Purely Periodic $\beta$-Expansions
with Pisot Unit Bases over Laurent Series

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Abstract. The present paper deals with $\beta$-expansions in algebraic function fields. If $\beta$ is a Pisot unit, we characterise the elements whose $\beta$-expansion is purely periodic. In order to pursue this characterisation, we introduce a variant of the Rauzy fractal.

1. Introduction

$\beta$-expansions of real numbers were introduced by Rényi [14]. Since then, their arithmetic, diophantine and ergodic properties have been extensively studied by several authors (cf. for instance [1, 4, 5, 13, 16]). Let $\mathbb{F}$ be a finite field. It is well known that the field of formal Laurent series $\mathbb{F}((y^{-1}))$ has many analogies to the $\mathbb{R}$. In [6, 15], $\beta$-expansions for Laurent series have been introduced. Furthermore, the sets with eventually periodic and finite expansions have been completely characterised. These characterisations are unknown in the real case. In the forthcoming papers [10, 11], metrical results are established and the relation to continued fractions is studied.
It is the aim of the present paper to characterise the set of elements having purely periodic expansions when $\beta$ is a Pisot unit. A corresponding analysis for $\beta$-expansions of real numbers was done in [2, 7, 8].

This paper is organised as follows. In section two, we will define the field of formal Laurent series over a finite field as well as the analogues to Pisot and Salem numbers. In section three, we will define the $\beta$-expansion algorithm for Laurent series. In section four is devoted we introduce a set of Laurent series which is an analogue of the well known Rauzy fractal. Finally, in section five, we characterise the set of Laurent series having purely periodic expansions.

2. Pisot and Salem elements in $\mathbb{F}((y^{-1}))$

Let $\mathbb{F}$ be a finite field with $q$ elements, let $\mathbb{F}[y]$ be the ring of polynomials with coefficients in $\mathbb{F}$ and $\mathbb{F}(y)$ be the field of fractions. Define a non archimedean absolute value on $\mathbb{F}(y)$ by $|0| = 0$ and

$$|\frac{f}{g}| = q^{\deg(f) - \deg(g)}$$

for $f, g \in \mathbb{F}[y] \setminus \{0\},$

where $\deg$ denotes the degree of a polynomial. The completion of $\mathbb{F}(y)$ with respect to $|\cdot|$ is the field $\mathbb{F}((y^{-1}))$ of Laurent series,

$$\alpha = \sum_{n=-\infty}^{h} a_n y^n, \text{ such that } a_n \in \mathbb{F}, \quad a_h \neq 0.$$ 

Here $h \in \mathbb{Z}$ is called the degree of $\alpha$ and $|\alpha| = q^h$. Thus $|y| = q$.

An element $\alpha \in \mathbb{F}((y^{-1}))$ is called irrational if $\alpha \neq \mathbb{F}(y)$. Every element $\alpha \in \mathbb{F}((y^{-1}))$ has a unique (Artin) decomposition

$$\alpha = \lfloor \alpha \rfloor + (\alpha - \lfloor \alpha \rfloor),$$

with $|\alpha| \in \mathbb{F}[y]$ and $|\alpha - \lfloor \alpha \rfloor| < 1$. For $x = (\xi_1, \ldots, \xi_d)^\top \in \mathbb{F}((y^{-1}))^d$ and $M = (\mu_{ij})_{i,j=1}^{d} \in \mathbb{F}((y^{-1}))^{d \times d}$, let

$$\|x\| = \max_{i=1}^{d} |\xi_i| \quad \text{and} \quad \|M\| = \max_{i,j=1}^{d} |\mu_{ij}|.$$ 

Then the matrix norm is compatible with the vector norm, i.e. $\|Mx\| \leq \|M\||x||$. For $a = (\alpha_1, \ldots, \alpha_d)^\top \in \mathbb{F}((y^{-1}))^d$ and $r \in \mathbb{R}_+$, let

$$D(a, r) = \{x \in \mathbb{F}((y^{-1}))^d : \|x - a\| < r\}.$$ 

In contrast to the real case, $D(a, r)$ is closed, since $|\cdot|$ can attain only a discrete set of values. Note that if $b \in D(a, r)$, then $D(a, r) = D(b, r)$, i.e. every point of a disc is a center.

Remark 2.1. In the following, we will denote polynomials by roman, Laurent series by greek, vectors and matrices by bold letters.
Proposition 2.2. [12, Chapter II, Theorem (4.8)]. Let \( K \) be complete with respect to \( | \cdot | \) and \( L/K \) be an algebraic extension of degree \( m \). Then \( | \cdot | \) has a unique extension to \( L \) defined by

\[
|a| = \sqrt[m]{|N_{L/K}(a)|},
\]

and \( L \) is complete with respect to this extension.

We apply Proposition 2.2 to algebraic extensions of \( \mathbb{F}((y^{-1})) \). Therefore, since \( \mathbb{F}[y] \subset \mathbb{F}((y^{-1})) \), every algebraic element over \( \mathbb{F}[y] \) can be valuated. However, since \( \mathbb{F}((y^{-1})) \) is not algebraically closed, such an element need not necessarily be a Laurent series. For a full characterisation of the algebraic closure of \( \mathbb{F}[y] \), we refer to Kedlaya [9].

