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Composition Followed and Proceeded by Differentiation between α-Bloch Spaces

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Abstract. In this paper, we consider linear operators $C_\varphi D$ and DC_φ acting between α-Bloch spaces and little α-Bloch spaces, where C_φ and D are composition and differentiation operators respectively. In fact we characterise those holomorphic self-maps of $\mathbb{D}$, that induce bounded and compact $C_\varphi D$ and DC_φ between Bloch-type spaces.

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1. Introduction

Let $\mathbb{D}$ be the open unit disk in the complex plane $\mathbb{C}$. Denote by $H(\mathbb{D})$, the space of holomorphic functions on $\mathbb{D}$. For a holomorphic map φ of $\mathbb{D}$ such that $\varphi(\mathbb{D}) \subset \mathbb{D}$, we can define linear operators

$$C_\varphi D f = (f' \circ \varphi) \quad \text{and} \quad DC_\varphi f = (f \circ \varphi)', \quad (f \in H(\mathbb{D})),$$

where C_φ and D are composition and differentiation operators respectively. For general background on composition operators, we refer [**2**] and [**7**] and references therein. Recently, several authors have studied $C_\varphi D$ and DC_φ on some spaces of analytic functions. For more information on these operators, one can refer to [**4**] and [**8**]. The main theme of this paper is to study these operators between α-Bloch spaces and the little α-Bloch spaces. The plan of the rest of the paper is as follows. In the next section we introduce α-Bloch spaces and the little α-Bloch spaces. Section 3 is devoted to characterise boundedness and compactness of $C_\varphi D$ and DC_φ between α-Bloch spaces whereas boundedness and compactness of $C_\varphi D$ and DC_φ between little α-Bloch spaces is tackled in section 4.

2 Preliminaries

In this section we will concentrate on those aspects of the α-Bloch spaces and little α-Bloch spaces that will be needed throughout this paper.

Let $0 < \alpha < \infty$. A function f holomorphic in $\mathbb{D}$ is said to belong to the α-Bloch space $\mathcal{B}^\alpha$ if

$$\sup_{z\in\mathbb{D}}(1-|z|^2)^\alpha|f'(z)| < \infty$$

and to the little α-Bloch space $\mathcal{B}_0^\alpha$ if

$$\lim_{|z|\to 1}(1-|z|^2)^\alpha|f'(z)| = 0.$$

It is well known that $\mathcal{B}^\alpha$ is a Banach space under the norm

$$||f||_{\mathcal{B}^\alpha} = |f(0)| + \sup_{z\in\mathbb{D}}(1-|z|^2)^\alpha|f'(z)| = |f(0)| + s(f)$$

and $\mathcal{B}_0^\alpha$ is a closed subspace of $\mathcal{B}^\alpha$. Note that $\mathcal{B}^1 = \mathcal{B}$ and $\mathcal{B}_0^1 = \mathcal{B}_0$ are the usual Bloch space and the usual little Bloch space respectively.

Two quantities a and b are said to be comparable, denoted by $a \approx b$, if there exist two positive constants C_1 and C_2 such that $C_1 a \le b \le C_2 a$.

Next result is an alternate characterisation of the α-Bloch spaces and little α-Bloch spaces (see [**1**]).

Theorem 2.1. [**1**] *Let $1 \le \alpha < \infty$. Then for $f \in H(\mathbb{D})$ following are equivalent:*

$$s(f) \approx |f'(0)| + \sup_{z\in\mathbb{D}}(1-|z|^2)^{\alpha+1}|f''(z)|.$$

Further $f \in \mathcal{B}_0^\alpha$ $(\alpha \ge 1)$ if and only if

$$\lim_{|z|\to 1}(1-|z|^2)^{\alpha+1}|f''(z)| = 0.$$

To be precise, the above theorem is shown in [**1**] for the case $\alpha = 1$, however the same proof given there works for $\alpha > 1$.

The following Lemma describes the compact subsets of $\mathcal{B}_0^\alpha$.

Lemma 2.2 [**6**] *Let $K \subset \mathcal{B}_0^\alpha$. Then K is compact if and only if K is closed, bounded and satisfies*

$$\lim_{|z|\to 1}\sup_{z\in K}(1-|z|^2)^\alpha|f'(z)| = 0.$$

For general background on Bloch spaces and little Bloch spaces, one may consult [**1**] [**3**] [**9**] [**10**] and references therein. Madigan and Matheson [**5**] characterised the boundedness and compactness of composition operators on $\mathcal{B}$ and $\mathcal{B}_0$.

3 Boundedness and Compactness of $C_\varphi D$ and DC_φ between α-Bloch spaces

Theorem 3.1. *Let $\alpha \geq 1$ and $\beta > 0$ be two real numbers and φ be a holomorphic self-map of $\mathbb{D}$. Then $C_\varphi D$ maps $\mathcal{B}^\alpha$ boundedly into $\mathcal{B}^\beta$ if and only if*

$$\sup_{z\in\mathbb{D}} \frac{(1-|z|^2)^\beta}{(1-|\varphi(z)|^2)^{\alpha+1}}|\varphi'(z)| < \infty. \tag{3.1}$$

Proof. First suppose that (3.1) holds. Then for arbitrary $z \in \mathbb{D}$, we have

$$\begin{aligned}(1-|z|^2)^\beta|(C_\varphi Df)'(z)| &= (1-|z|^2)^\beta|f''(\varphi(z))||\varphi'(z)| \\ &\leq C_\alpha \frac{(1-|z|^2)^\beta}{(1-|\varphi(z)|^2)^{\alpha+1}}|\varphi'(z)| \parallel f \parallel_{\mathcal{B}^\alpha},\end{aligned}$$

and consequently, $C_\varphi D$ maps $\mathcal{B}^\alpha$ boundedly into $\mathcal{B}^\beta$. Conversely, suppose that $C_\varphi D$ maps $\mathcal{B}^\alpha$ boundedly into $\mathcal{B}^\beta$. Fix a point $z_0 \in \mathbb{D}$ and let $w = \varphi(z_0)$. Consider the function f_w given by $f_w(z) = (1-|w|^2)/2^{\alpha+1}(1-\overline{w}z)^\alpha$. Then $f_w \in \mathcal{B}^\alpha$ and $\parallel f_w \parallel_{\mathcal{B}^\alpha} \leq 1$. Moreover

$$f_w'(z) = \frac{1-|w|^2}{2^{\alpha+1}(1-\overline{w}z)^{\alpha+1}}(\alpha\overline{w}) \qquad \text{and} \qquad f_w''(z) = \frac{\alpha(\alpha+1)(\overline{w})^2(1-|w|^2)}{2^{\alpha+1}(1-\overline{w}z)^{\alpha+2}}.$$

