \to 0$ as $n \to \infty$. Hence by Lemma 5 (ii), we have $||x_n - x|| \to 0$ as $n \to \infty$. ∎

Now, we will give a main results of this paper as following theorems. These are geometric properties of $ces\,[(p_n),(q_n)]$.

Theorem 2. *The space $ces\,[(p_n),(q_n)]$ has the property Kadec-Klee (H-property).*

Proof. Let $x \in S(ces\,[(p_n),(q_n)])$ and $(x_n) \subseteq B(ces\,[(p_n),(q_n)])$ such that $||x_n|| \to 1$ and $x_n \xrightarrow{w} x$ as $n \to \infty$. From the Lemma 3 (iii), we have $\sigma(x) = 1$, so it follows from the Lemma 5 (i) that $\sigma(x_n) \to \sigma(x)$ as $n \to \infty$. Since $x_n \xrightarrow{w} x$ and the i^{th}-coordinate mapping $\pi_i : ces\,[(p_n),(q_n)] \to \mathbb{R}$ defined by $\pi_i(x) = x_i$ is continuous linear function on $ces\,[(p_n),(q_n)]$, it follows that $x_n(i) \to x(i)$ as $n \to \infty$ for all $i \in N$. Thus we obtain by the Lemma 6 that $x_n \to x$ as $n \to \infty$. ∎

Theorem 3. *The space $ces\,[(p_n),(q_n)]$ is not rotund(R).*

Proof. For the proof of this theorem we shall give a counter example. Let $x = (1,0,0,0,0,...)$ and $y = (0,1,1,0,0,...)$ be two sequences in $ces\,[(p_n),(q_n)]$. It is easy to see that $||x||_{ces[(p_n),(q_n)]} = 1$ and $||y||_{ces[(p_n),(q_n)]} = 1$. For the $ces\,[(p_n),(q_n)]$ not posses rotund property, we may show that $||\frac{x+y}{2}|| \neq 1$. This follows from standard calculations easily. Consequently, we obtain that the sequence space $ces\,[(p_n),(q_n)]$ has not R-property. ∎

Remark 1. *If q_n=1, for all k, $ces\,[(p_n),(q_n)]$ reduces to the space ces(p) studied by Lim[7]. In [5] Sanhan and Suantai show that the ces(p) is not rotund. Also another way of to say that not rotundity of $ces\,[(p_n),(q_n)]$ is follows from the above inclusion. And we also have from relations between ces(p) and $ces\,[(p_n),(q_n)]$, the $ces\,[(p_n),(q_n)]$ is not LUR.*

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