Definition 2.3. Let \( U \) be the set of algebraic integers \( \alpha \) over \( \mathbb{F}[y] \), such that \( |\alpha| > 1 \) and \( |\tilde{\alpha}| \leq 1 \) for all remaining conjugates. The set \( U \) contains \( \mathbb{F}[y] \setminus \mathbb{F} \) and is therefore non-empty.

Definition 2.4. Let \( S \) be the set of elements \( \beta \in U \), such that \( |\tilde{\beta}| < 1 \) for all remaining conjugates. This set is called the Pisot set of \( \mathbb{F}((y^{-1})) \).

By considering the Newton polygon of the minimal polynomial of an element \( \alpha \in U \) (resp. \( \beta \in S \)), we obtain the following characterisation.

Theorem 2.5. [3, Theorem 12.1.1]. An element \( \alpha \) (resp. \( \beta \)) belongs to \( U \) (resp. \( S \)) if and only if its minimal polynomial can be written as

\[
p(x, y) = x^d - a_1 x^{d-1} - \cdots - a_{d-1} x - a_d \quad \text{with} \quad a_i \in \mathbb{F}[y],
\]

such that \( |a_1| = |\alpha| > 1 \) (resp. \( |a_1| = |\beta| > 1 \)) and \( |a_j| \leq |\alpha| \) (resp. \( |a_j| < |\beta| \)) for \( j \geq 2 \).

In the following Theorem, a method to compute the coefficients of Pisot or Salem elements is presented.

Theorem 2.6. [15]. Let \( \beta \) be a Pisot or Salem element and

\[
p(x, y) = x^n - a_1 x^{n-1} - \cdots - a_n, \quad a_i \in \mathbb{F}[y]
\]

be its minimal polynomial. Then \( \deg \beta = \deg a_1 \) and the recurrence

\[
\beta_1 = a_1 \quad \beta_{k+1} = a_1 + \frac{a_2}{\beta_k} + \cdots + \frac{a_n}{\beta_k^{n-1}} \quad \text{for} \ k \geq 1
\]

fulfills

\[
\lim_{k \to \infty} \beta_k = \beta \quad \text{and thus} \quad S \subset U \subset \mathbb{F}((y^{-1})).
\]

This implies that an element \( \alpha \in U \) is necessarily separable over \( \mathbb{F}(y) \). We obtain the following theorem by applying Minkowski’s theorem to the field of formal power series.

Theorem 2.7. [3, Theorem 12.1.2]. Every finite separable extension of \( \mathbb{F}(y) \) that is included in \( \mathbb{F}((y^{-1})) \) can be generated by a Pisot element.
3. β-EXPANSIONS

Let $\beta, \xi \in \mathbb{F}((y^{-1}))$ with $|\beta| > 1, |\xi| < 1$. The $\beta$-transformation $\tau$ on $D(0,1)$ is given by the mapping

$$\tau : D(0,1) \rightarrow D(0,1), \quad \xi \mapsto \beta \xi - \lfloor \beta \xi \rfloor.$$ 

By iterating this map and considering its trajectory

$$\xi \xrightarrow{d_1} \tau(\xi) \xrightarrow{d_2} \tau^2(\xi) \xrightarrow{d_3} \ldots$$

with $d_j = \lfloor \beta \tau^{j-1}(\xi) \rfloor$, we obtain an expansion

$$\xi = d_1 \beta^{-1} + d_2 \beta^{-2} + \cdots.$$ 

We will call the sequence $d_\beta(\xi) = \bullet d_1 d_2 \cdots$ the $\beta$-expansion of $\xi$, where “$\bullet$” indicates the decimal point. Thus

$$d_i \in D = \{ f \in \mathbb{F}[y] : \deg f < \deg \beta \}.$$ 

The $d_i$ are called digits of $\xi$ and $D$ is called digit set.

If $\xi$ is an element with $|\xi| \geq 1$, then there is a unique $k \in \mathbb{N}$ such that $|\beta|^k \leq |\xi| < |\beta|^{k+1}$. Hence $|\xi/\beta^{k+1}| < 1$ and we can represent $\xi$ by shifting $d_\beta(\xi/\beta^{k+1})$ by $k$ digits to the left. Therefore, if $d_\beta(\xi) = \bullet d_1 d_2 d_3 \cdots$, then $d_\beta(\xi) = d_1 \beta d_2 \beta \cdots$.

In contrast to the real case, there is no carry occurring, when we add two digits. Therefore, if $\xi, \eta \in \mathbb{F}((y^{-1}))$, we have $d_\beta(\xi + \eta) = d_\beta(\xi) + d_\beta(\eta)$ digitwise.

We say that $d_\beta(\xi)$ is finite when $d_i = 0$ for all sufficiently large $i$. This is the case when there is an integer $i \geq 0$ such that $\tau^i(x) = 0$. If $d_\beta(\xi) = d_\ell d_{\ell-1} \cdots d_m$, let $\deg_\beta(\xi) = \ell$ and $\ord_\beta(\xi) = m$.