Since $C_\varphi D$ maps $\mathcal{B}^\alpha$ boundedly into $\mathcal{B}^\beta$, so there exists a constant $C > 0$ such that $\parallel C_\varphi D f_w \parallel_{\mathcal{B}^\beta} \leq C \parallel f_w \parallel_{\mathcal{B}^\alpha} \leq C$, for all $w \in \mathbb{D}$. Hence for all $z \in \mathbb{D}$, we have $(1-|z|^2)^\beta |f_w''(\varphi(z))||\varphi'(z)| \leq C$. In particular, when $z = z_0$, we have

$$(1-|z_0|^2)^\beta \frac{\alpha(\alpha+1)|\varphi(z_0)|^2(1-|\varphi(z_0)|^2)}{2^{\alpha+1}(1-|\varphi(z_0)|^2)^{\alpha+2}}|\varphi'(z_0)| \leq C.$$

Thus

$$\frac{(1-|z_0|^2)^\beta}{2^{\alpha+1}(1-|\varphi(z_0)|^2)^{\alpha+1}}|\varphi(z_0)|^2|\varphi'(z_0)| \leq \frac{C}{\alpha(\alpha+1)}. \tag{3.2}$$

Let $K = \{z_0 \in \mathbb{D} : |\varphi(z_0)| \leq r\}$. With K as defined above the equation (3.2) gives

$$\sup_{z\notin K}\Big[\frac{(1-|z_0|^2)^\beta}{(1-|\varphi(z_0)|^2)^{\alpha+1}}|\varphi'(z_0)|\Big] \leq \frac{2^{\alpha+1}C}{\alpha(\alpha+1)r^2},$$

whence $\sup_{z\notin K}\{\Big[\cdots\Big] : z \notin K\}$ is bounded, and $\sup_{z\in K}\{\Big[\cdots\Big] : z \notin K\}$ is certainly bounded, whence (3.1). This completes the proof.

Theorem 3.2 *Let $\alpha \geq 1$ and $\beta > 0$ be two real numbers. Let φ be a holomorphic self-map of $\mathbb{D}$ such that $C_\varphi D$ maps $\mathcal{B}^\alpha$ boundedly into $\mathcal{B}^\beta$. Then $C_\varphi D$ maps $\mathcal{B}^\alpha$ compactly into $\mathcal{B}^\beta$ if and only if*

$$\lim_{|\varphi(z)|\to 1} \frac{(1-|z|^2)^\beta}{(1-|\varphi(z)|^2)^{\alpha+1}}|\varphi'(z)| = 0. \tag{3.3}$$

Proof. Let $\{f_n\}$ be a bounded sequence in $\mathcal{B}^\alpha$ that converges to zero uniformly on compact subsets of $\mathbb{D}$. Then we have to show that $\parallel C_\varphi D f_n \parallel_{\mathcal{B}^\beta} \to 0$ as $n \to \infty$. Let $M = \sup_n \parallel f_n \parallel_{\mathcal{B}^\alpha} < \infty$. Given $\epsilon > 0$, there exist an $r \in (0,1)$

such that if $|\varphi(z)| > r$, then $((1-|z|^2)^\beta/(1-|\varphi(z)|^2)^{\alpha+1})|\varphi'(z)| < \epsilon$. Using Theorem 2.1, we have for $|\varphi(z)| > r$,

$$\begin{aligned}(1-|z|^2)^\beta|(C_\varphi Df_n)'(z)| &= (1-|z|^2)^\beta|f_n''(\varphi(z))||\varphi'(z)| \\ &\leq C_\alpha(1-|z|^2)^\beta\frac{\| f_n \|_{\mathcal{B}^\alpha}}{(1-|\varphi(z)|^2)^{\alpha+1}}|\varphi'(z)| \\ &< \epsilon M C_\alpha.\end{aligned}$$

for all n. On the other hand since $f_n'' \to 0$ uniformly on $\{w : |w| \leq r\}$, there exists an n_0 such that if $|\varphi(z)| \leq r$ and $n \geq n_0$, then $|f_n''(\varphi(z))| < \epsilon$. Moreover, by (3.1), we have $A = \sup_{z\in\mathbb{D}}(1-|z|^2)^\beta|\varphi'(z)| < \infty$. Thus

$$(1-|z|^2)^\beta|(C_\varphi Df_n'(z)| \leq (1-|z|^2)^\beta|\varphi'(z)||f_n''(\varphi(z))| < \epsilon A.$$

The above arguments, together with the fact that $C_\varphi Df_n(0) = f_n'(\varphi(0)) \to 0$ as $n \to \infty$, yields that $\| C_\varphi Df_n \|_{\mathcal{B}^\beta} \to 0$ as $n \to \infty$. Hence $C_\varphi D$ maps $\mathcal{B}^\alpha$ compactly into $\mathcal{B}^\beta$.

Conversely, suppose that (3.3) does not hold. Then there exists a positive number λ and a sequence $\{z_m\}$ in $\mathbb{D}$ such that $|\varphi(z_m)| \to 1$ and

$$\frac{(1-|z_m|^2)^\beta}{(1-|\varphi(z_m)|^2)^{\alpha+1}}|\varphi'(z_m)| \geq \lambda,$$

for all m. For each m define $f_m(z) = (1-|\varphi(z_m)|^2)/2^{\alpha+1}(1-\overline{\varphi(z_m)}z)^\alpha$. Then $f_m \in \mathcal{B}^\alpha$ and $\| f_m \|_{\mathcal{B}^\alpha} \leq 1$. Since $C_\varphi D$ maps $\mathcal{B}^\alpha$ compactly into $\mathcal{B}^\beta$ and f_n is a norm bounded sequence that converges to zero uniformly on compact subsets of $\mathbb{D}$, it follows that a subsequence of $\{C_\varphi Df_m\}$ tends to zero in $\mathcal{B}^\beta$. On the other hand

$$\begin{aligned}\| C_\varphi Df_m \|_{\mathcal{B}^\beta} &\geq (1-|z_m|^2)^\beta|(C_\varphi Df_m)'(z_m)| \\ &= (1-|z_m|^2)^\beta|f_m''(z_m)||\varphi'(z_m)| \\ &= \frac{\alpha(\alpha+1)|\varphi(z_m)|^2(1-|z_m|^2)^\beta}{(1-|\varphi(z_m)|^2)^{\alpha+1}}|\varphi'(z_m)| \\ &\geq \alpha(\alpha+1)|\varphi(z_m)|^2\lambda,\end{aligned}$$

which is absurd and hence we are done.

