

Embedding of the Fundamental Topological Group in the Inverse Limit of Discrete Groups

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Abstract

Let X be a topological space. In this paper we show that if $\pi_1^{top}(X)$, the topological fundamental group of X , is compact then it can be embedded in the inverse limit of discrete groups.

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Introduction

The theory of topological fundamental groups is in the early stages. In [4],[2],[5],[1] the properties of these groups are investigated. For the spaces that are complicated both locally and globally, the topology of $\pi_1(X, p)$ can be more interesting [3],[4],[7]. If X and Y have the same homotopy type then $\pi_1(X, p)$ and $\pi_1(Y, b)$ are isomorphic and homeomorphic [1, corollary 3.4]. The topology of fundamental groups has the capacity to distinguish the homotopy type of X and Y when the algebra fails to do so [6, theorem 3]. Analysis of algebraic properties of fundamental groups often leads to the conclusion that X has certain property. For example if $\pi_1(X, p)$ is non-abelian then X is not

a topological group. It is possible to reach the same conclusion by analyzing topological properties of this group [8, theorem 5]. Recall that the Hawaiian earring HE , is the subspace of the plane consisting of the infinite family of circles X_i of radius $1/i$ and center $(1/i, 0)$, mutually tangent to the y-axis at the origin. In [1] Bliss uses Hawaiian earring and proves some nice general results on topological fundamental groups and their generalized covering spaces. Bliss claims that $\pi_1(HE, p)$ can be embedded in the inverse limit of free groups but Fable [7] shows that such a claim is false.

The intend of this paper is to show that under compactness $\pi_1^{top}(X)$ can be embedded in the inverse limit of discrete groups. In section 1, we recall the definition of topological fundamental group $\pi_1^{top}(X)$ and some of its properties. In section 2 , we show that if $\pi_1^{top}(X)$ is compact then it can be embedded in the inverse limit of discrete groups.

1 Topological fundamental groups

In this section we recall the definition of topological fundamental group and some of its properties [1]. By S^1 we mean the unit circle.

Let (X, x) be a pointed space. The space of continuous based maps $Hom((S^1, 1), (X, x))$ can be given the compact-open topology ; this topology has as its subbase $(K, U) = \{f : (S^1, 1) \rightarrow (X, x); f(K) \subseteq U\}$ where K ranges over all compact subsets of S^1 and U ranges over all open subset of X . We have a surjection $Hom((S^1, 1), (X, x)) \rightarrow [(S^1, 1), (X, x)] = \pi_1(X, x)$, which allows us to define a quotient topology on $\pi_1(X, x)$. When $\pi_1(X, x)$ is a topological space , it is denoted by $\pi_1^{top}(X, x)$. With this topology $\pi_1^{top}(X, x)$ is a topological group[1]:

Lemma 1.1 *Under concatenation of paths $\pi_1^{top}(X, x)$ is a topological group*

Proof. Let S^1 be the unit circle and $(a, b) = \{e^{i\theta}; a < \theta < b\}$. We show that the map $\pi_1^{top}(X, x) \rightarrow \pi_1^{top}(X, x), [f] \mapsto [f]^{-1}$ is continuous. Let $K^{-1} = \{x^{-1}; x \in K\}$. A path α is in $\cap_i(K_i, U_i)$ if and only if the inverse is in $\cap_i(K_i^{-1}, U_i)$. Hence the inverse map pulls opens back to opens. So it is continuous. Now we show that multiplication is continuous. Consider the

commutative diagram

$$\begin{array}{ccccc}
 Hom((S^1, 1), (X, x)) & \times & Hom((S^1, 1), (X, x)) & \xrightarrow{\tilde{m}} & Hom((S^1, 1), (X, x)) \\
 \downarrow & & \downarrow & & \downarrow p \\
 \pi_1^{top}(X, x) & \times & \pi_1^{top}(X, x) & \xrightarrow{m} & \pi_1^{top}(X, x)
 \end{array}$$