In the sequel, we will use the following notations:

$$\text{Fin}(\beta) = \{ \xi \in \mathbb{F}((y^{-1})) : d_\beta(\xi) \text{ is finite} \} \quad \text{and} \quad \text{Pur}(\beta) = \{ \xi \in D(0,1) : d_\beta(\xi) \text{ is purely periodic} \}.$$

**Lemma 3.1** ([15, Lemma 5.1]). Let $\beta$ be an arbitrary element of $\mathbb{F}((y^{-1}))$ with $\deg \beta > 0$, and let $\xi \in \mathbb{F}[y, \beta^{-1}]$ have purely a periodic $\beta$-expansion. Then, $\xi \in \mathbb{F}[y, \beta]$.

Therefore, in order to study purely periodic elements, we can confine ourselves to elements of $\mathbb{F}[y, \beta]$. Let $\beta$ be a Pisot series with the minimal polynomial (2.1). Set

$$\rho_{d-j} = \frac{a_{j+1}}{\beta} + \cdots + \frac{a_d}{\beta^{d-j}} = \beta^j - a_1 \beta^{j-1} - \cdots - a_j \quad \text{for} \quad 0 \leq j \leq d.$$ 

Note that

$$\rho_0 = \beta^d - a_1 \beta^{d-1} - \cdots - a_d = 0,$$

$$\rho_1 = \frac{a_d}{\beta} = \beta^{d-1} - a_1 \beta^{d-2} - \cdots - a_{d-1} \quad \text{and}$$

$$\rho_d = \frac{a_1}{\beta} + \cdots + \frac{a_d}{\beta^d} = 1.$$
Then $|\rho_i|$ for $1 \leq i \leq d - 1$ and
\[ x^d - a_1 x^{d-1} - \cdots - a_d = (x - \beta)(x^{d-1} + \rho_{d-1} x^{d-2} + \cdots + \rho_1). \]
Abusing the notation a little, we denote by $\mathbf{r}$ the vector $(\rho_1, \ldots, \rho_d)^\top$ and the set \( \{\rho_1, \ldots, \rho_d\} \).
Since $\mathbf{r}$ is a base of $\mathbb{F}[y, \beta]$ over $\mathbb{F}[y]$, the coordinates with respect to $\mathbf{r}$ can be computed from the coordinates with respect to $\{1, \beta, \ldots, \beta^{d-1}\}$ by a linear system of equations. In base $\mathbf{r}$, multiplication with $\beta$ is realised by the matrix $B_r = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 & \vdots \\ 0 & \cdots & \cdots & 0 & 1 \\ a_d & \cdots & \cdots & \cdots & a_1 \end{pmatrix}$.
For $\xi \in \mathbb{F}[y, \beta]$, let $x_r = (x_1, \ldots, x_n)^\top$ be the vector of coordinates of $\xi$ with respect to $\mathbf{r}$. Thus,
\[ (\beta \xi)_r = B_r x_r. \]
The $\beta$-transformation with respect to $\mathbf{r}$ takes the form
\[ \tau_r : \mathbb{F}[y]^d \to \mathbb{F}[y]^d, \quad x \mapsto B_r x - [B_r \cdot \mathbf{r}] e_d, \]
where $e_d$ denotes the $d$'th unit vector. Then
\[ (\tau(\xi))_r = \tau_r(x_r), \]
which shows that the following diagram is commutative:
\[ \begin{array}{ccc} \mathbb{F}[y, \beta] & \xrightarrow[\tau_r]{} & \mathbb{F}[y, \beta] \\
(\cdot)_r \downarrow & & \downarrow (\cdot)_r \\
\mathbb{F}[y]^d & \xrightarrow[\tau]{} & \mathbb{F}[y]^d. \end{array} \]
Substituting $B_r$, $\mathbf{r}$ and $e_d$ into (3.1), we can express $\tau_r$ as follows:
\[ \tau_r : (x_1, \ldots, x_d)^\top \mapsto (x_2, \ldots, x_{d-1}, -[x_2 r_1 + \cdots + x_d r_{d-1}])^\top. \]

