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Covering Properties Related to Special Base Properties

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Abstract. We introduce the notions of (n, κ)-metacompact, $(< \omega, \kappa)$-metacompact (for infinite cardinal κ), OIF-metacompact and δ - OIF - metacompact. Each of these covering properties extends a base property, and we show that for Moore spaces, the base property is often equivalent to the covering property (with $\kappa = \omega$). We show that for GO spaces, $(< \omega, \kappa)$-metacompact implies $(1, \kappa^+)$-metacompact. Furthermore, we demonstrate that the above is true for all spaces under the axiom known as CECA [3].

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1. Introduction

We define (n, κ)-metacompact, $(< \omega, \kappa)$-metacompact, OIF-metacompact and δ-OIF-metacompact. We determine that for GO spaces and $\kappa = \omega$, all four of these are equivalent to paracompact. We also establish that the properties do not necessarily coincide even if the space is required to be monotonically normal. For Moore spaces, the covering property is often equivalent to the related base property.

We begin with the definitions of the related base properties.

Definition.

1. An *n-weakly uniform base* $\mathcal{B}$ for the space X is a base so that given any subset A of X with $|A| = n$, the set $\{B \in \mathcal{B} : A \subseteq B\}$ is finite.
2. A $< \omega$*-weakly uniform base* $\mathcal{B}$ is a base so that given any infinite subset A of X there is a finite subset F of A so that $\{B \in \mathcal{B} : F \subseteq B\}$ is finite. [3]

3. A collection of sets is called *open-in-finite* (OIF) if each open set is contained in only finitely many elements of the collection [2]. So if $\mathcal{B}$ is an OIF-base, then if, $\{B_i : i < \omega\} \subseteq \mathcal{B}$ then $\left(\bigcap_{i<\omega} B_i\right)^{\circ} = \emptyset$.
4. A collection $\mathcal{C}$ is called δ-open-in-finite (δ-OIF), if $\overline{\left(\bigcap_{i<\omega} C_i\right)}^{\circ} = \emptyset$ for each $\{C_i : i < \omega\}$. [1]

Definition. Let $n < \omega$ and κ be an infinite cardinal.

1. A space X is (n, κ)-*metacompact* if every open cover $\mathcal{U}$ has an open refinement $\mathcal{V}$ so that for each $A \subseteq X$ such that $|A| = n$, then $|\{V \in \mathcal{V} : A \subseteq V\}| < \kappa$.
2. A space X is $(< \omega, \kappa)$-*metacompact* if every open cover $\mathcal{U}$ has an open refinement $\mathcal{V}$ so that for every infinite set $A \subseteq X$, there is a finite set B so that $|\{V \in \mathcal{V} : B \subseteq V\}| < \kappa$.
3. A space X is *OIF-metacompact* if every open cover $\mathcal{U}$ has an open refinement $\mathcal{V}$ that is an open-in-finite collection.
4. A space X is δ-*OIF-metacompact* if every open cover $\mathcal{U}$ has an open refinement $\mathcal{V}$ that is a δ-open-in-finite collection.

There are a few immediate observations. To begin, (n, ω)-metacompact is n-metacompact and $(< \omega, \omega)$-metacompact is $< \omega$-metacompact (defined in [3]). If a space has a $< \omega$-weakly uniform base then it is $< \omega$-metacompact, since for any open cover, a refinement consisting of elements of a $< \omega$-weakly uniform base will witness the $< \omega$-metacompact property. Also, $(1, \omega_1)$-metacompact is the same as meta-Lindelöf. For a fixed κ and if $n < m < \omega$ then, (n, κ)-metacompact implies (m, κ)-metacompact which implies $(< \omega, \kappa)$-metacompact. Since each δ-OIF collection is OIF, we know that δ-OIF-metacompact implies OIF-metacompact. Also, clearly any OIF space (resp. δ-OIF space) will be OIF-metacompact (resp. δ-OIF-metacompact).

