

# Strong Compactness and P-Closedness in Smooth L-Fuzzy Topological Spaces

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## Abstract

In this work, we introduce definitions of smooth strong compactness and smooth P-closedness for arbitrary L-fuzzy sets in smooth L-fuzzy topological spaces and study some of their properties. We also investigate relations between these concepts and the other definitions of compactness in smooth L-fuzzy spaces.

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## 1. Introduction

Sostak [14] introduced the fundamental concept of a fuzzy topological structure, as an extension of both classical topology and fuzzy topology, in the sense that not only objects are fuzzified, but also the axiomatics. In [15, 16], Sostak gave some rules and showed how such an extension can be realized. Chattopadhyay et al.[4, 5] and Ramadan [11] have redefined the same concept called as “smooth fuzzy topology” by Ramadan [11].

The notion of compactness is one of the most important concept in general topology. Therefore, the problem of generalization of the classical compactness

to fuzzy topological spaces has been intensively discussed over the past 30 years. Many papers on fuzzy compactness have been published and various kinds of fuzzy compactness have been presented and studied. Among these compactness, the fuzzy compactness in  $L$ -fuzzy topological spaces introduced by Kudri [8] possesses more satisfactory properties than others.

Aygün et al.[2] introduced the notion of  $L$ -fuzzy compactness in smooth  $L$ -fuzzy topological spaces as a generalization of  $L$ -fuzzy compactness introduced by Kudri [8]. Based on this definition different kinds of compactness in smooth  $L$ -fuzzy topological spaces have been defined and studied [12].

In ordinary topology strong compactness have been defined by Atia et al.[1]. The concepts of strong compactness and  $P$ -closedness have been presented in  $[0,1]$ -fuzzy topological spaces by Nanda [10] and Zahran [17], respectively. Kudri and Warner have defined strong compactness for arbitrary fuzzy sets in  $L$ -fuzzy topological spaces. Aygün and Kudri [3] have given the definition of  $P$ -closedness for arbitrary fuzzy sets in  $L$ -fuzzy topological spaces.

In the present paper, we introduce smooth strong compactness and smooth  $P$ -closedness in smooth  $L$ -fuzzy topological spaces along the line of smooth  $L$ -fuzzy compactness defined by Aygün et al.[2]. We obtain different characterizations of these concepts and study some of their properties and give the relations among these concepts and the other definitions of compactness in smooth  $L$ -fuzzy topological spaces.

## 2. Preliminaries

Throughout this paper  $X$  will be a non-empty ordinary set and  $L = L(\leq, \vee, \wedge, ')$  will denote a fuzzy lattice, i.e. completely distributive lattice with a smallest element 0 and a largest element 1 ( $1 \neq 0$ ) and with an order reversing involution  $a \rightarrow a'(a \in L)$ . We shall denote by  $L^X$  the lattice of all  $L$ -fuzzy sets on  $X$ . We assume that the reader is familiar with all other standard notations and definitions of the fuzzy set theory.

**Definition 2. 1 [7]:** An element  $p$  of a lattice  $L$  is called prime if and only if  $p \neq 1$  and whenever

$$a, b \in L \text{ with } a \wedge b \leq p \text{ then } a \leq p \text{ or } b \leq p.$$

The set of all prime elements of a lattice  $L$  will be denoted by  $pr(L)$ .

**Definition 2. 2 [7]:** An element  $\alpha$  of a lattice  $L$  is called coprime or irreducible if and only if  $\alpha \neq 0$  and whenever

$$a, b \in L \text{ with } a \vee b \geq \alpha \text{ then } a \geq \alpha \text{ or } b \geq \alpha.$$

The set of all coprime elements of a lattice  $L$  will be denoted by  $M(L)$ .

**Definition 2. 3 [11, 14]:** A smooth  $L$ -fuzzy topology on  $X$  is a map  $t : L^X \rightarrow L$  satisfying the following three axioms:

- 1)  $t(0) = t(1) = 1$
- 2)  $t(f \wedge g) \geq t(f) \wedge t(g)$  for every  $f, g \in L^X$

$$\mathbf{3)} \quad t \left( \bigvee_{i \in J} f_i \right) \geq \bigwedge_{i \in J} t(f_i) \quad \text{for every family } (f_i)_{i \in J} \text{ in } L^X$$

The pair  $(X, t)$  is called a smooth  $L$ -fuzzy topological space.

For every  $f \in L^X$ ,  $t(f)$  is called the degree of openness of the fuzzy subset  $f$ .

While an  $L$ -fuzzy topology on  $X$  is an ordinary subset of  $L^X$ , a smooth  $L$ -fuzzy topology on  $X$  is a fuzzy subset of  $L^X$ .

**Definition 2. 4 [11, 14]:** Let  $(X, t)$  be a smooth  $L$ -fuzzy topological space. The map  $c : L^X \rightarrow L$  defined by  $c(f) = t(f')$  for every  $f \in L^X$  is called the degree of closedness on  $X$ , where  $f'$  is the complement of the  $L$ -fuzzy set  $f$ . For  $f \in L^X$ ,  $c(f)$  is called the degree of closedness of the  $L$ -fuzzy set  $f$ .