Theorem 3.3 *Let $\alpha \geq 1$ and $\beta > 0$ be two real numbers and φ be a holomorphic self-map of $\mathbb{D}$. Then DC_φ maps $\mathcal{B}^\alpha$ boundedly into $\mathcal{B}^\beta$ if and only if*

$$(i)\ \sup_{z\in\mathbb{D}}\frac{(1-|z|^2)^\beta|\varphi'(z)|^2}{(1-|\varphi(z)|^2)^{\alpha+1}} < \infty \quad \textit{and} \quad (ii)\ \sup_{z\in\mathbb{D}}\frac{(1-|z|^2)^\beta|\varphi''(z)|}{(1-|\varphi(z)|^2)^{\alpha}} < \infty.$$

Proof. First suppose that

$$M = \sup_{z\in\mathbb{D}}\frac{(1-|z|^2)^\beta|\varphi'(z)|^2}{(1-|\varphi(z)|^2)^{\alpha+1}} < \infty \quad \text{and} \quad N = \sup_{z\in\mathbb{D}}\frac{(1-|z|^2)^\beta|\varphi''(z)|}{(1-|\varphi(z)|^2)^{\alpha}} < \infty.$$

For arbitrary $z \in \mathbb{D}$, we have

$$\begin{aligned}(1-|z|^2)^\beta|(DC_\varphi f)'(z)| &= (1-|z|^2)^\beta|(f\circ\varphi)''(z)| \\ &= (1-|z|^2)^\beta[|\varphi'(z)|^2|f''(\varphi(z))| + |f'(\varphi(z))||\varphi''(z)|] \\ &\leq \Big(C_\alpha \frac{(1-|z|^2)^\beta|\varphi'(z)|^2}{(1-|\varphi(z)|^2)^{\alpha+1}} + \frac{(1-|z|^2)^\beta|\varphi''(z)|}{(1-|\varphi(z)|^2)^\alpha}\Big) \parallel f \parallel_{\mathcal{B}^\alpha} \\ &\leq (C_\alpha M + N) \parallel f \parallel_{\mathcal{B}^\alpha}\end{aligned}$$

and consequently, $DC_\varphi f$ maps $\mathcal{B}^\alpha$ boundedly into $\mathcal{B}^\beta$.
Conversely, suppose that DC_φ maps $\mathcal{B}^\alpha$ boundedly into $\mathcal{B}^\beta$. Then taking $f(z) = z$ in $\mathcal{B}^\alpha$, we get $\varphi' \in \mathcal{B}^\beta$. Again taking $f(z) = z^2/2$ in $\mathcal{B}^\alpha$, we get $(1-|z|^2)^\beta|(\varphi'(z))^2 + \varphi(z)\varphi''(z)| \leq M$. Since $\varphi' \in \mathcal{B}^\beta$ and $|\varphi(z)| < 1$, we get $\sup_{z\in\mathbb{D}}(1-|z|^2)^\beta|\varphi'(z)|^2 < \infty$. Fix $\lambda \in \mathbb{D}$ and consider the function f_λ defined by

$$f_\lambda(z) = \Big\{\frac{(1-|\varphi(\lambda)|^2)^2}{(1-\overline{\varphi(\lambda)}z)^{\alpha+1}} - \frac{(\alpha+1)(1-|\varphi(\lambda)|^2)}{\alpha(1-\overline{\varphi(\lambda)}z)^\alpha}\Big\} \qquad (z \in \mathbb{D}).$$

Then

$$f'_\lambda(z) = \overline{\varphi(\lambda)}\Big\{\frac{(\alpha+1)(1-|\varphi(\lambda)|^2)^2}{(1-\overline{\varphi(\lambda)}z)^{\alpha+2}} - \frac{(\alpha+1)(1-|\varphi(\lambda)|^2)}{(1-\overline{\varphi(\lambda)}z)^{\alpha+1}}\Big\}.$$

An easy calculation yields that $(1-|z|^2)^\alpha|f'_\lambda(z)| \leq 3(\alpha+1)2^{\alpha+1}$ and $|f_\lambda(0)| \leq 1 + (\alpha+1)/\alpha$. Thus we have $M = \sup\{\parallel f_\lambda \parallel_{\mathcal{B}^\alpha}: \lambda \in \mathbb{D}\} \leq (1 + (\alpha+1)/\alpha + 3(\alpha+1)2^{\alpha+1}$. Moreover $f'_\lambda(\varphi(\lambda)) = 0$. Again

$$f''_\lambda(z) = \Big(\frac{(\alpha+1)(\alpha+2)(1-|\varphi(\lambda)|^2)^2}{(1-\overline{\varphi(\lambda)}z)^{\alpha+3}} - \frac{(\alpha+1)^2(1-|\varphi(\lambda)|^2)}{(1-\overline{\varphi(\lambda)}z)^{\alpha+2}}\Big)\overline{(\varphi(\lambda))}^2.$$

and so

$$f''_\lambda(\varphi(\lambda)) = \frac{(\alpha+1)}{(1-|\varphi(\lambda)|^2)^{\alpha+1}}\overline{(\varphi(\lambda))^2}$$

Since DC_φ maps $\mathcal{B}^\alpha$ boundedly into $\mathcal{B}^\beta$, so we can find a constant $C > 0$ such that $\parallel DC_\varphi f_\lambda \parallel_{\mathcal{B}^\beta} \leq C \parallel f_\lambda \parallel_{\mathcal{B}^\alpha} \leq CM$. Hence

$$(1-|z|^2)^\beta|f''_\lambda(\varphi(z))(\varphi'(z))^2 + f'_\lambda(\varphi(z))\varphi''(z)| \leq CM$$

for all $z \in \mathbb{D}$. In particular

$$(1-|\lambda|^2)^\beta|f''_\lambda(\varphi(\lambda))(\varphi'(\lambda))^2 + f'_\lambda(\varphi(\lambda))\varphi''(\lambda)| \leq CM$$

and so

$$(\alpha+1)\frac{(1-|\lambda|^2)^\beta}{(1-|\varphi(\lambda)|^2)^{\alpha+1}}|\varphi(\lambda)|^2|\varphi'(\lambda)|^2 \leq CM.$$

Thus for fixed δ_1, $0 < \delta_1 < 1$, we have

$$\sup\Big\{\frac{(1-|\lambda|^2)^\beta}{(1-|\varphi(\lambda)|^2)^{\alpha+1}}|\varphi'(\lambda)|^2 : \lambda \in \mathbb{D}, |\varphi(\lambda)| > \delta_1\Big\} < \infty. \tag{3.4}$$