Proposition 1. *If X is an OIF-metacompact space with character κ, then X is $(1, \kappa^+)$-metacompact.*

Proof. Let $\mathcal{U}$ be an arbitrary open cover, and let $\mathcal{V}$ be the OIF-metacompact refinement of $\mathcal{U}$. Then since the character of X is κ, we know that each $x \in X$ is contained in not more than κ many elements of $\mathcal{V}$, because if x is contained in κ^+ many members of $\mathcal{V}$, then there is some member of the local base for x that is contained in infinitely many members of $\mathcal{V}$. □

Corollary 2. *Every space that is OIF-metacompact and first countable is metaLindelöf.*

Proposition 3. *For GO spaces OIF-metacompact implies paracompact.*

Proof. Let $\mathcal{U}$ be an open cover of X an OIF-metacompact space, and let $\mathcal{V}$ be an OIF-refinement of $\mathcal{U}$. We may assume that $\mathcal{V}$ consists of convex subsets of

X. Then by Bennet and Lutzer [4] we may assume that $\mathcal{V}$ is a σ-point-finite collection. In fact, they show that if $\mathcal{V}_0 = \{V \in \mathcal{V} : |V| = 1\}$ and for $n \geq 0$, $\mathcal{V}_{n+1}$ is the collection of maximal subsets of $\mathcal{V} \setminus \bigcup_{0 \leq k \leq n} \mathcal{V}_k$, then each $\mathcal{V}_n$ is star-finite (e.g. each member of $\mathcal{V}_n$ meets only finitely many other members of $\mathcal{V}_n$). Then $\mathcal{V}_0 \cup \mathcal{V}_1$ covers the same set as all of $\mathcal{V}$ and is locally finite. □

Proposition 4. *For GO spaces* $(< \omega, \kappa)$*-metacompact implies* $(1, \kappa^+)$*- metacompact.*

Proof. Let X be a $(< \omega, \kappa)$-metacompact GO space, and let $\prec$ be the order on X. Let $\mathcal{U}$ be an open cover of X. Then let $\mathcal{V}$ be a refinement of $\mathcal{U}$ witnessing $(< \omega, \kappa)$-metacompact. We may assume that the elements of $\mathcal{V}$ are convex. We define $\mathcal{V}_0$ and $\mathcal{V}_1$, $\mathcal{V}_0 = \{V \in \mathcal{V} : |V| = 1\}$ and $\mathcal{V}_1 = \mathcal{V} \setminus \mathcal{V}_0$. Obviously, $\mathcal{V}_0$ is point-finite.

We wish to show that each point of X is contained in only κ many elements of $\mathcal{V}_1$. For contradiction, assume $b \in X$ and that $\mathcal{B} = \{V \in \mathcal{V}_1 : b \in V\}$ has cardinality κ^+. Then the sets $\{\inf(V) : V \in \mathcal{B}\}$ and $\{\sup(V) : V \in \mathcal{B}\}$ cannot both have cardinality less than or equal to κ. However, all the supremums are $\prec$-greater than b and all the infimums are $\prec$-less than b.

Assume that the set of supremums is of size κ^+, call this set S. Then we will show that there is a $x \in X$ so that the cardinality of the set of supremums in S $\prec$-less than x is at least κ.

For each $x \in S$, let $L(x) = \{y \in S : y \prec x\}$ and $R(x) = \{y \in S : x \prec y\}$. These are the "left" and "right" sides of x. Notice that $L(x) \cup R(x) \cup \{x\} = S$, therefore for each x we have $\max\{|L(x)|, |R(x)|\} = \kappa^+$. Next, define $L = \{x \in S : |L(x)| \geq \kappa\}$ and $R = \{x \in S : |R(x)| \geq \kappa\}$; then clearly $L \cup R = S$. Finally, we note that $|S \setminus L| \leq \kappa$. If $|S \setminus L| = \kappa^+$, then because κ^+ is regular, no sequence of κ many elements of $S \setminus L$ is cofinal in $S \setminus L$. Therefore, for any κ sized subset of $S \setminus L$, there is an upper bound z in $S \setminus L$. But then $|L(z)| \geq \kappa$, contradiction. Similarly, $S \setminus R$ is also a set of cardinality not more than κ. Hence, $L \cap R \neq \emptyset$.

