The Fractal Dimension of the Family of Trinomial Arcs $G(p, k, r, n)$

Kaoutar Lamrini Uahabi

F.A.R. Blvd., 49, Apartment N. 9
Nador 62000, Morocco
lamrinika@yahoo.fr

Abstract

We consider the subset $G$ of the unit disk that is the union of the trinomial arcs $G(p, k, r, n)$, these continuous arcs being solutions of the equation $z^n = \alpha z^k + (1 - \alpha)$, where $z = \rho e^{i\theta}$ is a complex number, $n$ and $k$ are two integers such that $0 < k < n$ and $\alpha \in (0, 1)$. Using the fact that $\rho(\theta)$ is a monotonic function, we will prove that the fractal dimension of $G$ is 3/2.

Mathematics Subject Classification: 14H45, 26A48, 28A80, 30C15.

Keywords: area, curve length, feasible angle, fractal dimension, Minkowski sausage, monotonic function, trinomial arcs, trinomial equation.

1 Preliminaries

The family of trinomial arcs $G(p, k, r, n)$ introduced and studied in [4] is the set of roots of the trinomial equation

$$z^n = \alpha z^k + (1 - \alpha),$$

where $z = \rho e^{i\theta}$ is a complex number, $n$ and $k$ are two integers such that $0 < k < n$, $p$ and $r$ are two integers satisfying some conditions and $\alpha \in (0, 1)$. These arcs are continuous in functions of $\alpha$ as $\alpha$ varies between 0 and 1 and can be expressed in polar coordinates $(\rho, \theta)$ by a function $\rho(\theta)$ verifying the equation

$$\rho^{n-k} \sin n\theta - \rho^n \sin(n-k)\theta = \sin k\theta.$$

As $0 < \alpha < 1$, we are interested in those $\theta$ for which :

$$\text{sign}(\sin n\theta) = \text{sign}(\sin k\theta) = -\text{sign}(\sin(n-k)\theta).$$
Definition 1.1 An angle \( \theta \) which fulfills (3) will be called a \((n,k)\)-feasible angle for the equation (1) with \( 0 < \alpha < 1 \).

In [4], the trinomial arcs \( G(p,k,r,n) \) are defined as follows:

Definition 1.2 If \( n \) is an integer greater than or equal to 3, so \( G(p,k,r,n) \) is the set of roots of equation (1) with \( 0 < \alpha < 1 \) and the feasible angles belong to \([(2p+1)\pi/k,2\pi r/n] \), where \( p \) is a nonnegative integer, \( r \) is an integer such that \( r \geq p+1 \) and \( k \) is an integer verifying \((2p+1)n/2r < k < (2p+1)n/(2r-1)\).

These arcs appear in Figure 1 and exist in view of the next lemma of [4]:

Lemma 1.3 If \( n \) is an integer greater than or equal to 3, \( k \) is an integer such that \( 0 < k < n \) and \( 0 < \alpha < 1 \), then in the trinomial equation (1) with \((2p+1)n/2r < k < (2p+1)n/(2r-1) \), where \( p \) is a nonnegative integer, \( r \) is an integer such that \( r \geq p+1 \), any angle of \([(2p+1)\pi/k,2\pi r/n] \) is feasible. In particular, for each trinomial arc \( G(p,k,r,n) \), we have \( \sin n\theta < 0, \sin k\theta < 0 \) and \( \sin(n-k)\theta > 0 \) for any \( \theta \) in the interval \([(2p+1)\pi/k,2\pi r/n] \).

Figure 1: Trinomial arcs \( G(p,k,r,n) \) inside the upper half unit disk.

We set \( G \) as the union of all trinomial arcs \( G(p,k,r,n) \). Our main interest in this work is to compute the fractal dimension of \( G \), which is the number:

\[
\Delta(G) = \limsup_{\varepsilon \to 0} (2 - \log |G(\varepsilon)|_2 / \log \varepsilon),
\]

where \( G(\varepsilon) \) is the set of points whose distance to \( G \) is smaller than \( \varepsilon \) and \( |G(\varepsilon)|_2 \) is the plane Lebesgue measure of \( G(\varepsilon) \).

There is one basic remark: From the stability of the fractal dimension \( \Delta \), i.e., \( \Delta(A \cup B) = \max\{\Delta(A),\Delta(B)\} \) and since \( G \) is symmetric with respect the real axis, we will compute only the fractal dimension \( \Delta(G) \) of the arcs \( G(p,k,r,n) \) on the upper half unit disk.
The rest of this paper is organized as follows. Section 2 contains some preliminary material concerning the length of a curve $\rho = f(\theta)$ when $f$ is monotonic and the area of the Minkowski sausage of a curve of finite length. In Section 3, we define the fractal dimension in the plane. Using the main result of [4] stating that $\rho(\theta)$ is an increasing function for the arcs $G(p, k, r, n)$, we will show in Section 4 that the dimension of $G$ is $3/2$.