From Definition 2. 3 and Definition 2. 4 it is easy to see that the mapping  $c$  has the following properties:

$$\mathbf{1)} \quad c(0) = c(1) = 1$$

$$\mathbf{2)} \quad c(f \vee g) \geq c(f) \wedge c(g) \quad \text{for every } f, g \in L^X$$

$$\mathbf{3)} \quad c \left( \bigwedge_{i \in J} f_i \right) \geq \bigwedge_{i \in J} c(f_i) \quad \text{for every family } (f_i)_{i \in J} \text{ in } L^X$$

**Definition 2. 5 [6]:** Let  $(X, t)$  be a smooth  $L$ -fuzzy topological space and  $f \in L^X$ . The smooth interior and the smooth closure of  $f$  are defined as follows:

$$\overset{\circ}{f} := \bigvee \{g \in L^X \mid g \leq f \text{ and } t(g) > 0\}$$

$$\overline{f} := \bigwedge \{g \in L^X \mid g \geq f \text{ and } t(g') > 0\}$$

**Lemma 2. 6 [6]:** Let  $(X, t)$  be a smooth  $L$ -fuzzy topological space and  $f, g \in L^X$ .

$$\mathbf{a)} \quad f \leq g \Rightarrow \overset{\circ}{f} \leq \overset{\circ}{g} \text{ ve } \overline{f} \leq \overline{g}$$

$$\mathbf{b)} \quad (\overline{f})' = (f')^\circ, (\overset{\circ}{f})' = \overline{(f')}$$

$$\mathbf{c)} \quad t(f) > 0 \Rightarrow f = \overset{\circ}{f}$$

$$\mathbf{d)} \quad c(f) > 0 \Rightarrow f = \overline{f}$$

**Definition 2. 7 [11]:** Let  $(X, t_X)$  and  $(Y, t_Y)$  be a smooth  $L$ -fuzzy topological spaces. The function  $\varphi : X \rightarrow Y$  is called smooth continuous iff  $t_X(\varphi^{-1}(f)) \geq t_Y(f)$  for all  $f \in L^Y$

**Definition 2. 8 [2]:** Let  $(X, t)$  be a smooth  $L$ -fuzzy topological space and  $g \in L^X$ .  $g$  is called smooth compact iff for every  $p \in pr(L)$  and every collection

$(f_i)_{i \in J}$  of  $L$ -fuzzy sets with  $t(f_i) \not\leq p$  for each  $i \in J$  and  $\left( \bigvee_{i \in J} f_i \right) (x) \not\leq p$

for all  $x \in X$  such that  $g(x) \geq p'$ , there exists a finite subset  $F$  of  $J$  with

$\left( \bigvee_{i \in F} f_i \right) (x) \not\leq p$  for all such that  $x \in X$  such that  $g(x) \geq p'$ .

**Definition 2. 9 [12]:** Let  $(X, t)$  be a smooth  $L$ -fuzzy topological space and  $g \in L^X$ .  $g$  is called smooth almost compact iff for every  $p \in pr(L)$  and every collection  $(f_i)_{i \in J}$  of  $L$ -fuzzy sets with  $t(f_i) \not\leq p$  for each  $i \in J$  and

$\left( \bigvee_{i \in J} f_i \right) (x) \not\leq p$  for all  $x \in X$  such that  $g(x) \geq p'$ , there exists a finite subset

$F$  of  $J$  with  $\left(\bigvee_{i \in F} \bar{f}_i\right)(x) \not\leq p$  for all such that  $x \in X$  such that  $g(x) \geq p'$ .

### 3. Proposed definitions

**Definition 3. 1:** Let  $(X, t)$  be a smooth  $L$ -fuzzy topological space and  $f \in L^X$ .

a)  $f$  is called smooth pre-open iff  $f \leq \overset{\circ}{\bar{f}}$

b)  $f$  is called smooth pre-open iff  $\bar{f} \leq f$

If an  $L$ -fuzzy set  $f$  is both smooth pre-open and smooth pre-closed, then the  $L$ -fuzzy set  $f$  is called smooth pre-clopen set.

**Lemma 3. 2:** Let  $(X, t)$  be a smooth  $L$ -fuzzy topological space. If  $f \in L^X$  smooth pre-open set then  $f' \in L^X$  smooth pre-closed.

**Proof:** Let  $f$  is a smooth pre-open set so  $f \leq \bar{f}$ . Then  $\left(\overset{\circ}{\bar{f}}\right)' \leq f' \Rightarrow ((\bar{f})')^- \leq f' \Rightarrow (f')^{\circ-} \leq f'$ . Hence  $f' \in L^X$  smooth pre-closed.

**Definition 3. 3:** Let  $(X, t)$  be a smooth  $L$ -fuzzy topological space and  $f \in L^X$ . The smooth pre-interior and the smooth pre-closure of  $f$  are defined as follows:

$$\circ f = \bigvee \{g \in L^X \mid g \leq f \text{ and } g \text{ is smooth pre-open}\}$$

$$-f = \bigwedge \{g \in L^X \mid g \geq f \text{ and } g \text{ is smooth pre-closed}\}$$

From the definitions we obtain  $f \leq -f \leq \bar{f}$  and  $\overset{\circ}{f} \leq \circ f \leq f$ .

If  $f \in L^X$  smooth pre-open then  $\circ f = f$  and if  $f \in L^X$  smooth pre-closed then  $-f = f$ .