Choose $x \in L \cap R$ and let $\mathfrak{A} = \{V \in \mathcal{B} : x \prec \sup(V)\}$. Then because the elements of $\mathfrak{A}$ are convex, all contain b and have supremums $\prec$-greater than x. So for all $V \in \mathfrak{A}$ we have $[b, x] \subseteq V$, contradicting $(< \omega, \kappa)$-metacompact.

If the infimums are uncountable, the proof is analogous.

Therefore, $\mathcal{V}_1$ is point-$\leq \kappa$. So $\mathcal{V}_0 \cup \mathcal{V}_1$ is a $(1, \kappa^+)$ open refinement of $\mathcal{U}$. □

It is known that for GO spaces, meta-Lindelöf implies paracompact. Naturally, metacompact implies both $(< \omega, \omega)$-metacompact and δ-OIF-metacompact, so we have the following corollary.

Corollary 5. *For GO spaces, the following are equivalent:*

a) X *is paracompact,*
b) X *is* δ*-OIF-metacompact,*
c) X *is OIF-metacompact,*

d) X is $(<\omega,\omega)$-metacompact,
e) X is (n,ω)-metacompact for each $n<\omega$.

It seems natural to wonder if these covering properties coincide for all spaces. We will show that there is a space which is monotonically normal, δ-OIF-metacompact and yet is not paracompact or metacompact. Before we can present that example, we need a few more results. The following proposition has a routine proof.

Proposition 6. *The (n,κ)-metacompact and $(<\omega,\kappa)$-metacompact properties are hereditary for closed sets.*

Corollary 7. *If X is a monotonically normal $(<\omega,\omega)$-metacompact space, then X is paracompact.*

Proof. If X is not paracompact, then X contains a closed subspace C that is homeomorphic to a stationary subset of an uncountable cardinal. Then since C is closed, C is $(<\omega,\omega)$-metacompact. Also, C is a GO space therefore, C is paracompact, contradiction. □

Corollary 8. *For GO spaces, OIF-metacompact is hereditary for closed sets.*

The following theorem can be found in [5].

Theorem 9. *A space is monotonically normal if and only if for each open set U and $x\in U$, one can assign an open set U_x containing x such that $U_x\cap V_y\neq\emptyset$ implies $x\in V$ or $y\in U$.*

Theorem 10. *Every space X is a closed subset of a δ-OIF space $O(X)$ such that if X is monotonically normal, then so is $O(X)$.*

Proof. We follow the construction found in [2]. The authors show that $O(X)$ is OIF and if X is T_1, then X is a G_δ subset of $O(X)$. They also state that $O(X)$ has the same separation axioms as X, but monotonically normal is not mentioned.

Let X be a space with base $\mathcal{A}$ and let $F[\mathcal{A}]$ be the set of all finite subsets of $\mathcal{A}$. Let $Y=\{\langle p,\mathcal{F}\rangle : p\in X \text{ and } \mathcal{F}\in F[\mathcal{A}]\}$. For each $U\in\mathcal{A}$, define $f(U)=\{\langle p,\mathcal{F}\rangle\in Y : p\in U\in\mathcal{F}\}$ and let $S(U)=U\cup f(U)$. The space is $O(X)=X\cup Y$ where the topology is generated by the subbase $\mathcal{S}=\{S(U) : U\in\mathcal{A}\}\cup\{\{x\} : x\in Y\}$.