2 Some preliminary results

In this section, we recall some intermediate results on the length of a monotonic curve in polar coordinates and the area of its Minkowski sausage.

2.1 A bound for the curve length

Let $C$ be a curve defined by:

$$x(\theta) = \rho(\theta) \cos \theta, \quad y(\theta) = \rho(\theta) \sin \theta,$$

where $\theta_1 \leq \theta \leq \theta_2$.

![Figure 2: A monotonic arc C.](image)

We assume that $\rho(\theta_1) = R_1$ and $\rho(\theta_2) = R_2$. Figure 2 illustrates a possible situation. Theorem 2 of [1] gives us the following result:

**Theorem 2.1** If $\rho(\theta)$ is monotonic, then $C$ has a finite length $L$ such that

$$L \leq \max(R_2, R_1) |\theta_2 - \theta_1| + |R_2 - R_1|.$$

2.2 A bound for the area of the Minkowski sausage

Let $F$ be a subset of the plane $\mathbb{R}^2$ and $D$ the unit disk of $\mathbb{R}^2$. For $\varepsilon > 0$, the Minkowski sausage of $F$ is the set $F(\varepsilon) = F + \varepsilon D = \{z + w : z \in F, |w| < \varepsilon\}$. The area of this sausage which is the Lebesgue measure of $F(\varepsilon)$ will be denoted by $|F(\varepsilon)|_2$. An example of a Minkowski sausage is sketched in Figure 3.

In [1], Theorem 3 gives a bound for the area of the Minkowski sausage of a curve as follows:
**Theorem 2.2** Let $C$ be a curve of finite length $L(C)$. Then, for any $\varepsilon > 0$, 

$$|C(\varepsilon)|_2 \leq 2\varepsilon L(C) + \pi\varepsilon^2.$$ 

Figure 3: The Minkowski sausage of a curve in the plane.

### 3 Fractal dimension in the plane

Now, we give some definitions of the fractal dimension as it can be found in the book of Falconer [2] or in that of Tricot [6]. Let $F$ be a bounded subset of the plane $\mathbb{R}^2$. The *upper fractal dimension* $\Delta(F)$ and the *lower fractal dimension* $\delta(F)$ of $F$ are respectively the following numbers:

$$\Delta(F) = \limsup_{\varepsilon \to 0} \left( 2 - \log |F(\varepsilon)|_2 / \log \varepsilon \right)$$

and

$$\delta(F) = \liminf_{\varepsilon \to 0} \left( 2 - \log |F(\varepsilon)|_2 / \log \varepsilon \right),$$

where $F(\varepsilon)$ is the Minkowski sausage of $F$ and $|F(\varepsilon)|_2$ is its Lebesgue measure. If $\Delta(F) = \delta(F)$, then the *dimension* of $F$ is this common number, which is sometimes called the *fractal dimension* or the *Minkowski dimension*.

According to [2], p. 44, we can state the two following properties:

**(i)** The upper and lower fractal dimensions are monotonic, i.e. if $E$ and $F$ are two subsets of $\mathbb{R}^2$ such that $E \subseteq F$, then

$$\delta(E) \leq \delta(F) \quad \text{and} \quad \Delta(E) \leq \Delta(F).$$

**(ii)** The upper dimension is *finitely* stable, i.e.

$$\Delta(E \cup F) = \max\{\Delta(E), \Delta(F)\},$$

though the lower dimension is not.
An example of the fact that the lower fractal dimension is not stable is given in [6], pp. 32 - 33. For that, in all what follows, we will only use the upper fractal dimension $\Delta$.

Moreover, it is often convenient to replace, under some conditions, the continuous variable $\varepsilon$ by a sequence $\varepsilon_n$ going to 0. Thus, the dimension can be computed by using the next lemma of [1].

**Lemma 3.1** Let $(\varepsilon_n)$ be a sequence of positive real numbers that converges to 0. If the sequence $\log \varepsilon_n / \log \varepsilon_{n+1}$ converges to 1, then

$$\Delta(F) = \limsup_{n \to +\infty} (2 - \log |F(\varepsilon_n)| / \log \varepsilon_n).$$

**Remark 3.2** Examples of sequences that might be used with Lemma 3.1 are $(1/[n+1]^2)$, $(1/n^2)$ or $(1/n^q)$ where $q > 0$.