For $\lambda \in \mathbb{D}$ such that $|\varphi(\lambda)| \leq \delta_1$, we have

$$\frac{1-|\lambda|^2)^\beta}{(1-|\varphi(\lambda)|^2)^{\alpha+1}}|\varphi'(\lambda)|^2 \leq \frac{1}{(1-\delta_1^2)^{\alpha+1}}(1-|\lambda|^2)^\beta|\varphi'(\lambda)|^2.$$

Since $\varphi' \in \mathcal{B}^\beta$, we have

$$\sup \frac{(1-|\lambda|^2)^\beta}{(1-|\varphi(\lambda)|^2)^{\alpha+1}}|\varphi'(\lambda)|^2 : \lambda \in \mathbb{D}, |\varphi(\lambda)| \leq \delta_1 < \infty. \tag{3.5}$$

Consequently, by (3.4) and (3.5), we have

$$\sup_{\lambda\in\mathbb{D}}\Big\{\frac{(1-|\lambda|^2)^\beta}{(1-|\varphi(\lambda)|^2)^{\alpha+1}}|\varphi'(\lambda)|^2\Big\} < \infty.$$

Next for fixed $\lambda \in \mathbb{D}$, consider the function

$$f_\lambda(z) = \frac{(\alpha+1)(1-|\varphi(\lambda)|^2)^3}{(\alpha+3)(1-\overline{\varphi(\lambda)}z)^{\alpha+2}} - \frac{1-|\varphi(\lambda)|^2}{(1-\overline{\varphi(\lambda)}z)^{\alpha+1}}$$

Then

$$f'_\lambda(z) = \Big(\frac{(\alpha+1)(\alpha+2)(1-|\varphi(\lambda)|^2)^3}{(\alpha+3)(1-\overline{\varphi(\lambda)}z)^{\alpha+3}} - \frac{(\alpha+1)(1-|\varphi(\lambda)|^2)^2}{(1-\overline{\varphi(\lambda)}z)^{\alpha+2}}\Big)\overline{\varphi(\lambda)}$$

Thus an easy calculation yields that $(1-|z|^2)^\alpha|f'_\lambda(z)| \leq (\alpha+1)(5\alpha+11)2^{\alpha+1}$ and $|f_\lambda(0)| \leq 4(3\alpha+5)/(\alpha+3)$ and so, we have $M = \sup\{\| f_\lambda \|_{\mathcal{B}^\alpha} : \lambda \in \mathbb{D}\} \leq 4(3\alpha+5)/(\alpha+3) + (\alpha+1)(5\alpha+11)2^{\alpha+1}$. Also

$$f''_\lambda(z) = (\alpha+1)(\alpha+2)\Big(\frac{(1-|\varphi(\lambda)|^2)^3}{(1-\overline{\varphi(\lambda)}z)^{\alpha+4}} - \frac{(1-|\varphi(\lambda)|^2)^2}{(1-\overline{\varphi(\lambda)}z)^{\alpha+3}}\Big)(\overline{\varphi(\lambda)})^2.$$

Thus

$$f''(\varphi(\lambda)) = 0 \quad \text{and} \quad f'_\lambda(\varphi(\lambda)) = -\frac{(\alpha+1)}{(\alpha+3)(1-|\varphi(\lambda)|^2)^\alpha}\overline{\varphi(\lambda)}.$$

Now we can find a constant $C > 0$ such that

$$C \geq (1-|\lambda|^2)^\beta|f''_\lambda(\varphi(\lambda))(\varphi'(\lambda))^2 + f'_\lambda(\varphi(\lambda))\varphi''(\lambda)|$$

and hence

$$\frac{(\alpha+1)(1-|\lambda|^2)^\beta}{(\alpha+3)(1-|\varphi(\lambda)|^2)^\alpha}|\varphi(\lambda)||\varphi''(\lambda)| \leq C'.$$

Thus for fixed δ_2, $0 < \delta_2 < 1$, we have

$$\sup\Big\{\frac{(1-|\lambda|^2)^\beta}{(1-|\varphi(\lambda)|^2)^\alpha}|\varphi''(\lambda)| : \lambda \in \mathbb{D}, |\varphi(\lambda)| > \delta_2\Big\} < \infty. \tag{3.6}$$

For $\lambda \in \mathbb{D}$ such that $|\varphi(\lambda)| \leq \delta_2$, we have

$$\frac{(1-|\lambda|^2)^\beta}{(1-|\varphi(\lambda)|^2)^\alpha}|\varphi''(\lambda)| \leq \frac{1}{(1-{\delta_2}^2)^\alpha}(1-|\lambda|^2)^\beta|\varphi''(\lambda)|$$

Since $\varphi' \in \mathcal{B}_0^\beta$, we have

$$\sup\left\{\frac{(1-|\lambda|^2)^\beta}{(1-|\varphi(\lambda)|^2)^\alpha}|\varphi''(\lambda)| : \lambda \in \mathbb{D}, |\varphi(\lambda)| \leq \delta_2\right\} < \infty. \tag{3.7}$$

Consequently, by (3.6) and (3.7) we have

$$\sup_{\lambda\in\mathbb{D}} \frac{(1-|\lambda|^2)^\beta}{(1-|\varphi(\lambda)|^2)^\alpha}|\varphi''(\lambda)| < \infty.$$

This completes the proof.

Theorem 3.4. *Let $\alpha \geq 1$, $\beta > 0$ be two real numbers and φ be a holomorphic self map of $\mathbb{D}$ such that DC_φ maps $\mathcal{B}^\alpha$ boundedly into $\mathcal{B}^\beta$. Then DC_φ maps $\mathcal{B}^\alpha$ compactly into $\mathcal{B}^\beta$ if and only if*

$$(i) \lim_{|\varphi(z)|\to 1} \frac{(1-|z|^2)^\beta |\varphi'(z)|^2}{(1-|\varphi(z)|^2)^{\alpha+1}} = 0 \quad \textit{and} \quad (ii) \lim_{|\varphi(z)|\to 1} \frac{(1-|z|^2)^\beta |\varphi''(z)|}{(1-|\varphi(z)|^2)^{\alpha}} = 0.$$

Proof. Let $\{f_n\}$ be a bounded sequence in $\mathcal{B}^\alpha$ that converges to zero uniformly on compact subsets of $\mathbb{D}$. Then $\| DC_\varphi f_n \|_{\mathcal{B}^\beta} \to 0$ as $n \to \infty$. Let $M = \sup_n \| f_n \|_{\mathcal{B}^\alpha} < \infty$. Given $\epsilon > 0$, there exist an $r \in (0,1)$ such that if $|\varphi(z)| > r$, then

$$\frac{(1-|z|^2)^\beta}{(1-|\varphi(z)|^2)^{\alpha+1}}|\varphi'(z)|^2 < \epsilon \quad \text{and} \quad \frac{(1-|z|^2)^\beta}{(1-|\varphi(z)|^2)^{\alpha}}|\varphi''(z)|^2 < \epsilon.$$