4. The embedding in $\mathbb{F}((y^{-1}))^{d-1}$

Let $\beta$ be a Pisot series with the minimal polynomial (2.1). If $\mathbf{v}_1 = (\beta^{-d+1}, \ldots, \beta^{-1}, 1)^\top$, then $B_r \mathbf{v}_1 = \beta \mathbf{v}_1$. Thus, $\beta$ is an eigenvalue of $B_r$ and the corresponding eigenvector is $\mathbf{v}_1$. Since $|\beta| > 1$, the corresponding eigenspace $V$ is expanding.
Let $e_i$ be the $i$'th unit vector and $p_i = -\beta e_{i-1} + e_i$ for $2 \leq i \leq d$. Then, $p_i \perp v_1$ and \( \{v_1, p_2, \ldots, p_d\} \) is a base of the vectorspace $\mathbb{F}((y^{-1}))^d$ over $\mathbb{F}((y^{-1}))$. Let $P$ be the subspace spanned by the $p_i$. Then $\dim V = 1$, $\dim P = d - 1$,
\[ \mathbb{F}((y^{-1}))^d = V \oplus P. \]
and \( P \) is contracting under \( B_r \). According to this direct sum, we define two natural projections \( \nu : \mathbb{F}((y^{-1}))^d \to V \) and \( \pi : \mathbb{F}((y^{-1}))^d \to P \). It is easy to see that
\[
B_r \circ \pi = \pi \circ B_r \quad \text{and} \quad B_r \circ \nu = \nu \circ B_r.
\]
Define the base \( t \) of \( \mathbb{F}((y^{-1}))^d \) by its inverse
\[
t^{-1} = \begin{pmatrix}
\beta^{-d+1} & \cdots & \cdots & \beta^{-1} & 1 \\
-\beta & 1 & 0 & \cdots & 0 \\
0 & \ddots & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & \ddots & \vdots \\
0 & \cdots & 0 & -\beta & 1
\end{pmatrix} = \begin{pmatrix}
v_1^\top \\
p_2^\top \\
\vdots \\
p_d^\top
\end{pmatrix}.
\]

**Remark 4.1.** This definition of \( t \) looks somewhat awkward. It was chosen, since the entries of \( t \) are rather complicated and not needed explicitly.

Thus, the coordinates with respect to \( t \) are given by \( z_t = t^{-1}z_r \) and multiplication with \( \beta \) is realised by
\[
B_t = t^{-1}B_r t = \begin{pmatrix}
\beta & \frac{1}{\beta} - \rho_1 & \cdots & \cdots & \frac{1}{\beta} - \rho_{d-1} \\
0 & 0 & 1 & 0 & \cdots & 0 \\
\vdots & \vdots & \ddots & \ddots & \ddots & \ddots \\
\vdots & \vdots & \ddots & \ddots & \ddots & \ddots \\
0 & 0 & \cdots & 0 & 1 & -\rho_1 \\
0 & -\rho_1 & \cdots & \cdots & \cdots & \cdots & -\rho_{d-1}
\end{pmatrix}.
\]
This relation is proved by verifying that \( t^{-1}B_r = B_t t^{-1} \).

In base \( t \), the projection \( \pi \) is realised by setting the first component to zero.

If we are working with vectors in \( P \), we will skip this component, i.e.
\[
\pi((z_1, z_2, \ldots, z_d)^\top)_t = (z_2, \ldots, z_d)^\top.
\]

Let \( \mathcal{M} = \{ \pi(f)_t : f \in \mathcal{D} \} \). Note that \( \pi(1)_t = (0, \ldots, 0, 1)^\top \). Let
\[
R = \begin{pmatrix}
0 & 1 & 0 & \cdots & 0 \\
\vdots & \ddots & \ddots & \ddots & \vdots \\
\vdots & \vdots & \ddots & \ddots & \vdots \\
0 & \cdots & 0 & 1 \\
-\rho_1 & \cdots & \cdots & \cdots & \cdots & -\rho_{d-1}
\end{pmatrix}.
\]
Then, since \(|\rho_i| < 1 \) for \( 1 \leq i \leq d - 1 \), it follows that \( \|R^{d-1}\| < 1 \).

**Lemma 4.2.** The projection image \( \pi(\mathbb{F}[y, \beta]) \) is dense in \( P \) or equivalently, \( \pi(\mathbb{F}[y, \beta])_t \) is dense in \( \mathbb{F}((y^{-1}))^{d-1} \).

**Proof.** Let \( r > 0 \) and \( a = (\alpha_1, \ldots, \alpha_{d-1})^\top \in \mathbb{F}((y^{-1})) \). The image \( \pi(\mathbb{F}[y, \beta])_t \) contains the vectors
\[
(-\beta x_1 + x_2, \ldots, -\beta x_{d-1} + x_d)^\top.
\]
where $x_i \in \mathbb{F}[y]$. This set is dense, since $\beta$ is irrational. \hfill \Box

**Lemma 4.3** (cf. [1, Lemma 1]). Let $A > 0$ and

$$X(i) = \{ \xi \in \text{Fin}(\beta) : |\xi| \leq A, \text{ord}_\beta(\xi) = -i \}.$$  

Then,

$$\lim_{i \to \infty} \min_{\xi \in X(i)} \| \pi(\xi)_i \| = \infty.$$  

**Proof.** Assume that there exist a constant $B$ and an infinite sequence of $\xi^{(i)} \in \mathbb{F}[y, \beta]$, such that $\| \pi(\xi^{(i)})_i \| \leq B$,

$$\lim_{i \to \infty} \text{ord}_\beta(\xi^{(i)}) = -\infty \quad \text{and} \quad |\xi^{(i)}| \leq A. \quad (4.2)$$  

If $\xi^{(i)} = x^{(i)}_1 \rho_1 + \cdots + x^{(i)}_d \rho_d$ with $x^{(i)}_j \in \mathbb{F}[y]$, then,

$$\| \pi(\xi^{(i)})_i \| = \max\{|\beta x^{(i)}_1 - x^{(i)}_2|, \ldots, |\beta x^{(i)}_{d-1} - x^{(i)}_d|\} \leq B.$$  

We distinguish two cases.