First, we show that $\mathcal{S}$ generates a δ-OIF base. For any collection $\{B_i : i<\omega\}\subseteq\mathcal{S}$ we may consider $\bigcap_{i<\omega}B_i=\bigcap_{j<\omega}S(U_j)=\bigcap_{j<\omega}(U_j\cup f(U_j))=\bigcap_{j<\omega}U_j\cup\bigcap_{j<\omega}f(U_j)$. Note that $\bigcap_{j<\omega}f(U_j)=\emptyset$. Therefore, $\overline{\bigcap_{j<\omega}S(U_j)}=\overline{\bigcap_{j<\omega}U_j}$, and because the points of Y are isolated $Y\cap\overline{\bigcap_{j<\omega}U_j}=\emptyset$. However, Y is dense in $O(X)$, so $\overline{\bigcap_{j<\omega}U_j}^{\circ}=\emptyset$.

Now we show that if X is monotonically normal then $O(X)$ is as well. Let O be open in $O(X)$, and let $x \in O$. There is a basic open set containing x and contained in O, which has the form $\bigcap_{i<n_x} S(U_i^x)$. If $x \in Y$, then let $O_x = \{x\}$. If $x \in X$, then $x \in \bigcap_{i<n_x} U_i^x = O^x$. Since X is monotonically normal, there is an assigned set O_x^x as in Theorem 9. In $O(X)$ let $O_x = S(O_x^x) \cap O$. We need to check that if V is open in $O(X)$ and $y \in V$ then $O_x \cap V_y \neq \emptyset$ implies $x \in V$ or $y \in O$.

Assume that $O_x \cap V_y \neq \emptyset$. There are essentially two cases; either both points x and y are contained in X or not.

1. If $y \in Y$, then $O_x \cap V_y \neq \emptyset$ and $V_y = \{y\}$ implies $y \in O_x \subseteq O$. Similarly, if $x \in Y$, then $x \in V_y \subseteq V$.
2. If $x, y \in X$, then let $V_y = S(V_y^y) \cap V$ be assigned to V. Let $\bigcap_{i<n_y} V_i^y$ be the basic open set containing y and used to define V_y. Therefore, $O_x \cap V_y = S(O_x^x) \cap S(V_y^y) \cap O \cap V$. This means that $O_x^x \cap V_y^y \neq \emptyset$, therefore $y \in \bigcap_{i<n_x} U_i^x$ or $x \in \bigcap_{i<n_y} V_i^y$. Without loss of generality, let $y \in \bigcap_{i<n_x} U_i^x$. Then $y \in \bigcap_{i<n_x} U_i^x \subset \bigcap_{i<n_x} S(U_i^x) \subseteq O$.

So we see that $O(X)$ is monotonically normal. □

Corollary 11. *There is a monotonically normal space that is OIF, hence OIF-metacompact, but not paracompact.*

Proof. Let $X = \omega_1$ with the order topology. Then X is monotonically normal, since it is an ordered space, but not paracompact. Form $O(X)$, as in Theorem 10, and then $O(X)$ is OIF, hence OIF-metacompact. This cannot be a paracompact space, because all closed subspaces of paracompact spaces are paracompact, and X is such a closed subspace. □

This is also an example of a space that is OIF-metacompact but not metacompact or $(< \omega, \omega)$-metacompact. For if it were $(< \omega, \omega)$-metacompact, then X would be as well, and therefore X would be paracompact.

It is already known that metacompact Moore spaces are OIF and have a n-weakly uniform base for each $n < \omega$, but we are able to be more precise in the next theorem. We are able to say, informally, that for Moore spaces "the base property is the same as the covering property".

Theorem 12. *Suppose X is a Moore space. Then we have the following.*

1. *X has an OIF base if and only if X is OIF-metacompact.*
2. *X has an δ-OIF base if and only if X is δ-OIF-metacompact.*
3. *For $n \geq 2$, X has an n-weakly uniform base if and only if X is (n, ω)-metacompact.*