**Lemma 3. 4:** Let  $(X, t)$  be a smooth  $L$ -fuzzy topological space and  $f \in L^X$ .

a) If  $t(f) \not\leq p$  for every  $p \in pr(L)$  then  $f$  is smooth pre-open.

b) If  $c(f) \not\leq p$  for every  $p \in pr(L)$  then  $f$  is smooth pre-closed.

c)  $(\circ f)' = - (f')$  and  $(-f)' = \circ (f')$

**Proof:**

a) If  $t(f) \not\leq p$  for every  $p \in pr(L)$  then  $t(f) > 0$ . From Lemma 2. 6.(c)  $f = \overset{\circ}{f}$

and since  $f \leq \bar{f}$  we have  $\overset{\circ}{f} = f \leq \bar{f}$ . Hence  $f$  smooth pre-open.

b) This is similar to the proof of (a).

c) Let  $f \in L^X$ .

$$\begin{aligned} (\circ f)' &= \left(\bigvee \{g \in L^X \mid g \leq f \text{ and } g \text{ is smooth pre-open}\}\right)' \\ &= \bigwedge \{g' \in L^X \mid f' \leq g' \text{ and } g' \text{ is smooth pre-closed}\} \\ &= - (f') \end{aligned}$$

**Definition 3. 6:** Let  $(X, t_X)$  and  $(Y, t_Y)$  be a smooth  $L$ -fuzzy topological spaces. A function  $\varphi : (X, t_X) \rightarrow (Y, t_Y)$  is called

a) smooth pre-irresolute iff  $\varphi^{-1}(g)$  is smooth pre-open in  $(X, t_X)$  for every smooth pre-open  $g$  in  $(Y, t_Y)$ .

b) smooth weakly pre-irresolute iff  $\varphi^{-1}(g) \leq \circ(\varphi^{-1}(-g))$  for every smooth pre-open set  $g$  in  $(Y, t_Y)$ .

**Definition 3. 7:** Let  $(X, t_X)$  be a smooth  $L$ -fuzzy topological space and  $g \in L^X$ .

$g$  is called smooth strong compact iff for every  $p \in pr(L)$  and every collection  $(f_i)_{i \in J}$  of smooth pre-open  $L$ -fuzzy sets with  $\left(\bigvee_{i \in J} f_i\right)(x) \not\leq p$  for all  $x \in X$  such that  $g(x) \geq p'$ , there exists a finite subset  $F$  of  $J$  with  $\left(\bigvee_{i \in F} f_i\right)(x) \not\leq p$  for all  $x \in X$  such that  $g(x) \geq p'$ .

If  $g$  is the whole space  $X$ , we say that the smooth  $L$ -fuzzy space  $(X, t)$  is smooth strong compact.

In the crisp case of  $t$ ; means  $t : L^X \rightarrow \{0, 1\}$ , smooth strong compactness coincides with the strong compactness in  $L$ -fuzzy spaces introduced by S. R. T. Kudri [9].

**Definition 3. 8:** Let  $(X, t)$  be a smooth  $L$ -fuzzy topological space and  $g \in L^X$ .

$g$  is called smooth  $P$ -closed iff for every  $p \in pr(L)$  and every collection  $(f_i)_{i \in J}$  of smooth pre-open  $L$ -fuzzy sets with  $\left(\bigvee_{i \in J} f_i\right)(x) \not\leq p$  for all  $x \in X$  such that  $g(x) \geq p'$ , there exists a finite subset  $F$  of  $J$  with  $\left(\bigvee_{i \in F} f_i\right)(x) \not\leq p$  for all  $x \in X$  such that  $g(x) \geq p'$ .

If  $g$  is the whole space  $X$ , we say that the smooth  $L$ -fuzzy space  $(X, t)$  is smooth  $P$ -closed.

In the crisp case of  $t$ ; means  $t : L^X \rightarrow \{0, 1\}$ , smooth  $P$ -closedness coincides with the  $P$ -closedness in  $L$ -fuzzy spaces introduced by H. Aygün [3].

#### 4. Other Characterizations

**Theorem 4. 1:** Let  $(X, t)$  be a smooth  $L$ -fuzzy topological space and  $g \in L^X$ .  $g$  is smooth strong compact iff for every  $\alpha \in M(L)$  and every collection  $(f_i)_{i \in J}$  of smooth pre-closed  $L$ -fuzzy sets with  $\left(\bigwedge_{i \in J} f_i\right)(x) \not\geq \alpha$  for all  $x \in X$  such that  $g(x) \geq \alpha$ , there exists a finite subset  $F$  of  $J$  with  $\left(\bigwedge_{i \in F} f_i\right)(x) \not\geq \alpha$  for all such that  $x \in X$  such that  $g(x) \geq \alpha$ .

**Proof:** This immediately follows from Definition 3. 7.

**Theorem 4. 2:** Let  $(X, t)$  be a smooth  $L$ -fuzzy topological space and  $g \in L^X$ .  $g$  is smooth strong compact iff for every  $p \in pr(L)$  and every collection  $(f_i)_{i \in J}$  of smooth pre-closed  $L$ -fuzzy sets with  $\left(\bigvee_{i \in J} f_i \vee g'\right)(x) \not\leq p$  for all  $x \in X$  there exists a finite subset  $F$  of  $J$  with  $\left(\bigvee_{i \in F} f_i \vee g'\right)(x) \not\leq p$  for all such that

$x \in X$ .

**Proof:** Necessity: Let  $p \in pr(L)$  and let  $(f_i)_{i \in J}$  be a collection of smooth pre-open  $L$ -fuzzy sets with  $\left(\bigvee_{i \in J} f_i \vee g'\right)(x) \not\leq p$  for all  $x \in X$ . Then  $\left(\bigvee_{i \in J} f_i\right)(x) \not\leq p$  for all  $x \in X$  with  $g(x) \geq p'$ .