(i) If $|x^{(i)}_j|$ is bounded for all $j$, then, $\xi^{(i)}$ can take only finitely many values, which contradicts (4.1).

(ii) Assume that there is a $j$ such that $|x^{(i)}_j|$ is unbounded. W.l.o.g., we can assume that $\lim_{i \to \infty} |x^{(i)}_j| = \infty$. Otherwise, we choose an appropriate subsequence. If $j > 1$, then,

$$|\beta x^{(i)}_j - x^{(i)}_j| = |x^{(i)}_j||\beta x^{(i)}_{j-1}/x^{(i)}_j - 1| \leq B.$$  

From $\lim_{i \to \infty} |x^{(i)}_j| = \infty$ and

$$|\beta x^{(i)}_{j-1}/x^{(i)}_j - 1| \leq B/|x^{(i)}_j| \quad \text{follows that} \quad \lim_{i \to \infty} \beta x^{(i)}_{j-1}/x^{(i)}_j = 1.$$  

Analogously, if $j < d$, then, $\lim_{i \to \infty} x^{(i)}_{j+1}/(\beta x^{(i)}_j) = 1$. It follows by induction that

$$(x^{(i)}_1, \ldots, x^{(i)}_d) = x^{(i)}_1 \cdot (1, \beta, \ldots, \beta^{d-1}) + (K^{(i)}_1, \ldots, K^{(i)}_d)$$  

with $\lim_{i \to \infty} K^{(i)}_j/x^{(i)}_1 = 0$. Thus $|\xi^{(i)}| = |x^{(i)}_1 \cdot \beta^{d-1}|$, for $i$ large enough. Since $|x^{(i)}_1| \to \infty$, this contradicts (4.2). \hfill \Box

**Definition 4.4.** For $\omega = \omega_0 \omega_{\ell-1} \cdots \omega_m$, define

$$S_\omega = \{ \xi \in \text{Fin}(\beta) : d_\beta(\xi) = b_0 b_{\ell-1} \cdots b_{\ell+1} \omega_\ell \cdots \omega_m \}$$  

and $T_\omega = \pi(S_\omega)_1$. A tile is a set $T_\omega$ with $\deg_\beta(\omega) = -1$ and a subtile is a set $T_\omega$ with $\deg_\beta(\omega) \geq -1$. We define

$$S_* = \{ \xi \in \text{Fin}(\beta) : \text{ord}_\beta(\xi) \geq 0 \}. $$
Thus, if $\xi \in S_*$, then, $d_\beta(\xi) = d_k \cdots d_0 \cdot$. Let

$$T_\ast = \pi(S_\ast)_t = \left\{ \sum_{j \geq 0} R^j d_j \mid d_j \in \mathcal{M} \right\}.$$ 

We will call $T_\ast$ the central tile.

**Proposition 4.5.** If $\beta$ is a Pisot series, then,

$$\mathbb{F}((y^{-1}))^{d-1} = \bigcup_{\omega \in \text{Fin}(\beta)} \bigcup_{\deg_\beta(\omega) < 0} T_\omega \quad \text{and} \quad T_\ast = \bigcup_{\omega \in \text{Fin}(\beta)} \bigcup_{\deg_\beta(\omega) \geq 0} T_\omega.$$ 

**Proof.** By the assumption, we have

$$\mathbb{F}[y, \beta] \subset \mathbb{F}[y, \beta^{-1}] = \bigcup_{\omega \in \text{Fin}(\beta)} S_\omega.$$ 

Applying $\pi(\cdot)_t$ and taking the closure, we obtain the result. The relation for $T_\ast$ follows similarly. \hfill $$

**Theorem 4.6** (cf. [1, Theorem 2]). Let $\beta$ be a Pisot element. For each $\xi \in S_\ast$, we have that $(\pi(\xi))_t$ is an inner point of $T_\ast$. Especially, the origin is an inner point of $T_\ast$, i.e., there exist $r > 0$ such that $D(0, r) \subset T_\ast$.