Thus for $z \in \mathbb{D}$ such that $|\varphi(z)| > r$ we have

$$\begin{aligned}(1-|z|^2)^\beta |(DC_\varphi f_n)'(z)| &= (1-|z|^2)^\beta (|\varphi'(z)|^2 |f_n''(\varphi(z))| + |f_n'(\varphi(z))||\varphi''(z)|) \\ &\leq \left(C_\alpha \frac{(1-|z|^2)^\beta |\varphi'(z)|^2}{(1-|\varphi(z)|^2)^{\alpha+1}} + \frac{(1-|z|^2)^\beta |\varphi''(z)|}{(1-|\varphi(z)|^2)^{\alpha}}\right) \| f_n \|_{\mathcal{B}^\alpha} \\ &< \epsilon M(C_\alpha + 1)\end{aligned}$$

for all n. On the other hand since f_n' and f_n'' converges uniformly on $\{w : |w| \leq r\}$, there exist an n_0 such that if $|\varphi(z)| \leq r$ and $n \geq n_0$, then $|f_n'(\varphi(z))| < r$ and $|f_n''(\varphi(z))| < \epsilon$. Also conditions (1) and (2) of Theorem 3.3 implies that

$$A = \sup_{z\in\mathbb{D}} (1-|z|^2)^\beta |\varphi'(z)|^2 < \infty \quad \text{and} \quad B = \sup_{z\in\mathbb{D}} (1-|z|^2)^\beta |\varphi''(z)| < \infty.$$

Thus we deduce that

$$\begin{aligned}(1-|z|^2)^\beta |(DC_\varphi f_n)'(z)| &\leq (1-|z|^2)^\beta (|\varphi'(z)|^2 |f_n''(\varphi(z))| + |f_n'(\varphi(z))||\varphi''(z)|) \\ &\leq (A+B)\epsilon.\end{aligned}$$

The above arguments together with the fact that $DC_\varphi f_n(0) = f_n'(\varphi(0)) \to 0$ as $n \to \infty$ yields that $\| DC_\varphi f_n \|_{\mathcal{B}^\beta} \to 0$ as $n \to \infty$.

Conversely, suppose that DC_φ maps $\mathcal{B}^\alpha$ compactly into $\mathcal{B}^\beta$. Let $\{z_n\}$ be a sequence in $\mathbb{D}$ such that $|\varphi(z_n)| \to 1$ as $n \to \infty$. Let

$$f_n(z) = \frac{(1-|\varphi(z_n)|^2)^2}{(1-\overline{\varphi(z_n)}z)^{\alpha+1}} - \frac{(\alpha+1)(1-|\varphi(z_n)|^2)}{\alpha(1-\overline{\varphi(z_n)}z)^{\alpha}}$$

for $z \in \mathbb{D}$. Then as in Theorem 3.3, $f_n \in \mathcal{B}^\alpha$, f_n is norm bounded in $\mathcal{B}^\alpha$ and $f_n \to 0$ uniformly on compact subsets of $\mathbb{D}$. Moreover

$$f_n'(z) = \overline{\varphi(z_n)}\Big\{\frac{(\alpha+1)(1-|\varphi(z_n)|^2)^2}{(1-\overline{\varphi(z_n)}z)^{\alpha+2}} - \frac{(\alpha+1)(1-|\varphi(z_n)|^2)}{(1-\overline{\varphi(z_n)}z)^{\alpha+1}}\Big\}$$

$$f_n''(z) = \Big\{\frac{(\alpha+1)(\alpha+2)(1-|\varphi(z_n)|^2)^2}{(1-\overline{\varphi(z_n)}z)^{\alpha+3}} - \frac{(\alpha+1)^2(1-|\varphi(z_n)|^2)}{(1-\overline{\varphi(z_n)}z)^{\alpha+2}}\Big\}(\overline{\varphi(z_n)})^2.$$

Note that

$$f_n'(\varphi(z_n)) = 0 \quad \text{and} \quad f_n''(\varphi(z_n)) = \frac{(\alpha+1)}{(1-|\varphi(z_n)|^2)^{\alpha+1}}(\overline{\varphi(z_n)})^2.$$

Since DC_φ maps $\mathcal{B}^\alpha$ compactly into $\mathcal{B}^\beta$, it follows that $\| DC_\varphi f_n \|_{\mathcal{B}^\beta} \to 0$ as $n \to \infty$. Thus

$$\| DC_\varphi f_n \|_{\mathcal{B}^\beta} \geq \frac{(1-|z_n|^2)^\beta(\alpha+1)|\varphi(z_n)|^2|\varphi'(z_n)|^2}{(1-|\varphi(z_n)|^2)^{\alpha+1}}$$

implies that

$$\lim_{|\varphi(z_n)|\to 1} \frac{(1-|z_n|^2)^\beta}{(1-|\varphi(z_n)|^2)^{\alpha+1}}|\varphi'(z_n)|^2 = 0.$$

Next for $\{z_n\} \in \mathbb{D}$ such that $|\varphi(z_n)| \to 1$ consider the function

$$g_n(z) = \frac{(\alpha+1)(1-|\varphi(z_n)|^2)^3}{(\alpha+3)(1-\overline{\varphi(z_n)}z)^{\alpha+2}} - \frac{(1-|\varphi(z_n)|^2)^2}{(1-\overline{\varphi(z_n)}z)^{\alpha+1}}.$$

Again as in Theorem 3.3, $g_n \in \mathcal{B}^\alpha$, g_n is norm bounded in $\mathcal{B}^\alpha$ and $g_n \to 0$ uniformly on compact subsets of $\mathbb{D}$. Moreover

$$g_n'(z) = \Big\{\frac{(\alpha+1)(\alpha+2)}{\alpha+3}\frac{(1-|\varphi(z_n)|^2)^3}{(1-\overline{\varphi(z_n)}z)^{\alpha+3}} - \frac{(\alpha+1)(1-|\varphi(z_n)|^2)^2}{(1-\overline{\varphi(z_n)}z)^{\alpha+3}}\Big\}\overline{\varphi(z_n)}$$

and

$$g_n''(z) = (\alpha+1)(\alpha+2)\Big\{\frac{(1-|\varphi(z_n)|^2)^3}{(1-\overline{\varphi(z_n)}z)^{\alpha+4}} - \frac{(1-|\varphi(z_n)|^2)^2}{(1-\overline{\varphi(z_n)}z)^{\alpha+3}}\Big\}(\overline{\varphi(z_n)})^2.$$