Since  $g$  is smooth strong compact there exists a finite subset  $F$  of  $J$  with

$$\left(\bigvee_{i \in F} f_i\right)(x) \not\leq p \text{ for all } x \in X \text{ such that } g(x) \geq p'.$$

Take an arbitrary  $x \in X$ .

If  $g'(x) \leq p$  then we have  $g'(x) \vee \left(\bigvee_{i \in F} f_i\right)(x) = \left(\bigvee_{i \in F} f_i \vee g'\right)(x) \not\leq p$ , because  $\left(\bigvee_{i \in F} f_i\right)(x) \not\leq p$ .

If  $g'(x) \leq p$  then we have  $g'(x) \vee \left(\bigvee_{i \in F} f_i\right)(x) = \left(\bigvee_{i \in F} f_i \vee g'\right)(x) \not\leq p$ .

Thus we have  $\left(\bigvee_{i \in F} f_i \vee g'\right)(x) \not\leq p$  for all  $x \in X$ .

Sufficiency: Let  $p \in pr(L)$  and let  $(f_i)_{i \in J} \subseteq L^X$  be a collection of smooth pre-open  $L$ -fuzzy sets with  $\left(\bigvee_{i \in J} f_i\right)(x) \not\leq p$  for all  $x \in X$  with  $g(x) \geq p'$ .

Hence  $\left(\bigvee_{i \in F} f_i \vee g'\right)(x) \not\leq p$  for all  $x \in X$ .

From the hypothesis there exists a finite subset  $F$  of  $J$  such that

$$\left(\bigvee_{i \in F} f_i \vee g'\right)(x) \not\leq p \text{ for all } x \in X.$$

Then  $\left(\bigvee_{i \in F} f_i\right)(x) \not\leq p$  for all  $x \in X$  with  $g(x) \geq p'$ .

Hence  $g$  is smooth strong compact.

**Theorem 4. 3:** Let  $(X, t)$  be a smooth  $L$ -fuzzy topological space and  $g \in L^X$ .  $g$  is smooth  $P$ -closed iff for every  $\alpha \in M(L)$  and every collection  $(f_i)_{i \in J}$  of smooth pre-closed  $L$ -fuzzy sets with  $\left(\bigwedge_{i \in J} f_i\right)(x) \not\geq \alpha$  for all  $x \in X$  such that  $g(x) \geq \alpha$ , there exists a finite subset  $F$  of  $J$  with  $\left(\bigwedge_{i \in F} f_i\right)(x) \not\geq \alpha$  for all such that  $x \in X$  such that  $g(x) \geq \alpha$ .

**Proof:** This immediately follows from Definition 3. 8. and Lemma 3. 4.(c).

**Theorem 4. 4:** Let  $(X, t)$  be a smooth  $L$ -fuzzy topological space and  $g \in L^X$ .  $g$  is smooth  $P$ -closed iff for every  $p \in pr(L)$  and every collection  $(f_i)_{i \in J}$  of smooth pre-closed  $L$ -fuzzy sets with  $\left(\bigvee_{i \in J} f_i \vee g'\right)(x) \not\leq p$  for all  $x \in X$  there

exists a finite subset  $F$  of  $J$  with  $\left(\bigvee_{i \in F} f_i \vee g'\right)(x) \not\leq p$  for all  $x \in X$ .

**Proof:** This is similar to the proof of Theorem 4.2.

### 5. Some Properties

**Proposition 5. 1:** Let  $(X, t)$  be a smooth  $L$ -fuzzy topological space and  $g, h \in L^X$ . If  $g$  and  $h$  are smooth strong compact then  $g \vee h$  is smooth strong compact as well.

**Proof:** Let  $g$  and  $h$  be smooth strong compact  $L$ -fuzzy sets. Let  $p \in pr(L)$  and let  $(f_i)_{i \in J}$  be a family of pre-open sets with  $\left(\bigvee_{i \in J} f_i\right)(x) \not\leq p$  for all  $x \in X$  such that  $(g \vee h)(x) \geq p'$ .

If  $(g \vee h)(x) \geq p'$  then  $g(x) \geq p'$  or  $h(x) \geq p'$  because  $p \in pr(L)$  and we always have that if  $g(x) \geq p'$  or  $h(x) \geq p'$  then  $(g \vee h)(x) \geq p'$ .

So from the smooth strong compactness of  $g$  and  $h$ , there are finite subsets  $F$  and  $E$  of  $J$  with  $\left(\bigvee_{i \in F} f_i\right)(x) \not\leq p$  for all  $x \in X$  such that  $g(x) \geq p'$  and

$\left(\bigvee_{i \in E} f_i\right)(x) \not\leq p$  for all  $x \in X$  such that  $h(x) \geq p'$ . Then,  $\left(\bigvee_{i \in F \cup E} f_i\right)(x) \not\leq p$

for all  $x \in X$  such that  $g(x) \geq p'$  or  $h(x) \geq p'$ . Thus  $\left(\bigvee_{i \in F \cup E} f_i\right)(x) \not\leq p$  for all  $x \in X$  with  $(g \vee h)(x) \geq p'$ . Hence  $g \vee h$  is smooth strong-compact.