where the vertical maps are projections, \tilde{m} is concatenation of paths, and m is multiplication. If $U \subset \pi_1^{top}(X, x)$ is open then $(p \times p)^{-1}m^{-1}(U) = \tilde{m}^{-1}p^{-1}(U)$ is open. By the definition of a quotient map $m^{-1}(U)$ is open. Now let $(K, U) \subset Hom((S^1, 1), (X, x))$ be a basic open set. let $K_1 = [0, \pi]$ and $K_2 = [\pi, 2\pi]$. Then $\tilde{m}^{-1}(K, U) = (K_1, U) \times (K_2, U)$ which is open.

proposition 1.2 *let $f : (X, x) \rightarrow (Y, y)$ be a continuous map of based spaces. The homomorphism $f_* : \pi_1^{top}(X, x) \rightarrow \pi_1^{top}(Y, y)$ is continuous.*

Proof. Let X be a space and $x, y \in X$, γ a path with $\gamma(0) = x, \gamma(1) = y$. Now f induces the map $f_{\#} : Hom((S^1, 1), (X, x)) \rightarrow Hom((S^1, 1), (X, x))$ given by $f_{\#}[\alpha] = [f^{-1} * \alpha * f]$ where $\alpha : (S^1, 1) \rightarrow (X, x)$, $[\alpha]$ denotes the homotopy class of α and $*$ denotes composition of paths. The induced map is continuous. For if (K, U) is open in $Hom((S^1, 1), (X, x))$ then $f_{\#}^{-1}(K, U) = (K, f^{-1}(U))$ which is a basic open set in $Hom((S^1, 1), (X, x))$. Hence $f_{\#}$ is continuous. Consider the following commutative diagram:

$$\begin{array}{ccc}
 Hom((S^1, 1), (X, x)) & \xrightarrow{f_{\#}} & Hom((S^1, 1), (X, y)) \\
 \downarrow p & & \downarrow p \\
 \pi_1^{top}(X, x) & \xrightarrow{f_*} & \pi_1^{top}(Y, y)
 \end{array}$$

where the vertical maps are projections. By a similar argument as in lemma 1.1, f_* is continuous.

Remark . π_1^{top} is a functor from the category of spaces to the category of topological groups [1, corollary 3.4].

2 The main result

In this section we show that if $\pi_1^{top}(X, x)$ is compact then there is an embedding of topological fundamental group X in the inverse limit of discrete groups. The argument is similar to [1].

In [1] a new category is introduced whose spaces are built up out of finite-dimensional cells like CW complexes, but that need not satisfy a global compatibility condition.

Definition 2.1 . Let A be a closed subset of a space X . Then X is said to be obtained from A by adjoining n -cells if we can partition $X - A$ into open subspaces U_i^n such that we have maps $f_i : E^n \rightarrow X$ taking interior of E^n homeomorphically onto U_i^n , and mapping the boundary S^{n-1} of E^n onto the boundary δU_i^n in X of U_i^n .

The objects of this new category are as follows:

Definition 2.2 . A relative CF-complex (X, A) consists of a space X , a closed subspace A , and a sequence of closed subspaces $A = (X, A)^{-1} \subset (X, A)^0 \subset (X, A)^1 \subset \dots$ such that $(X, A)^k$ is obtained from $(X, A)^{k-1}$ by attaching k -cells. $X = \cup_k (X, A)^k$, and the boundary of each cell is contained in the union of finitely many open cells of lower dimension.

By [1, lemma 2.2 and 2.3] the topological fundamental group of a CF complex depends only on the 2-skeleton. In this case we consider the complexes that are metrizable.