Thus

$$g_n'(\varphi(z_n)) = -\frac{(\alpha+1)}{(\alpha+3)}\frac{1}{(1-|\varphi(z_n)|^2)^\alpha}\overline{\varphi}(z_n) \quad \text{and} \quad g_n''(\varphi(z_n)) = 0.$$

Since DC_φ maps $\mathcal{B}^\alpha$ compactly into $\mathcal{B}^\beta$, so

$$\| DC_\varphi g_n \|_{\mathcal{B}^\beta} \geq \frac{(\alpha+1)}{(\alpha+3)}\frac{(1-|z_n|^2)^\beta}{(1-|\varphi(z_n)|^2)^\alpha}|\varphi''(z_n)|$$

and hence

$$\lim_{|\varphi(z)|\to 1} \frac{(1-|z|^2)^\beta}{(1-|\varphi(z)|^2)^\alpha}|\varphi''(z)| = 0$$

This completes the proof.

4 Boundedness and Compactness of $C_\varphi D$ and DC_φ between little α-Bloch spaces

In this section, we consider the operators $C_\varphi D$ and DC_φ acting between little α-Bloch spaces $\mathcal{B}_0^\alpha$ and $\mathcal{B}_0^\beta$.

Theorem 4.1. *Let $\alpha \geq 1$, $\beta > 0$ be two real numbers and φ be a holomorphic self map of $\mathbb{D}$. Then $C_\varphi D$ maps $\mathcal{B}_0^\alpha$ boundedly into $\mathcal{B}_0^\beta$ if and only if the following conditions are satisfied*

(i) $\sup_{z\in\mathbb{D}} \frac{(1-|z|^2)^\beta}{(1-|\varphi(z)|^2)^{\alpha+1}}|\varphi'(z)| < \infty$ *and* (ii) $\varphi \in \mathcal{B}_0^\beta$.

Proof. First suppose that $C_\varphi D$ maps $\mathcal{B}_0^\alpha$ boundedly into $\mathcal{B}_0^\beta$. Then (i) can be proved exactly in the same way as in the proof of the Theorem 3.1. By taking $f(z) = z^2/2$ in $\mathcal{B}_0^\alpha$, we get $\varphi \in \mathcal{B}_0^\beta$ which is (ii).

Next, suppose that (i) and (ii) are satisfied. Take any $\varepsilon > 0$. Let $f \in \mathcal{B}_0^\alpha$. Then by Theorem 2.1, there is $\delta_1 \in (0,1)$ such that for any $z \in \mathbb{D}$, $|z| > \delta_1$, we have $|f''(z)| < \varepsilon/(1-|z|^2)^{\alpha+1}$. Thus for $|\varphi(z)| > \delta_1$, by (i), we can find a constant $M > 0$ such that

$$(1-|z|^2)^\beta |f''(\varphi(z))\varphi'(z)| < \varepsilon|\varphi'(z)|\frac{(1-|z|^2)^\beta}{(1-|\varphi(z)|^2)^{\alpha+1}} \leq \varepsilon M. \tag{4.1}$$

On the other hand, since by (ii) $\varphi \in \mathcal{B}_0^\beta$, so for above ε, there is $\delta_2 \in (0,1)$ such that $|z| > \delta_2$ implies that $(1-|z|^2)^\beta|\varphi'(z)| < \varepsilon$. Thus for $|\varphi(z)| \leq \delta_1$, if $|z| > \delta_2$, we have a constant $N > 0$ such that

$$(1-|z|^2)^\beta|\varphi'(z)f''(\varphi(z))| < C_\alpha||f||_{\mathcal{B}^\alpha}|\varphi'(z)|\frac{(1-|z|^2)^\beta}{(1-\delta_1^2)^{\alpha+1}} \leq \varepsilon N. \tag{4.2}$$

By combining (4.1) and (4.2), we see that whenever $|z| > \delta_2$, we have

$$(1-|z|^2)^\beta|\varphi'(z)f''(\varphi(z))| \leq max(M,N)\varepsilon$$

which means

$$\lim_{|z|\to 1}(1-|z|^2)^\beta|(C_\varphi Df)'(z)| = 0.$$

Thus $C_\varphi Df \in B_0^\beta$. By Closed Graph Theorem DC_φ maps $\mathcal{B}_0^\alpha$ boundedly into $\mathcal{B}_0^\beta$.

Theorem 4.2 *Let $\alpha \geq 1$, $\beta > 0$ be two real numbers and φ be a holomorphic self map of $\mathbb{D}$. Then $C_\varphi D$ maps $\mathcal{B}_0^\alpha$ compactly into $\mathcal{B}_0^\beta$ if and only if*

$$\lim_{|z|\to 1}\frac{(1-|z|^2)^\beta}{(1-|\varphi(z)|^2)^{\alpha+1}}|\varphi'(z)| = 0. \tag{4.3}$$

Proof. By Lemma 2.2, the set $\{C_\varphi Df : f \in \mathcal{B}_0^\alpha, ||f||_{\mathcal{B}^\alpha} \leq 1\}$ has compact closure in $\mathcal{B}_0^\beta$ if and only if

$$\lim_{|z|\to 1}\sup\{(1-|z|^2)^\beta|(C_\varphi Df)'(z)| : f \in \mathcal{B}_0^\alpha, ||f||_{\mathcal{B}^\alpha} \leq 1\} = 0. \tag{4.4}$$

Suppose that $f \in \mathcal{B}_0^\alpha$ is such that $||f||_{\mathcal{B}^\alpha} \le 1$, and (4.3) is satisfied. Then

$$\begin{aligned}(1-|z|^2)^\beta|(C_\varphi Df)'(z)| &= (1-|z|^2)^\beta|\varphi'(z)f''(\varphi(z))|\\ &\le \frac{(1-|z|^2)^\beta}{(1-|\varphi(z)|^2)^{\alpha+1}}|\varphi'(z)|.\end{aligned}$$

By (4.2) above inequality implies (4.4). Hence $C_\varphi D$ maps $\mathcal{B}_0^\alpha$ compactly into $\mathcal{B}_0^\beta$.

Conversely, suppose that $C_\varphi D$ maps $\mathcal{B}_0^\alpha$ compactly into $\mathcal{B}_0^\beta$. Using the same test as in the proof of Theorem 3.2, we see that

$$\lim_{|\varphi(z)|\to 1}\frac{(1-|z|^2)^\beta}{(1-|\varphi(z)|^2)^{\alpha+1}}|\varphi'(z)| = 0. \tag{4.5}$$

Since $C_\varphi D$ maps $\mathcal{B}_0^\alpha$ boundedly into $\mathcal{B}_0^\beta$, Theorem 4.1 implies that $\varphi \in B_0^\beta$. It is easy to show that $\varphi \in B_0^\beta$ and (4.5) is equivalent to (4.4).