**Proposition 5. 2:** Let  $(X, t)$  be a smooth  $L$ -fuzzy topological space and  $g, h \in L^X$ . If  $g$  is a smooth strong compact and  $h$  is a smooth pre-closed  $L$ -fuzzy set then  $g \wedge h$  is smooth strong compact.

**Proof:** Let  $g$  be a smooth strong compact and  $h$  be a smooth pre-closed  $L$ -fuzzy set.

Let  $p \in pr(L)$  and  $(f_i)_{i \in J} \subseteq L^X$  be a family of smooth pre-open  $L$ -fuzzy sets with  $\left(\bigvee_{i \in J} f_i\right)(x) \not\leq p$  for all  $x \in X$  such that  $(g \wedge h)(x) \geq p'$ .

Since  $h$  is smooth pre-closed  $L$ -fuzzy set,  $h'$  is smooth pre-open  $L$ -fuzzy set.

Thus  $\beta = (f_i)_{i \in J} \cup \{h'\}$  is a family of smooth pre-open  $L$ -fuzzy sets in  $(X, t)$

with  $\left(\bigvee_{k \in \beta} k\right)(x) \not\leq p$  for all  $x \in X$  with  $g(x) \geq p'$ .

In fact, for each  $x \in X$  with  $g(x) \geq p'$ , if  $h(x) \geq p'$  then  $(g \wedge h)(x) \geq p'$

which implies that  $\left(\bigvee_{i \in J} f_i\right)(x) \not\leq p$ , thus  $\left(\bigvee_{k \in \beta} k\right)(x) \not\leq p$  and if  $h(x) \not\geq p'$

then  $h'(x) \not\leq p$  which implies that  $\left(\bigvee_{k \in \beta} k\right)(x) \not\leq p$ .

From the smooth strong compactness of  $g$  there is a finite subfamily  $\beta_0$  of  $\beta$ ,

say  $\beta_0 = \{f_1, f_2, \dots, f_n, h'\}$  with  $\left(\bigvee_{k \in \beta_0} k\right)(x) \not\leq p$  for all  $x \in X$  with  $g(x) \geq p'$ .

Then  $\left(\bigvee_{i=1}^n f_i\right)(x) \not\leq p$  for all  $(g \wedge h)(x) \geq p'$ . In fact, if  $(g \wedge h)(x) \geq p'$  then

$g(x) \geq p'$ , hence  $\left(\bigvee_{k \in \beta_0} k\right)(x) \not\leq p$ . Therefore, there is a  $k \in \beta_0$  such that

$k(x) \not\leq p$ . Moreover,  $h(x) \geq p'$ , that is,  $h'(x) \leq p$ , so  $\left(\bigvee_{i=1}^n f_i\right)(x) \not\leq p$  for all  $x \in X$  with  $(g \wedge h)(x) \geq p'$ . Hence  $g \wedge h$  is smooth strong compact.

**Corollary 5. 3:** Let  $(X, t)$  be a smooth  $L$ -fuzzy topological space. If  $g$  is a smooth strong compact  $L$ -fuzzy set, then each smooth pre-closed set contained in  $g$  is smooth strong compact as well.

**Proposition 5. 4:** Let  $(X, t)$  be a smooth  $L$ -fuzzy topological space and  $g, h \in L^X$ . If  $g$  and  $h$  are smooth P-closed  $L$ -fuzzy sets then  $g \wedge h$  is smooth P-closed as well.

**Proof:** This is similar to the proof of Proposition 5. 1.

**Proposition 5. 5:** Let  $(X, t)$  be a smooth  $L$ -fuzzy topological space and  $g, h \in L^X$ . If  $g$  is a smooth P-closed  $L$ -fuzzy set and  $h$  is a smooth pre-closed  $L$ -fuzzy set then  $g \wedge h$  is smooth P-closed.

**Proof:** Let  $g$  be a smooth P-closed  $L$ -fuzzy set and  $h$  be a smooth pre-closed  $L$ -fuzzy set.

Let  $p \in pr(L)$  and let  $(f_i)_{i \in J}$  be a family of smooth pre-open  $L$ -fuzzy sets with  $\left(\bigvee_{i \in J} f_i\right)(x) \not\leq p$  for all  $x \in X$  with  $(g \wedge h)(x) \geq p'$ .

If  $h$  is smooth pre-closed  $L$ -fuzzy set then  $h'$  is smooth pre-open.

Thus,  $\beta = (f_i)_{i \in J} \cup \{h'\}$  is a family of smooth pre-open  $L$ -fuzzy sets in

$(X, t)$  with  $\left(\bigvee_{k \in \beta} k\right)(x) \not\leq p$  for all  $x \in X$  with  $g(x) \geq p'$ . In fact, for each

$x \in X$  with  $g(x) \geq p'$ , if  $h(x) \geq p'$  then  $(g \wedge h)(x) \geq p'$  which implies that

$\left(\bigvee_{i \in J} f_i\right)(x) \not\leq p$ , thus  $\left(\bigvee_{k \in \beta} k\right)(x) \not\leq p$ . If  $h(x) \not\geq p'$  then  $h'(x) \not\leq p$  which

implies that  $\left(\bigvee_{k \in \beta} k\right)(x) \not\leq p$ . From the smooth P-closedness of  $g$  there is

a finite subfamily  $\beta_0$  of  $\beta$ , say  $\beta_0 = \{f_1, f_2, \dots, f_n, h'\}$  with  $\left(\bigvee_{k \in \beta_0} k\right)(x) \not\leq p$

for all  $x \in X$  with  $g(x) \geq p'$ . Then  $\left(\bigvee_{i=1}^n f_i\right)(x) \not\leq p$  for all  $x \in X$  with

$(g \wedge h)(x) \geq p'$ .