### 4 Fractal dimension of the family $G$

Let $G(p, k, r, n)$ be a trinomial arc. From Lemma 1.3, the interval of feasible angles $\theta$ is $[(2p + 1)\pi/k, 2\pi r/n]$, where $n$ is an integer greater than or equal to 3, $p$ is a nonnegative integer, $r$ is an integer such that $r \geq p + 1$ and $k$ is an integer such that $(2p + 1)n/2r < k < (2p + 1)n/(2r - 1)$.

First of all, estimate the function $\rho(\theta)$ at the bounds $(2p + 1)\pi/k$ and $2\pi r/n$. For that, consider the equation $\rho^{n-k} \sin n\theta - \rho^n \sin (n-k)\theta = \sin k\theta$ given by (2). If we put $\theta = (2p + 1)\pi/k$ in this equation, we obtain that $\rho^{n-k}(\rho^k + 1) \sin((2p + 1)n/k) = 0$. Since $(2r - 1)\pi < (2p + 1)n/k < 2\pi r$, we get $\sin((2p + 1)n/k) \neq 0$. Then, as $\rho \geq 0$, it follows that $\rho ((2p + 1)\pi/k) = 0$. Moreover, if we put $\theta = 2\pi r/n$ in equation (2), we get the equality $(\rho^n - 1) \sin(2\pi rk/n) = 0$. But, from the fact that $(2p + 1)\pi < 2\pi rk/n < 2(p + 1)\pi$, we deduce that $\sin(2\pi rk/n) \neq 0$ and that $\rho(2\pi r/n) = 1$.

Hence, any trinomial arc $G(p, k, r, n)$ joins the two points $\rho = 0$, $\theta = (2p + 1)\pi/k$ and $\rho = 1$, $\theta = 2\pi r/n$.

Now, consider the subset $G$ of the unit disk that is the union of all trinomial arcs $G(p, k, r, n)$. Let us recall that the main purpose of this paper is to show that the fractal dimension $\Delta(G)$ is $3/2$.

From the equation (2), we can extract a function $\rho(\theta)$ defined on the interval $[(2p + 1)\pi/k, 2\pi r/n]$. Theorem 4.9 of [4] gives us the following main result for the trinomial arcs $G(p, k, r, n)$.

**Theorem 4.1** $\rho(\theta)$ is an increasing function on the interval of feasible angles $[(2p + 1)\pi/k, 2\pi r/n]$ for the trinomial arcs $G(p, k, r, n)$. 
It is very important that $\rho(\theta)$ is an increasing function for the trinomial arcs $G(p, k, r, n)$. By using this property and Theorem 2.1, we can obtain the next result.

**Proposition 4.2** For any integer $n$ greater than or equal to 3, the length of the trinomial arc $G(p, k, r, n)$ is smaller than $5/2$.

**Proof.** Let $G(p, k, r, n)$ be a trinomial arc and $L(G(p, k, r, n))$ its length. Applying Theorem 2.1 and Theorem 4.1, it yields that

$$L(G(p, k, r, n)) \leq \frac{2\pi r}{n} - \frac{(2p+1)\pi}{k} + 1.$$ 

Because $k < (2p+1)n/(2r-1)$, it follows that

$$L(G(p, k, r, n)) < \frac{\pi}{n} + 1 < 5/2.$$ 

Thus, we achieve the proof.

**Remark 4.3** In order to estimate the fractal dimension $\Delta(G)$, let us remark that it is possible to divide the union $G$ of the trinomial arcs $G(p, k, r, n)$ (see Figure 1) into two families of arcs, denoted by $G_1$ and $G_2$, as follows:

- $G_1$ is the union of the trinomial arcs $G(p, k, r, n)$ located in the first quadrant of the plane.

- $G_2$ is the union of the trinomial arcs $G(p, k, r, n)$ located in the second quadrant of the plane.

From the stability of the fractal dimension $\Delta$, it yields that

$$\Delta(G) = \Delta(G_1 \cup G_2) = \max\{\Delta(G_1), \Delta(G_2)\}.$$ 

Therefore, for estimate $\Delta(G)$, we will compute $\Delta(G_1)$. As for $\Delta(G_2)$, it can be obtained in a similar manner.

We end this section with the main result.

**Theorem 4.4** The fractal dimension $\Delta(G)$ of the union $G$ of all trinomial arcs $G(p, k, r, n)$ is $3/2$.

**Proof.** The present proof will be divided into two parts. The first part will show that $\Delta(G) \leq 3/2$ and the second part will prove that $\Delta(G) \geq 3/2$. In both cases, we will use Remark 4.3 and Lemma 3.1 with two distinct sequences $(\varepsilon_n)$. 