**Proof.** Any $\xi \in \text{Fin}(\beta)$ can be decomposed as $\xi = \xi_1 + \xi_2$, with $\text{ord}_\beta(\xi_1) > 0$ and $\deg_\beta(\xi_2) \leq 0$. Since $\beta$ is a Pisot series, there are absolute constants $B$ and $C$, such that

$$\|\pi(\xi_1)\| < C \quad \text{and} \quad |\xi_2| < B.$$ 

We apply Lemma 4.3 to $\xi_2$. It follows that there exists some $N$, such that $\text{ord}_\beta(\xi_2) \leq -N$ implies that $\|\pi(\xi_2)\| \geq B + C$. Since $\text{ord}_\beta(\xi) = \text{ord}_\beta(\xi_2)$, it follows that

$$\|\pi(\xi)\| \geq \|\pi(\xi_1)\| - \|\pi(\xi_2)\| \geq B.$$ 

Thus, we have proved that there exists an $N$, such that if $\text{ord}_\beta(\xi) \leq -N$, it follows that $\|\pi(\xi)\| \geq B$. If $\eta = \beta^{N-1} \xi$, then,

$$\text{ord}_\beta(\eta) < 0 \Rightarrow \|\pi(\eta)\| \geq B/\|\mathbb{R}^{-N+1}\|.$$ 

Let $r = B/\|\mathbb{R}^{-N+1}\|$. Thus, there are no elements $\pi(\eta)_t$ with $\eta \in \text{Fin}(\beta)$ and $\text{ord}_\beta(\eta) < 0$ in the disc $U = D(0, r)$. Since

$$\{\eta \in \text{Fin}(\beta) : \text{ord}_\beta(\eta) < 0\} = \mathbb{F}[y, \beta^{-1}] \setminus S_\ast,$$

it follows that

$$U \cap \pi(\mathbb{F}[y, \beta^{-1}] \setminus S_\ast)_t = \emptyset \quad \text{and thus,} \quad U \cap \mathbb{F}((y^{-1}))^{d-1} \setminus T_\ast = \emptyset.$$ 

From $U \subset \mathbb{F}((y^{-1}))^{d-1}$ follows that $U \subset T_\ast$ and therefore, the origin is an inner point of $T_\ast$. \hfill $$
Now let \( \xi \in S_\bullet \) an element with \( d_\beta(\xi) = d_k \cdots d_0 \). Let \( \omega = 0^{k+1} \). Since the origin is an inner point of \( T_\bullet \), there is an \( r > 0 \), such that \( \pi(0) \in D(0, r) \subset T_\omega \subset T_\bullet \). Then, 
\[
\pi(\xi) = \pi(\xi)_t + D(0, r) \subset \pi(\xi)_t + T_\omega = T_{d_k \cdots d_0} \subset T_\bullet.
\]
Thus \( \pi(\xi)_t \) is an inner point of \( T_\bullet \). \( \square \)

**Remark 4.7.** From the proof of Theorem 4.6 follows, that for every \( N \geq 1 \) there exists \( B > 0 \), such that from \( \|\pi(\xi)_t\| < B \) follows that \( \text{ord}_\beta(\xi) > -N \).

**Theorem 4.8.** Every \( z \in \mathbb{F}( (y^{-1}) ) \) admits a representation
\[
\sum_{j=\ell}^{\infty} R^j c_j \quad \text{with} \quad c_j \in \mathcal{M}.
\]

**Proof.** Since \( \pi(\mathbb{F}[y, \beta])_t = \pi(\text{Fin}(\beta))_t \) is dense in \( \mathbb{F}( (y^{-1}) ) \), there exists a sequence \( \zeta^{(k)} \in \text{Fin}(\beta) \) with \( \lim_{k \to \infty} \pi(\zeta^{(k)})_t = z \). Therefore \( \|\pi(z^{(k)})_t\| < B \) and by Remark 4.7, there exists an \( M \in \mathbb{Z} \) such that \( \text{ord}_\beta(z^{(i)}) > M \) for all \( i \geq 1 \). Let \( z_k = \pi(\zeta^{(k)})_t \). Then, we can write \( z_k \) as follows
\[
z_k = R^M a_M^{(k)} + R^{M+1} a_{M+1}^{(k)} + R^{M+2} a_{M+2}^{(k)} + \cdots
\]
with \( a_j^{(k)} \in \mathcal{M} \). Take \( c_M \in \mathcal{M} \) such that \( c_M = a_M^{(k)} \) for infinitely many \( k \)'s. Such a \( c_M \) exists, since \( \mathcal{M} \) is a finite set. Let
\[
A_M = \{ k \in \mathbb{N} : c_M = a_M^{(k)} \}.
\]
Choose \( c_{M+1} \in \mathcal{M} \) such that \( c_{M+1} = a_{M+1}^{(k)} \) for infinitely many \( k \)'s in \( A_M \). Define
\[
A_{M+1} = \{ k \in A_M : c_{M+1} = a_{M+1}^{(k)} \}.
\]
Repeating this process, we construct a sequence
\[
A_M \supset A_{M+1} \supset A_{M+2} \cdots
\]
such that for all \( k \in A_\ell \), \( a_j^{(k)} = c_j \) for \( M \leq j \leq \ell \). Take \( k_1 < k_2 < \cdots \) such that \( k_1 \in A_M \), \( k_2 \in A_{M+1} \), \( \cdots \). Then,
\[
z_{k_j} = R^M c_M + \cdots + R^{M+j-1} c_{M+j-1} + R^{M+j} a_{M+j}^{(k_j+1)} + R^{M+j+1} a_{M+j+1}^{(k_j+2)} + \cdots.
\]
We obtain
\[
\sum_{i \geq M} R^i c_i = \lim_{j \to \infty} z_{k_j} = \lim_{j \to \infty} z_j = z,
\]
which proves the theorem. \( \square \)
Theorem 4.9. The central tile $T_\circ$ is a finite union of disks, i.e. there is an $r > 0$ and $a_1, \ldots, a_M \in \mathbb{F}((y^{-1}))^{d-1}$, such that

$$T_\circ = \bigcup_{i=1}^{M} D(a_i, r).$$

Proof. Take $r$ such that $D(0, r) \subset T_\circ$. Then there exists an $N \geq 0$ such that

$$\left\| \sum_{k=N}^{\infty} R^k d_k \right\| < r \quad \text{for all } d_k \in \mathcal{M}.$$