In fact, if  $(g \wedge h)(x) \geq p'$  then  $g(x) \geq p'$ , hence  $\left(\bigvee_{k \in \beta_0} \_k\right)(x) \not\leq p$ . Therefore, there is a  $k \in \beta_0$  such that  $\_k(x) \not\leq p$ . Moreover, if  $h(x) \geq p'$ , that is,  $h'(x) \leq p$ . Since  $h'$  is also smooth pre-closed, we have  $\_(h') = h'$  then  $(\_(h'))(x) = h'(x) \leq p$ . So  $\left(\bigvee_{i=1}^n \_f_i\right)(x) \not\leq p$  for all  $x \in X$  with  $(g \wedge h)(x) \geq p'$ . Hence  $g \wedge h$  is smooth P-closed.

**Corollary 5. 6:** Let  $(X, t)$  be a smooth  $L$ -fuzzy topological space. If  $g$  is a smooth P-closed  $L$ -fuzzy set, then each pre-closed  $L$ -fuzzy set contained in  $g$  is P-closed as well.

**Theorem 5. 7:** Let  $(X, t_X)$  and  $(Y, t_Y)$  be smooth  $L$ -fuzzy topological spaces and let  $\varphi : (X, t_X) \rightarrow (Y, t_Y)$  be a smooth pre-irresolute mapping with  $\varphi^{-1}(y)$  is finite for every  $y \in Y$ . If  $g$  is smooth strong compact  $L$ -fuzzy set in  $(X, t_X)$  then  $\varphi(g)$  is smooth strong compact  $L$ -fuzzy set in  $(Y, t_Y)$ .

**Proof:** Let  $p \in pr(L)$  and let  $(f_i)_{i \in J}$  be a family of smooth pre-open  $L$ -fuzzy sets in  $(Y, t_Y)$  with  $\left(\bigvee_{i \in J} f_i\right)(y) \not\leq p$  for all  $y \in Y$  such that  $\varphi(g)(y) \geq p'$ .

Since  $\varphi$  is smooth pre-irresolute mapping,  $(\varphi^{-1}(f_i))_{i \in J}$  is a family of smooth pre-open  $L$ -fuzzy sets in  $X$  with  $\left(\bigvee_{i \in J} \varphi^{-1}(f_i)\right)(x) \not\leq p$  for every  $x \in X$  such that  $g(x) \geq p'$ . Because, if  $g(x) \geq p'$  then  $\varphi(g)(\varphi(x)) \geq p'$ . So  $\left(\bigvee_{i \in J} \varphi^{-1}(f_i)\right)(x) = \left(\bigvee_{i \in J} f_i\right)(\varphi(x)) \not\leq p$ .

From the smooth strong compactness of  $g$  in  $(X, t_X)$ , there exists a finite subset  $F$  of  $J$  such that  $\left(\bigvee_{i \in F} \varphi^{-1}(f_i)\right)(x) \not\leq p$  for every  $x \in X$  with  $g(x) \geq p'$ .

We also have that  $\left(\bigvee_{i \in F} f_i\right)(y) \not\leq p$  for all  $y \in Y$  with  $\varphi(g)(y) \geq p'$ .

In fact if  $\varphi(g)(y) \geq p'$  then  $\bigvee_{x \in \varphi^{-1}(y)} g(x) \geq p'$  which implies that there is  $x \in X$

with  $g(x) \geq p'$  and  $\varphi(x) = y$ . So, we have

$$\left(\bigvee_{i \in F} f_i\right)(y) = \left(\bigvee_{i \in F} f_i\right)(\varphi(x)) = \left(\bigvee_{i \in F} \varphi^{-1}(f_i)\right)(x)$$

Hence,  $\left(\bigvee_{i \in F} f_i\right)(y) \not\leq p$  for all  $y \in Y$  such that  $\varphi(g)(y) \geq p'$

Hence  $\varphi(g)$  is smooth strong compact in  $(Y, t_Y)$ .

**Corollary 5. 8:** Let  $(X, t_X)$  and  $(Y, t_Y)$  be smooth  $L$ -fuzzy topological spaces and let  $\varphi : (X, t_X) \rightarrow (Y, t_Y)$  be a smooth pre-irresolute surjection. If  $(X, t_X)$  is smooth strong compact space then  $(Y, t_Y)$  is smooth strong compact as well.

**Proof:** This immediately follows from Theorem 5. 7.

**Theorem 5. 9:** Let  $(X, t_X)$  and  $(Y, t_Y)$  be smooth  $L$ -fuzzy topological

spaces and let  $\varphi : (X, t_X) \rightarrow (Y, t_Y)$  be a smooth pre-irresolute mapping with  $\varphi^{-1}(y)$  is finite for every  $y \in Y$ . If  $g$  is smooth P-closed  $L$ -Fuzzy set in  $(X, t_X)$  then  $\varphi(g)$  is smooth P-closed  $L$ -fuzzy set in  $(Y, t_Y)$ .