Remark. The conditions in Theorem 4.2 include the necessary and sufficient conditions for boundedness of $C_\varphi D$ from $\mathcal{B}_0^\alpha$ into $\mathcal{B}_0^\beta$.

Theorem 4.3. *Let $\alpha \ge 1$ and $\beta > 0$ be two real numbers. Then DC_φ maps $\mathcal{B}_0^\alpha$ boundedly into ${\mathcal{B}^\beta}_0$ if and only if the following conditions are satisfied.*

$$\text{(i)}\quad \sup_{z\in\mathbb{D}}\frac{(1-|z|^2)^\beta|\varphi'(z)|^2}{(1-|\varphi(z)|^2)^{\alpha+1}} < \infty,\quad (ii)\quad \sup_{z\in\mathbb{D}}\frac{(1-|z|^2)^\beta|\varphi''(z)|}{(1-|\varphi(z)|^2)^{\alpha}} < \infty,$$

$$\text{(iii)}\quad \varphi' \in \mathcal{B}_0^\beta \quad \textit{and} \quad (iv)\quad \lim_{|z|\to 1}(1-|z|^2)^\beta|\varphi'(z)|^2 = 0.$$

Proof. First suppose that DC_φ maps $\mathcal{B}_0^\alpha$ boundedly into $\mathcal{B}_0^\beta$. Then (i) and (ii) can be proved exactly in the same way as in the proof of the Theorem 3.1. By taking $f(z) = z$ in $\mathcal{B}_0^\alpha$, we get $\varphi' \in \mathcal{B}_0^\beta$ which is (ii). Again by taking $f(z) = z^2/2$ in $\mathcal{B}_0^\alpha$, we get $\lim_{|z|\to 1}(1-|z|^2)^\beta(|(\varphi'(z))^2 + \varphi(z)\varphi''(z)|) = 0$. Since $\varphi' \in \mathcal{B}_0^\beta$ and $|\varphi(z)| < 1$, we get $\lim_{|z|\to 1}(1-|z|^2)^\beta|\varphi'(z)|^2 = 0$, which is (iv).

Next, suppose that $(i)-(iv)$ are satisfied. Take any $\varepsilon > 0$. Let $f \in \mathcal{B}_0^\alpha$. Then by (2.2) there is $\delta_1 \in (0,1)$ such that for any $z \in \mathbb{D}$, $|z| > \delta_1$, we have $|f''(z)| < \varepsilon/(1-|z|^2)^{\alpha+1}$ Thus for $|\varphi(z)| > \delta_1$, by (i), we can find a constant $C_1 > 0$ such that

$$(1-|z|^2)^\beta|(\varphi'(z))^2 f''(\varphi(z))| < \varepsilon|\varphi'(z)|^2\frac{(1-|z|^2)^\beta}{(1-|\varphi(z)|^2)^{\alpha+1}} \le C_1\varepsilon \tag{4.6}$$

On the other hand, by (iv) there is $\delta_2 \in (0,1)$ such that $(1-|z|^2)^\beta|\varphi'(z)|^2 < \varepsilon$. Thus for $|\varphi(z)| \le \delta_1$, if $|z| > \delta_2$, we have a constant $C_2 > 0$ such that

$$(1-|z|^2)^\beta|(\varphi'(z))^2 f''(\varphi(z))| < C_\alpha||f||_{\mathcal{B}^\alpha}|\varphi'(z)|^2\frac{(1-|z|^2)^\beta}{(1-{\delta_1}^2)^{\alpha+1}} \le C_2\varepsilon \tag{4.7}$$

By combining (4.6) and (4.7), we see that whenever $|z| > \delta_2$, we have

$$(1-|z|^2)^\beta |(\varphi'(z))^2 f''(\varphi(z))| \leq max(C_1, C_2)\varepsilon. \tag{4.8}$$

Again, since $f \in \mathcal{B}_0^\alpha$, there is $\delta_3 \in (0,1)$ such that $|z| > \delta_3$ implies that $|f'(z)| < \varepsilon/(1-|z|^2)^\alpha$ Thus for $|\varphi(z)| > \delta_3$, by (ii), we can find a constant $C_3 > 0$ such that

$$(1-|z|^2)^\beta |\varphi''(z) f'(\varphi(z)| < \varepsilon |\varphi''(z)| \frac{(1-|z|^2)^\beta}{(1-|\varphi(z)|^2)^\alpha} \leq C_3\varepsilon \tag{4.9}$$

On the other hand, by (iii) $\varphi' \in \mathcal{B}_0^\beta$, so for above ε, there is $\delta_4 \in (0,1)$ such that $|z| > \delta_4$ implies that $(1-|z|^2)^\beta |\varphi''(z)| < \varepsilon$. Thus for $|\varphi(z)| \leq \delta_3$, if $|z| > \delta_4$, we have a constant $C_4 > 0$ such that

$$(1-|z|^2)^\beta |\varphi''(z) f'(\varphi(z))| < ||f||_{\mathcal{B}^\alpha} |\varphi''(z)| \frac{(1-|z|^2)^\beta}{(1-\delta_3^2)^\alpha} \leq C_4\varepsilon \tag{4.10}$$

By combining (4.9) and (4.10), we see that whenever $|z| > \delta_4$, we have

$$(1-|z|^2)^\beta |\varphi''(z) f'(\varphi(z))| \leq max(C_3, C_4)\varepsilon. \tag{4.11}$$

By combining (4.8) and (4.11), we have for $\delta = max(\delta_2, \delta_4)$, if $|z| > \delta$, there is a constant $C > 0$ such that

$$(1-|z|^2)^\beta (|\varphi'(z)|^2 |f''(\varphi(z))| + |f'(\varphi(z))||\varphi''(z)|) < \varepsilon C$$

which means that

$$\lim_{|z|\to 1} (1-|z|^2)^\beta |(DC_\varphi f)'(z)| = 0.$$

Thus $DC_\varphi \in \mathcal{B}_0^\beta$. The proof is complete.