Proof. For 1, suppose X has an OIF base $\mathcal{B}$ and let $\mathcal{U}$ be an open cover of X. Then refine $\mathcal{U}$ with elements of $\mathcal{B}$ to find an OIF-refinement of $\mathcal{U}$. This proves the forward direction, we now prove the reverse. Suppose that X has development $\mathcal{G} = (\mathcal{G}_i)_{i<\omega}$ and that X is OIF-metacompact. Then let $\mathcal{G}''_0$ be an OIF refinement of $\mathcal{G}_0$. For $\mathcal{G}''_i$ defined, let $\mathcal{G}'_{i+1} = \{V \cap U : V \in \mathcal{G}_{i+1}, U \in \mathcal{G}''_i\}$, then take $\mathcal{G}''_{i+1}$ to be an OIF refinement of $\mathcal{G}'_{i+1}$. Therefore, $\mathcal{G}'' = (\mathcal{G}''_i)_{i<\omega}$ is a development for X with each $\mathcal{G}''_i$ an OIF collection of open sets, and each $\mathcal{G}''_{i+1}$ a refinement of $\mathcal{G}''_i$. Therefore $\bigcup_{i<\omega} \mathcal{G}''_i$ is an base for X. To see that $\bigcup_{i<\omega} \mathcal{G}''_i$ is OIF, suppose that V is an open set from $\bigcup_{i<\omega} \mathcal{G}''_i$. Then for some $n < \omega$ we have that $V \in \mathcal{G}''_n$ and V is a subset of finitely many of the elements of each $\mathcal{G}''_i$ for $i \leq n$. If there are infinitely many sets from $\bigcup_{i>n} \mathcal{G}''_i$ containing V, then we may assume that every $\mathcal{G}''_i$ for $i > n$ has an element containing V. For each $i < \omega$, let U_i be an element of $\mathcal{G}''_i$ containing V. Then $(U_i)_{i<\omega}$ is a base for any point of V. This implies that V is a singleton, contradiction.

Based on the proof of 1, the proof for 2 should be clear.

For 3, assume that a space X has a (n, ω)-weakly uniform base $\mathcal{B}$ and let $\mathcal{U}$ be an open covering of X. Then any refinement of $\mathcal{U}$ consisting of elements of $\mathcal{B}$ witnesses the (n, ω)-metacompact property.

The reverse direction is much the same as 1. For each $i < \omega$, choose $\mathcal{G}''_i$ to be a (n, ω) refinement. Suppose that F is a subset of X having cardinality n, and that F is contained in infinitely many sets from $\bigcup_{i<\omega} \mathcal{G}''_i$. Then we may assume that there is an $U_i \in \mathcal{G}''_i$ so that $F \subseteq U_i$ for each $i < \omega$. Yet, $(U_i)_{i<\omega}$ is a base for some point of F, implying $|F| = 1$, contradiction. $\square$

Question: Does every $(< \omega, \omega)$-metacompact Moore space have a $< \omega$-weakly uniform base?

There is an example of a space that has each of these covering properties but does not have any of these base properties. Indeed, the example we present is paracompact.

Example. Suppose that X is the sequential fan with ω_1 many sequences all approaching the point ∞. Let $\mathcal{U}$ be any covering of X. Choose $U_\infty \in \mathcal{U}$ so that $\infty \in U_\infty$. Then let $\mathcal{V} = \{U_\infty\} \cup \{\{x\} : x \in X \setminus U_\infty\}$. Then $\mathcal{V}$ refines $\mathcal{U}$ and is locally finite.

2. Set Theoretic and Combinatorial Conditions

In [3] the axiom CECA was introduced and proven consistent with ZFC. The authors show that CECA is equivalent to GCH plus a weakening of $\square_\lambda$ for singular λ. So in particular, CECA holds in Gödels constructible universe. In the same paper, CECA is used to prove the following result.

Theorem 13. *Assume CECA. Suppose that σ, τ are regular infinite cardinals, and let $\langle A_\alpha\rangle_{\alpha<\kappa}$ be a sequence of sets such that for every $I \in [\kappa]^\tau$ there is $J \in [I]^{<\tau}$ so that $|\bigcap_{\alpha\in J} A_\alpha| < \sigma$. Then there exists $\langle A'_\alpha\rangle_{\alpha<\kappa}$ such that $|A'_\alpha| \le \sigma$ for each $\alpha < \kappa$ and the sequence $\langle A_\alpha \backslash A'_\alpha\rangle_{\alpha<\kappa}$ is point-$<\tau$.*

We have already established the next result for GO spaces without special set theoretic considerations; in [3] this result is given for $\kappa = \omega$.