**Proof:** Let  $p \in pr(L)$  and let  $(f_i)_{i \in J}$  be a family of smooth pre-open  $L$ -fuzzy sets in  $(Y, t_Y)$  with  $\left(\bigvee_{i \in J} f_i\right)(y) \not\leq p$  for all  $y \in Y$  such that  $\varphi(g)(y) \geq p'$ .

Since  $\varphi$  is smooth pre-irresolute mapping,  $(\varphi^{-1}(f_i))_{i \in J}$  is a family of smooth pre-open sets of  $X$  with  $\left(\bigvee_{i \in J} \varphi^{-1}(f_i)\right)(x) \not\leq p$  for every  $x \in X$  such that

$g(x) \geq p'$ , because if  $g(x) \geq p'$  then  $\varphi(g)(\varphi(x)) \geq p'$ . So  $\left(\bigvee_{i \in J} f_i\right)(\varphi(x)) = \left(\bigvee_{i \in J} \varphi^{-1}(f_i)\right)(x) \not\leq p$ . From the smooth P-closedness of  $g$  in  $(X, t_X)$ , there

exists a finite subset  $F$  of  $J$  with  $\left(\bigvee_{i \in F} \varphi^{-1}(f_i)\right)(x) \not\leq p$  for all  $x \in X$  such

that  $g(x) \geq p'$ . Then  $\left(\bigvee_{i \in F} -f_i\right)(y) \not\leq p$  for all  $y \in Y$  such that  $\varphi(g)(y) \geq p'$ . In

fact  $\varphi(g)(y) \geq p'$  then  $\bigvee_{x \in \varphi^{-1}(y)} g(x) \geq p'$  which implies that there is  $x \in X$  with

$g(x) \geq p'$  and  $\varphi(x) = y$ . Since  $-f_i$  is smooth pre-closed  $L$ -fuzzy set for each  $i \in J$  and  $\varphi$  is smooth pre-irresolute mapping,  $\varphi^{-1}(-f_i)$  is smooth pre-closed and so  $-(\varphi^{-1}(-f_i)) = \varphi^{-1}(-f_i)$  for each  $i \in J$ . Therefore, we have

$$\left(\bigvee_{i \in F} -f_i\right)(y) = \left(\bigvee_{i \in F} -f_i\right)(\varphi(x)) = \left(\bigvee_{i \in F} \varphi^{-1}(-f_i)\right)(x) = \bigvee_{i \in F} (\varphi^{-1}(-f_i))(x) \geq \bigvee_{i \in F} (\varphi^{-1}(f_i))(x)$$

Hence,  $\left(\bigvee_{i \in F} -f_i\right)(y) \not\leq p$ , for all  $y \in Y$  such that  $\varphi(g)(y) \geq p'$ . Consequently

$\varphi(g)$  is smooth P-closed in  $(Y, t_Y)$ .

**Corollary 5. 10:** Let  $(X, t_X)$  and  $(Y, t_Y)$  be smooth  $L$ -fuzzy topological spaces and let  $\varphi : (X, t_X) \rightarrow (Y, t_Y)$  be a smooth pre-irresolute surjection. If  $(X, t_X)$  is smooth P-closed space then  $(Y, t_Y)$  is smooth P-closed space as well.

**Proof:** This immediately follows from Theorem 5. 9.

**Theorem 5. 11:** Let  $(X, t_X)$  and  $(Y, t_Y)$  be smooth  $L$ -fuzzy topological spaces and let  $\varphi : (X, t_X) \rightarrow (Y, t_Y)$  be a smooth weakly pre-irresolute mapping with  $\varphi^{-1}(y)$  is finite for every  $y \in Y$ . If  $g$  is smooth strong compact  $L$ -fuzzy set in  $(X, t_X)$  then  $\varphi(g)$  is smooth P-closed  $L$ -fuzzy set in  $(Y, t_Y)$ .

**Proof:** Let  $p \in pr(L)$  and let  $(f_i)_{i \in J}$  be a family of smooth pre-open  $L$ -fuzzy sets in  $(Y, t_Y)$  such that  $\left(\bigvee_{i \in J} f_i\right)(y) \not\leq p$  for all  $y \in Y$  with  $\varphi(g)(y) \geq p'$ .

Since  $\varphi$  is smooth weakly pre-irresolute mapping,  $(\varphi^{-1}(f_i)) \leq \circ(\varphi^{-1}(-f_i))$  for each  $i \in J$ . Then  $\left(\bigvee_{i \in J} \varphi^{-1}(f_i)\right)(x) \not\leq p$  for all  $x \in X$  such that  $g(x) \geq p'$ , because if  $g(x) \geq p'$  then  $\varphi(g)(\varphi(x)) \geq p'$ . So  $\left(\bigvee_{i \in J} f_i\right)(\varphi(x)) = \left(\bigvee_{i \in J} \varphi^{-1}(f_i)\right)(x) \not\leq p$  and so we have that  $\left(\bigvee_{i \in J} \circ(\varphi^{-1}(-f_i))\right)(x) \geq \left(\bigvee_{i \in J} \varphi^{-1}(f_i)\right)(x) \not\leq p$  for all  $x \in X$  such that  $g(x) \geq p'$  and then  $\left(\bigvee_{i \in J} \circ(\varphi^{-1}(-f_i))\right)(x) \not\leq p$  for all  $x \in X$  such that  $g(x) \geq p'$ . Since  $(\circ(\varphi^{-1}(-f_i)))_{i \in J} \subseteq L^X$  pre-open L-fuzzy sets in  $(X, t_X)$  and  $g$  is smooth strong compact in  $(X, t_X)$ , there is a finite subset  $F$  of  $J$  such that  $\left(\bigvee_{i \in F} \circ(\varphi^{-1}(-f_i))\right)(x) \not\leq p$  for all  $x \in X$  for all  $g(x) \geq p'$ .