Every $x \in T_\circ$ has a representation

$$x = \sum_{k=0}^{N-1} R^k d_k + \sum_{k=N}^{\infty} R^k d_k = x_1 + x_2$$

with $x_2 \in D(0, r)$ and there are only finitely many vectors of the form $x_1$. Thus

$$x \in \bigcup_{(d_0, \ldots, d_{N-1})} D\left( \sum_{k=0}^{N-1} R^k d_k, r \right) =: V.$$

Thus $T_\circ \subset V$. On the other hand, since

$$D(0, r) \subset T_\circ = \left\{ \sum_{k=0}^{\infty} R^k d_k \mid d_k \in \mathcal{M} \right\}$$

and addition is performed digitwise without carry, every $x \in V$ has a representation

$$x = \sum_{k=0}^{\infty} R^k d_k \in T_\circ.$$

5. Purely periodic expansions

In this section, we characterise the set of numbers with purely periodic expansions. We mainly follow [7]. Let

$$\Omega = \{(w, u) = (\cdots w_1 w_0 \bullet u_1 u_2 \cdots) : w_i, u_i \in \mathcal{D}\}$$

be the set of two-sided sequences over $\mathcal{D}$ and $\sigma : \Omega \to \Omega$,

$$(\cdots w_1 w_0 \bullet u_1 u_2 \cdots) \mapsto (\cdots w_1 w_0 u_1 \bullet u_2 \cdots)$$

be the shift on $\Omega$. For $(w, u) \in \Omega$ we will call $w$ the backward part and $u$ the forward part. We will call $(\Omega, \sigma)$ the symbolic dynamical system associated with $\beta$. For a two-sided sequence $(w \bullet u)$, define

$$\rho(w, u) = \left( \sum_{i=1}^{\infty} u_i \beta^{-i}, -\sum_{i=0}^{\infty} R^i \pi(w_i) t \right) \in \mathbb{F}((y^{-1}))^d.$$
Let
\[ K = \{ \rho(w, u) : (w, u) \in \Omega \} \]
be the projection of \( \Omega \).

**Remark 5.1.** Clearly \( K \) is a bounded subset of \( \mathbb{F}((y^{-1}))^d \) and if \( \alpha = (a_1, \ldots, a_d)^\top \in K \), then \( |a_1| < 1 \).

We define the mapping
\[
\varphi : K \to K \quad \text{by} \quad \varphi \left( \begin{array}{c} \xi \\ \eta \end{array} \right) = I_\beta \left( \begin{array}{c} \xi \\ \eta \end{array} \right) - \lfloor \beta \xi \rfloor \cdot \mathbf{1},
\]
where \( \mathbf{1} \in \mathbb{F}((y^{-1}))^d \) denotes the vector \( \mathbf{1} = (1, 0, \ldots, 0, 1)^\top = (1, \pi(1)^\top) \) and \( I_\beta = \left( \begin{array}{c} \beta \\ 0 \end{array} \mathbf{R} \right) \).

**Proposition 5.2.** \((K, \varphi)\) is a realisation of \((\Omega, \sigma)\), i.e.
\[
\varphi \circ \rho = \rho \circ \sigma \quad \text{and} \quad \varphi(K) = K.
\]

**Proof.**
\[
\varphi(K) = \varphi(\rho(\Omega)) = \rho(\sigma(\Omega)) = \rho(\Omega) = K. \quad \square
\]

The last proposition implies that the following diagram is commutative:
\[
\begin{array}{ccc}
\Omega & \xrightarrow{\sigma} & \Omega \\
\rho \downarrow & & \rho \downarrow \\
K & \xrightarrow{\varphi} & K
\end{array}
\]

**Proposition 5.3.** For any \( \xi \in \mathbb{F}(y, \beta) \) with \( |\xi| < 1 \), follows that
\[
\varphi((\xi, \pi(\xi)^\top) = (\tau(\xi), \pi(\tau(\xi))^\top)
\]

**Proof.**
\[
\varphi \left( \begin{array}{c} \xi \\ \pi(\xi)^\top \end{array} \right) = I_\beta \left( \begin{array}{c} \xi \\ \pi(\xi)^\top \end{array} \right) - \lfloor \beta \xi \rfloor \cdot \mathbf{1}
= \left( \begin{array}{c} \beta \xi - \lfloor \beta \xi \rfloor \cdot \mathbf{1} \\ \mathbf{R} \pi(\xi)^\top - \lfloor \beta \xi \rfloor \cdot \pi(1)^\top \end{array} \right)
= \left( \begin{array}{c} \tau(\xi) \\ \pi(\tau(\xi))^\top \end{array} \right). \quad \square
\]

**Theorem 5.4.** Let \( \beta \) be a Pisot series which is a unit, i.e. \( a_d \in \mathbb{F}^\times \) in (2.1). Then, \( \xi \in \text{Pur}(\beta) \) if and only if \( \xi \in \mathbb{F}[y, \beta] \) and \( (\xi, \pi(\xi)^\top) \in K \).
Proof. The proof follows [7, Theorem 2.1].

