On Imaginary Quadratic Fields whose Class Numbers are Divisible by 3

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Abstract

The purpose in this paper is to give a theorem on the divisibility by 3 of the class numbers of the imaginary quadratic fields $\mathbb{Q}(\sqrt{-q})$.

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1. Introduction

It was proved by Nagel [7] that there exist infinitely quadratic number fields each with class number divisible by a given integer. Humbert [5] and Ankeny-Chowla [2] have proved this fact independently. Recently, P. Hartung has proved [4] that there is an infinite number of imaginary quadratic number field $Q(\sqrt{-q})$ whose class number is divisible by 3, where $q$ is a square free integer in the form of $t^2 - 4$ and $q \equiv 7 \pmod{12}$. Artin, Ankeny, Chowla Proved [1] that $H \equiv \frac{-u}{t}h(d) \pmod{3}$. Here, $t$ and $u$ are coefficients of the fundamental unit of $Q(\sqrt{d})$, $H$ is the class number of $Q(\sqrt{-q})$ and $h(d)$ is the class number of $Q(\sqrt{d})$.

In this note prove, by a congruence that above mentioned, the following:
2. Theorem

Let $d \equiv 1 \pmod{4}$ be a square free integer. If $d$ is expressible in the form of $d = 3q = 9p^2m^2 \mp 4m$, where $p$ and $m$ are integers, then the class number of $Q(\sqrt{-q})$ divisible by 3.

Proof. Since $d \equiv 1 \pmod{4}$ then the fundamental unit of $Q(\sqrt{d})$ is $\epsilon = \frac{t+\sqrt{d}}{2} > 1$ and its discriminant is $d = 3q$. We see that $t \equiv \mp 2 \pmod{m}$ from $t^2 - (9p^2m^2 \mp 4m)u^2 = 4$. We can take $t = 9p^2m \mp 2$ because the smallest solution of the equation $t^2 - (9p^2m^2 \mp 4m)u^2 = 4$ is $t = 9p^2m \mp 2$ and $u = 3p$. Hence the fundamental unit of $Q(\sqrt{9p^2m^2 \mp 4m})$ is

$$\epsilon = \frac{(9p^2m \mp 2) + 3p\sqrt{9p^2m^2 \mp 4m}}{2}.$$ 

Thus we observe that $h(-q) \equiv 0 \pmod{3}$ from $H \equiv \frac{-u}{t}h(d) \pmod{3}$.

References


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