We also have that  $\left(\bigvee_{i \in F} -f_i\right)(y) \not\leq p$  for all  $y \in Y$  such that  $\varphi(g)(y) \geq p'$ . In fact if  $\varphi(g)(y) \geq p'$  then  $\bigvee_{x \in \varphi^{-1}(y)} g(x) \geq p'$  which implies that there is  $x \in X$

with  $g(x) \geq p'$  and  $\varphi(x) = y$ . So, we have  $\left(\bigvee_{i \in F} -f_i\right)(y) = \left(\bigvee_{i \in F} -f_i\right)(\varphi(x)) = \left(\bigvee_{i \in F} \varphi^{-1}(-f_i)\right)(x) \geq \left(\bigvee_{i \in F} \circ(\varphi^{-1}(-f_i))\right)(x)$ .

Hence,  $\left(\bigvee_{i \in F} -f_i\right)(y) \not\leq p$  for all  $y \in Y$  with  $\varphi(g)(y) \geq p'$ . Consequently  $\varphi(g)$  is smooth P-closed in  $(Y, t_Y)$ .

**Corollary 5. 12:** Let  $(X, t_X)$  and  $(Y, t_Y)$  be smooth  $L$ -fuzzy topological spaces and let  $\varphi : (X, t_X) \rightarrow (Y, t_Y)$  be a smooth pre-irresolute surjection. If  $(X, t_X)$  is smooth strong compact space then  $(Y, t_Y)$  is smooth P-closed space as well.

**Proof:** This immediately follows from Theorem 5. 11.

**Proposition 5. 13:** Let  $(X, t)$  be a smooth  $L$ -fuzzy topological space and  $g \in L^X$ . Then the following implications hold:

$g$  is smooth strong compact  $\stackrel{(i)}{\Rightarrow} g$  is smooth P-closed  $\stackrel{(ii)}{\Rightarrow} g$  is smooth almost compact.

**Proof:** (i) Let  $p \in pr(L)$  and let  $(f_i)_{i \in J}$  be a family of smooth pre-open  $L$ -fuzzy sets with  $\left(\bigvee_{i \in J} f_i\right)(x) \not\leq p$  for all  $x \in X$  such that  $g(x) \geq p'$ . From the smooth strong compactness of  $g$ , there is a finite subset  $F$  of  $J$  with  $\left(\bigvee_{i \in F} f_i\right)(x) \not\leq p$  for all  $x \in X$  such that  $g(x) \geq p'$ . Since  $f_i \leq -f_i$  for each

$i \in J$  then  $\left(\bigvee_{i \in F} f_i\right)(x) \not\leq p$  for all  $x \in X$  such that  $g(x) \geq p'$ . So  $g$  is smooth P-closed.

(ii) Let  $p \in pr(L)$  and let  $(f_i)_{i \in J}$  be a collection of  $L$ -fuzzy sets with  $t(f_i) \not\leq p$  for each  $i \in J$  and  $\left(\bigvee_{i \in J} f_i\right)(x) \not\leq p$  for all  $x \in X$  such that  $g(x) \geq p'$ .

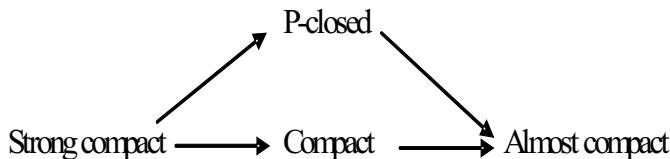
From Lemma 3. 4.(a),  $(f_i)_{i \in J}$  is a family of smooth pre-open  $L$ -fuzzy sets. By the hypothesis, there is a finite subset  $F$  of  $J$  with  $\left(\bigvee_{i \in F} f_i\right)(x) \not\leq p$  for all  $x \in X$  such that  $g(x) \geq p'$ . Since  $f_i \leq \bar{f}_i$  for each  $i \in J$  then we have  $\left(\bigvee_{i \in F} \bar{f}_i\right)(x) \not\leq p$  for all  $x \in X$  such that  $g(x) \geq p'$ . So  $g$  is smooth almost compact.

**Proposition 5. 14:** Let  $(X, t)$  be a smooth  $L$ -fuzzy topological space and  $g \in L^X$ . If  $g$  is smooth strong compact then  $g$  is smooth compact as well.

**Proof:** Let  $p \in pr(L)$  and let  $(f_i)_{i \in J}$  be a collection of  $L$ -fuzzy sets with  $t(f_i) \not\leq p$  for each  $i \in J$  and  $\left(\bigvee_{i \in J} f_i\right)(x) \not\leq p$  for all  $x \in X$  such that  $g(x) \geq p'$ .

From Lemma 3. 4.(a),  $(f_i)_{i \in J}$  is a family of smooth pre-open  $L$ -fuzzy sets. By the hypothesis, there is a finite subset  $F$  of  $J$  with  $\left(\bigvee_{i \in F} f_i\right)(x) \not\leq p$  for all  $x \in X$  such that  $g(x) \geq p'$ . So  $g$  is smooth compact.

From the above propositions we have the following diagram:



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