Theorem 14. *Assume CECA. Fix a regular infinite cardinal κ. If X is $(<\omega, \kappa)$-metacompact, then X is $(1, \kappa^+)$-metacompact.*

Proof. Let $\mathcal{U}$ be an open cover of X and let $\mathcal{V}$ be an $(<\omega, \kappa)$-metacompact refinement of $\mathcal{U}$. List $X = \{x_\alpha : \alpha < \lambda\}$ and for each $\alpha < \lambda$ let $A_\alpha = \{V \in \mathcal{V} : x_\alpha \in V\}$. For each infinite collection of points, there is some finite subcollection that is contained in less than κ-many elements of $\mathcal{V}$. Hence for each $I \in [\lambda]^\omega$ there is some finite subset J of I so that $|\bigcap_{\alpha\in J} A_\alpha| < \kappa$. We apply Theorem 13 with $\tau = \omega$ and $\sigma = \kappa$. Therefore for each $\alpha < \lambda$ there is a $A'_\alpha \in [A_\alpha]^{\le\kappa}$ such that $\langle A_\alpha \backslash A'_\alpha : \alpha < \kappa\rangle$ is point-finite on the set $\mathcal{V}$. For each $V \in \mathcal{V}$ let $I(V) = \{x_\alpha \in V : V \in A_\alpha \backslash A'_\alpha\}$. Each $I(V)$ is finite and if $x_\alpha \in V \backslash I(V)$, then $V \in A'_\alpha$. For each $\alpha < \lambda$ choose some $V_\alpha \in \mathcal{V}$ so that $x_\alpha \in V_\alpha$. Then for each $V \in \mathcal{V}$, define $V^* = (V \backslash I(V)) \cup \{x_\alpha \in I(V) : V = V_\alpha\}$, note that each $V^* \subseteq V$. Therefore $\mathcal{V}^* = \{V^* : V \in \mathcal{V}\}$ is an open refinement of $\mathcal{U}$. Such a $\mathcal{V}^*$ is a $(1, \kappa^+)$-metacompact refinement, because if $x_\alpha \in V$, then $V \in A'_\alpha \cup \{V_\alpha\}$, and this is a set of size $\le \kappa$. □

Now the question is raised whether the statement "$(<\omega, \kappa)$-metacompact implies $(1, \kappa^+)$-metacompact" is independent of ZFC. If the following combinatorial principal holds (and is consistent with MA + $\omega_3 \le 2^\omega$), then there is a space that is $(<\omega, \omega)$-metacompact and not $(1, \omega_1)$-metacompact.

(*) *If X is a set and $|X| = \omega_3$, then there exists a collection $\mathcal{H}$ of subsets of X and a partition $\{\mathcal{H}_n : n < \omega\}$ of $\mathcal{H}$ such that : (1) if $H_1, H_2 \in \mathcal{H}$ and $H_1 \ne H_2$, then $|H_1 \cap H_2| < \omega$; and (2) if $Y \subseteq X$ and $|Y| = \omega_3$, then for each $n \in \omega$, there exists $H \in \mathcal{H}_n$ such that $|Y \cap H| = \omega_2$.*

Theorem 15. *(MA + $\omega_3 \le 2^\omega$ + *) There is a space with a weakly uniform base that is not $(1, \omega_2)$-metacompact.*

Proof. Assume MA + $\omega_3 \le 2^\omega$. This construction is essentially due to [6]. Let S be a subset of the x-axis of size ω_3 and let K be a countable dense subset of the upper half plane of $\mathbb{R}^2$. Denote $K = \{p_n : n < \omega\}$. Then we let $X = S \cup K$. For $x \in S$, the neighborhoods are $B_n(x)$ define to be an open disk in the upper half plane of radius $1/n$ tangent to the axis at x intersected with K, together with $\{x\}$. The points of K are isolated. Now let $\mathcal{H} = \bigcup_{n<\omega} \mathcal{H}_n$

be the collection and partition satisfying condition (*). For each $n < \omega$ let $K'(n) = \{(p_n, H) : H \in \mathcal{H}_n\}$ and define $K' = \bigcup_{n<\omega} K'(n)$. Now let $X' = S \cup K'$.