Let X be the Hawaiian earring, the union of a sequence of simple closed curves S_n joint at a common point; $X = \cup_{i=1}^{\infty} S_n$. Consider the canonical homomorphism

$$\phi : \pi_1(X) \rightarrow \lim_{\leftarrow} (\cup_{i=1}^{\infty} S_n)$$

Bliss[1] claims that ϕ is an embedding. Fable [7, theorem 2.1] shows that $\pi_1^{top}(HE)$ does not embed in the inverse limit of free groups. In the following by a similar argument as in [1] we show that if X is a space such that $\pi_1^{top}(X)$ is compact then there exists an embedding into the inverse limit of discrete groups.

Remark. Let X be a 1-dimensional CF complex with a metric. For a positive number a , let X_a be the union of all 1-cells with diameter less than a . Then for any decreasing sequence of positive numbers a_1, a_2, \dots we have a map

$$\pi_1(X) \rightarrow \lim_{\leftarrow} (X/X_{a_i})$$

Thus we obtain a map

$$\pi_1(X) \rightarrow \lim_{\leftarrow} \pi_1(X/X_{a_i})$$

Theorem 2.3 *Let X be a one-dimensional metrizable CF complex such that $\pi_1^{top}(X)$ is compact. Let a_1, a_2, \dots be a decreasing sequence of positive numbers converging to zero and X_{a_1}, X_{a_2}, \dots be the decreasing family of subspaces of the 1-skeleton of X defined above. Then the map*

$$\phi : \pi_1(X) \rightarrow \lim_{\leftarrow} \pi_1(X/X_{a_i})$$

is an embedding

Proof. Let X be a metrizable 1-dimensional CF-complex and a_1, a_2, \dots be a decreasing sequence of positive numbers converging to zero . So we have the natural map $f : X \rightarrow \lim_{\leftarrow} (X/X_{a_1} \leftarrow X/X_{a_2} \leftarrow \dots)$. Now f induces

$$f_* : \pi_1^{top}(X) \rightarrow \pi_1^{top}(\lim_{\leftarrow} (X/X_{a_1} \leftarrow X/X_{a_2} \leftarrow \dots))$$

Let f_i be the i th projection . Consider

$$g : \pi_1^{top}(\lim_{\leftarrow} (X/X_{a_1} \leftarrow X/X_{a_2} \leftarrow \dots)) \rightarrow \lim_{\leftarrow} (\pi_1^{top}(X/X_{a_1}) \leftarrow \pi_1^{top}(X/X_{a_2}) \leftarrow \dots),$$

$$[\alpha] \mapsto ([f_1\alpha], [f_2\alpha], \dots)$$

and

$$\psi = g \circ f_*$$

$$\begin{aligned} \psi : \pi_1^{top}(X) &\rightarrow \lim_{\leftarrow} (X/X_{a_1} \leftarrow X/X_{a_2} \leftarrow \dots) \\ &\rightarrow \lim_{\leftarrow} (\pi_1^{top}(X/X_{a_1}) \leftarrow \pi_1^{top}(X/X_{a_2}) \leftarrow \dots) \end{aligned}$$

Since by assumption $\pi_1^{top}(X)$ is compact, it is enough to show that ψ is one to one and continuous.

Let $[\gamma] \in Ker\psi$. Then $\psi[\gamma] = [e], [e], \dots)$, ($[e]$ is the identity).

So

$$\begin{aligned} g \circ f_*[\gamma] &= ([e], [e], \dots), \\ g \circ [f \circ \gamma] &= g([(f_1 \circ \gamma), (f_2 \circ \gamma), \dots)]) = ([e], [e], \dots) \\ ([f_1 \circ \gamma], [f_2 \circ \gamma], \dots) &= ([e], [e], \dots) \end{aligned}$$

Hence $f_i \circ \gamma$ is homotopic to e for any i . So for any $i, j, i \neq j$, $f_i \circ \gamma \sim f_j \circ \gamma$ and $f_i \circ \gamma(S^1) \subset \cap X/X_{a_i}$. But $\cap X/X_{a_i}$ is discrete [1]. Hence $\gamma \sim e$, that is $Ker\psi = 0$. So ψ is one to one. By [1] , ψ is continuous. Therefore, ψ is an embedding.

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