(i) Suppose that $d_β(ξ) = \bullet a_1 \cdots a_L ∈ \text{Pur}(β)$. Then, $|ξ| < 1$. Furthermore

$$π(ξ)_t = (R^L - I)^{-1}(R^{L-1}π(a_1)_t + \cdots + Rπ(a_{L-1})_t + π(a_L)_t),$$

where $I$ denotes the unit matrix. Thus $(ξ, π(ξ)_t) ⊤ ∈ K$.

(ii) Suppose that $ξ ∈ F(y, β)$ and $(ξ, π(ξ)_t) ⊤ ∈ K$. Let $b ∈ F[y]$ such that $bξ ∈ F[y, β]$ and the degree of $b$ is minimal. Set

$$R_b = \{(ξ, π(ξ)_t) ⊤ : ξ ∈ b^{-1}F[y, β]\} ∩ K.$$

Then, since $K$ is bounded and $F[y, β]$ is discrete, $R_b$ is a finite set. We claim that

$$φ(R_b) ⊂ R_b.$$

For $(ξ, π(ξ)_t) ⊤ ∈ R_b$, it follows by Proposition 5.3, that

$$φ(((ξ, π(ξ)_t) ⊤) = (τ(ξ), π(τ(ξ))_t) ⊤.$$

Since $τ(ξ) = βξ - [βξ] ∈ b^{-1}F[y, β]$ and $φ(K) = K$, it follows that $φ(R_b) ⊂ R_b$.

Secondly, we claim that $φ$ is surjective on $R_b$. For $(ξ, π(ξ)_t) ⊤ ∈ R_b$, there exists at least one sequence $(w, u) ∈ Ω$ such that $ρ(w, u) = (ξ, π(ξ)_t) ⊤$.

Let $w_0$ be the first digit of $w$ and $η = β^{-1}(ξ + w_0)$. We claim that, $ρ ◦ σ^{-1}(w, u) = (η, π(η)_t) ⊤$. Let

$$ρ ◦ σ^{-1}(w, u) = ρ(\cdots w_2w_1 \bullet w_0u_1u_2 \cdots) = (t_1, \ldots, t_d) ⊤.$$

Then,

$$t_1 = \frac{w_0}{β} + \frac{u_1}{β^2} + \frac{u_2}{β^3} + \cdots = \frac{ξ + w_0}{β} = η$$

and

$$(t_2, \ldots, t_d) ⊤ = -π(w_1)_t - Rπ(w_2)_t - R^2π(w_3)_t - \cdots$$

$$= R^{-1}((-π(w_0)_t - Rπ(w_1)_t - R^2π(w_2)_t - \cdots) + π(w_0)_t)$$

$$= R^{-1}(ρ(ξ + w_0)_t) = π(η)_t$$

which proves the claim. So $ρ ◦ σ^{-1}(w, u) = (η, π(η)_t) ⊤ ∈ K$ and hence

$$φ((η, π(η)_t) ⊤) = (ξ, π(ξ)_t) ⊤.$$ 

On the other hand, since $β$ is an algebraic unit, it follows that $η ∈ b^{-1}F[y, β]$. So

$$(η, π(η)_t) ⊤ ∈ R_b ⊂ φ^{-1}(((ξ, π(ξ)_t) ⊤).$$

Thus $φ$ is surjective on $R_b$. Hence $φ|_{R_b}$ is one to one and therefore, there exists an integer $n$ such that

$$(ξ, π(ξ)_t) ⊤ = φ^n((ξ, π(ξ)_t) ⊤) = (τ^n(ξ), π(τ^n(ξ))_t) ⊤.$$ 

Hence $ξ = τ^n(ξ)$, i.e. $ξ ∈ \text{Pur}(β)$. □

Theorem 4.6 implies together with Theorem 5.4.
Corollary 5.5. There exists an $r > 0$ such that for all $\xi \in \mathbb{F}(y, \beta)$ with $\| (\xi, \pi(\xi)_t) \| < r$ follows that $\xi \in \text{Pur}(\beta)$.

Example 5.6. Let $\beta$ a Pisot element such that $\beta^2 + y\beta + 1 = 0$. Let

$$\xi = \overline{\beta+1} = \frac{\beta + 1}{\beta^2 + \beta + 1}.$$  

It is evident that $\xi \in \mathbb{F}(y, \beta)$.

References


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