In X' we define the open neighborhoods of $x \in S$ to be $B'_n(x) = \{x\} \cup \{(p_i, H) : (p_i, H) \in K'(i), p_i \in B_n(x)$ and $x \in H\}$. The points of K' are isolated. Let $\mathcal{B}_n = \{B'_n(x) : x \in S\} \cup \{\{q\} : q \in K'\}$, then define $\mathcal{B} = \bigcup_{n<\omega} \mathcal{B}'_n$. We claim that this is a weakly uniform base for X'. Suppose that x_1 and x_2 are both in K', and let $x_1 = (p_k, H_1)$ and $x_2 = (p_j, H_2)$. We have that $|H_1 \cap H_2| < \omega$; suppose that $y_0, y_1, \cdots, y_m$ list $H_1 \cap H_2$. Then for each y_i there is an n_i so that x_1 is not in $B'_{n_i}(y_i)$. Let $n = \max\{n_i : 0 \leq i \leq m\}$; then we have that $\{x_1, x_2\}$ is contained in less than $n \cdot m$ many sets from $\mathcal{B}$.

Next, we claim that X' has an open cover with no point $\leq \omega_1$ open refinement. Consider $\mathcal{B}'_1 = \{B'_1(x) : x \in S\} \cup \{\{q\} : q \in K'\}$, which is an open cover of X'. Suppose that $\mathcal{V}$ is an open point $\leq \omega_1$ open refinement of $\mathcal{B}'_1$. Then for each $x \in S$, there is an n_x so that $B'_{n_x}(x)$ is contained in some element of $\mathcal{V}$; then $\mathcal{U}' = \{B'_{n_x}(x) : x \in S\}$ must be point $\leq \omega_1$. Let $\mathcal{U} = \{B_{n_x}(x) : x \in S\}$. Since K is dense in X, there is p_i that is contained in ω_3 many sets from $\mathcal{U}$, say $\{B_{n_x}(x) : x \in Y\}$ for some subset Y of X of cardinality ω_3. So there exists $H \in \mathcal{H}_i$ so that $|H \cap Y| = \omega_2$ and the point (p_i, H) is in K'. For each $x \in Y \cap H$ we have $(p_i, H) \in G'_{n_x}(x)$. So (p_i, H) is contained in ω_2 many elements of $\mathcal{U}'$, which gives us our contradiction. □

References

[1] B. Bailey, δ-OIF spaces, *Questions Answers Gen. Topology* 24 (2006) 79-84

[2] Z. Balogh, H. Bennet, D. Burke, G. Gruenhage, D. Lutzer, J. Mashburn, OIF spaces, *Questions Answers Gen. Topology* 18 (2000)129-141.

[3] Z.T. Balogh, S.W. Davis, W. Just, S. Shelah, P.J. Szeptycki, Strongly almost disjoint sets and weakly uniform bases, *Trans. Amer. Math. Soc.* 352 (2000) 4971-4987.

[4] H. Bennett, D. Lutzer, Ordered spaces with special bases, *Fund. Math.* 158 (1998) 289-299.

[5] G. Gruenhage, Generalized metric spaces, in: K. Kunen, J. Vaughan (Eds.), Handbook of Set-Theoretic Topology, North-Holland, Amsterdam, 1984, pp. 423-501.

[6] M.L. Wage, Almost disjoint sets and Martin's axiom, *J. Symbolic Logic* 44 (1979) 3, 313-318.

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