Fractal dimension of a family of trinomial arcs

In order to show that $\Delta(G_1) \leq 3/2$, let $n$ be an integer greater than 3 and let $(n - 2)$ be the number of the $n$ first arcs $G(p, k, r, m)$ where $m \geq 3$. First, set $\varepsilon_n = 1/(n + 1)^2$. Let us remark that the area of the Minkowski sausage of $G_1$ is smaller than or equal to the sum of the areas of the sausages of the $(n - 2)$ first arcs $G(p, k, r, m)$ and the area of that of the curvilinear triangle $T$ with radius 1 and delimited by the angles $0$ and $2\pi/(n + 1)$. Thus,

$$|G_1(\varepsilon_n)|_2 \leq |T(\varepsilon_n)|_2 + \sum_{m=3}^{n} |G(p, k, r, m)(\varepsilon_n)|_2.$$

On the one hand, we have

$$|T(\varepsilon_n)|_2 \leq \frac{\pi}{(n+1)} + (2\varepsilon_n + \pi\varepsilon_n^2) + \left(\varepsilon_n \frac{2\pi}{(n+1)} + \frac{1}{2}\pi \varepsilon_n^2\right)$$

$$= \pi \sqrt{\varepsilon_n} \left[ 1 + \frac{2\sqrt{\varepsilon_n}}{\pi} (1 + \pi \sqrt{\varepsilon_n}) + \frac{3}{2} \varepsilon_n^{3/2} \right]$$

$$= O\left(\sqrt{\varepsilon_n}\right).$$

On the other hand, applying Theorem 2.2 and Proposition 4.2, we deduce that, for any trinomial arc $G(p, k, r, m)$, $m \geq 3$, we get

$$|G(p, k, r, m)(\varepsilon_n)|_2 \leq \varepsilon_n^2 (5 + \pi \varepsilon_n) \leq 5\varepsilon_n (1 + \varepsilon_n).$$

This yields that,

$$\sum_{m=3}^{n} |G(p, k, r, m)(\varepsilon_n)|_2 \leq 5(n - 2)\varepsilon_n (1 + \varepsilon_n)$$

$$\leq \frac{5}{(n+1)} (1 + \varepsilon_n)$$

$$= 5\sqrt{\varepsilon_n} (1 + \varepsilon_n) = O\left(\sqrt{\varepsilon_n}\right).$$

Consequently,

$$\Delta(G_1) = \limsup_{n \to +\infty} (2 - \log |G_1(\varepsilon_n)|_2 / \log \varepsilon_n)$$

$$\leq \limsup_{n \to +\infty} (2 - \log O\left(\sqrt{\varepsilon_n}\right) / \log \varepsilon_n) = 3/2.$$

Otherwise, using a similar reasoning, we obtain that

$$\Delta(G_2) = \limsup_{n \to +\infty} (2 - \log |G_2(\varepsilon_n)|_2 / \log \varepsilon_n) \leq 3/2.$$
Therefore, we deduce that
\[ \Delta(G) = \max\{\Delta(G_1), \Delta(G_2)\} \leq 3/2. \]

- Lower bound for \( \Delta(G) \)

Now, with a view to proving that \( \Delta(G_1) \geq 3/2 \), we set \( \varepsilon_m = 1/9m^2 \), where \( m > 3 \). Let us remark that the Minkowski sausage \( G_1(\varepsilon_m) \) contains the curvilinear triangle \( T' \) with radius 1/3 and delimited by the angles 0 and \( 2\pi/m \). Then, it follows that the area of \( G_1(\varepsilon_m) \) is larger than or equal to \( |T'|_2 = \pi/9m \). This implies that \( |G_1(\varepsilon_m)|_2 \geq \pi/9m \geq \sqrt{\varepsilon_m} \). Hence,
\[
\Delta(G_1) = \limsup_{m \to +\infty}(2 - \log|G_1(\varepsilon_m)|_2 / \log \varepsilon_m) \\
\geq \limsup_{m \to +\infty}(2 - \log(\sqrt{\varepsilon_m}) / \log \varepsilon_m) = 3/2.
\]

Proceeding with a similar manner, we deduce that \( \Delta(G_2) \geq 3/2 \). Therefore,
\[ \Delta(G) = \max\{\Delta(G_1), \Delta(G_2)\} \geq 3/2. \]

Thus, we achieve the proof.

References


Received: August 17, 2007