Theorem 4.4. Let $\alpha \geq 1$ and $\beta > 0$ be two real numbers. Then DC_φ maps $\mathcal{B}_0^\alpha$ compactly into $\mathcal{B}_0^\beta$ if and only if

$$(i) \lim_{|z|\to 1} \frac{(1-|z|^2)^\beta |\varphi'(z)|^2}{(1-|\varphi(z)|^2)^{\alpha+1}} = 0 \quad \text{and} \quad (ii) \lim_{|z|\to 1} \frac{(1-|z|^2)^\beta |\varphi''(z)|}{(1-|\varphi(z)|^2)^\alpha} = 0.$$

Proof. By Lemma 2.2, the set $\{DC_\varphi f : f \in \mathcal{B}_0^\beta, ||f||_{\mathcal{B}^\alpha} \leq 1\}$ has compact closure in $\mathcal{B}_0^\beta$ if and only if

$$\lim_{|z|\to 1} \sup\{(1-|z|^2)^\beta |(DC_\varphi f)'(z)| : f \in \mathcal{B}_0^\alpha, ||f||_{\mathcal{B}^\alpha} \leq 1\} = 0. \tag{4.12}$$

Suppose that $f \in \mathcal{B}_0^\alpha$ is such that $||f||_{\mathcal{B}^\alpha} \leq 1$, and φ satisfies (i) and (ii). Then

$$\begin{aligned}(1-|z|^2)^\beta |(DC_\varphi f)'(z)| &\leq (1-|z|^2)^\beta [|\varphi'(z)|^2 |f''(\varphi(z))| + |f'(\varphi(z))||\varphi''(z)|] \\ &\leq C_\alpha \Big(\frac{(1-|z|^2)^\beta |\varphi'(z)|^2}{(1-|\varphi(z)|^2)^{\alpha+1}} + \frac{(1-|z|^2)^\beta |\varphi''(z)|}{(1-|\varphi(z)|^2)^\alpha} \Big) \parallel f \parallel_{\mathcal{B}^\alpha}\end{aligned}$$

By (i) and (ii) above inequality implies (4.12). Hence DC_φ maps $\mathcal{B}_0^\alpha$ compactly into $\mathcal{B}_0^\beta$.

Conversely, suppose that DC_φ maps $\mathcal{B}_0^\alpha$ compactly into $\mathcal{B}_0^\beta$. Using the same test function as in the proof of Theorem 3.4, we see that

$$\lim_{|\varphi(z)|\to 1} \frac{(1-|z|^2)^\beta |\varphi'(z)|^2}{(1-|\varphi(z)|^2)^{\alpha+1}} = 0 \tag{4.13}$$

and

$$\lim_{|\varphi(z)|\to 1} \frac{(1-|z|^2)^\beta |\varphi''(z)|}{(1-|\varphi(z)|^2)^{\alpha}} = 0. \tag{4.14}$$

Since DC_φ maps $\mathcal{B}_0^\alpha$ boundedly into $\mathcal{B}_0^\beta$, Theorem 4.3 implies that $\varphi' \in \mathcal{B}_0^\beta$ and

$$\lim_{|z|\to 1} (1-|z|^2)^\beta |\varphi'(z)|^2 = 0 \tag{4.15}$$

It is easy to show that $\varphi' \in \mathcal{B}_0^\beta$ and (4.14) is equivalent to (i) and (4.13) and (4.15) is equivalent to (ii).

Remark. The conditions in Theorem 4.4 include the necessary and sufficient conditions for boundedness of DC_φ from $\mathcal{B}_0^\alpha$ into $\mathcal{B}_0^\beta$.

In the trivial case that $\varphi(z) = z$, our theorems give necessary and sufficient conditions for boundedness and compactness of the differentiation operator between α-Bloch spaces. It seems that the results for the boundedness and compactness of the differentiation operator between α-Bloch spaces has not appeared in the literature. Therefore we single these results as corollaries.

Corollary 1. *Let $\alpha \geq 1$ and $\beta > 0$ be two real numbers. Then the following are equivalent:*

(i) *D maps $\mathcal{B}^\alpha$ boundedly into $\mathcal{B}^\beta$;*
(ii) *D maps $\mathcal{B}_0^\alpha$ boundedly into $\mathcal{B}_0^\beta$;*
(iii) *$\alpha + 1 \leq \beta$.*

Corollary 2. *Let $\alpha \geq 1$ and $\beta > 0$ be two real numbers. Then the following are equivalent:*

(i) *D maps $\mathcal{B}^\alpha$ compactly into $\mathcal{B}^\beta$;*
(ii) *D maps $\mathcal{B}_0^\alpha$ compactly into $\mathcal{B}_0^\beta$;*
(iii) *$\alpha + 1 < \beta$.*

Before we give some examples, we state characterisations of boundedness and compactness of the C_φ between α-Bloch spaces, obtained by Ohno, Stroethoff and Zhao in [**6**], (see Corollaries 2.4 and 3.2).

Theorem 4.5. [**6**] *Let $\alpha \geq 1$ and $\beta > 0$ be two real numbers. Then C_φ maps $\mathcal{B}^\alpha$ compactly into $\mathcal{B}^\beta$ if and only if*

$$\sup_{z\in\mathbb{D}} \frac{(1-|z|^2)^\beta}{(1-|\varphi(z)|^2)^\alpha} |\varphi'(z)| < \infty.$$

Further, if C_φ maps $\mathcal{B}^\alpha$ boundedly into $\mathcal{B}^\beta$, then C_φ maps $\mathcal{B}^\alpha$ compactly into $\mathcal{B}^\beta$ if and only if

$$\lim_{|\varphi(z)|\to 1} \frac{(1-|z|^2)^\beta}{(1-|\varphi(z)|^2)^\alpha} |\varphi'(z)| = 0.$$

Example 1. Let $\varphi(z) = (1-z)/2$. Then $1 - |\varphi(z)|^2 \geq (1-|z|^2)/4$. Thus by Theorem 4.5, we obtain that C_φ maps $\mathcal{B}^\alpha$ boundedly (respectively compactly)into $\mathcal{B}^\beta$, when $\alpha \leq \beta$ (respectively $\alpha < \beta$).
Furthermore $C_\varphi D$ and DC_φ maps $\mathcal{B}^\alpha$ boundedly (respectively compactly) into $\mathcal{B}^\beta$, when $\alpha + 1 \leq \beta$ (respectively $\alpha + 1 < \beta$).
Example 2. Let $\varphi_\gamma(z) = 1 - (1-z)^\gamma, 0 < \gamma < 1$. Then $\varphi'_\gamma(z) = \gamma(1-z)^{\gamma-1}$. Again for z near to 1, $1 - |\varphi(z)|^2 \approx (1-z)^\gamma$. Thus by Theorem 4.5, C_φ maps $\mathcal{B}^\alpha$ boundedly (respectively compactly)into $\mathcal{B}^\beta$, when $\alpha - \gamma + 1 \leq \beta$ (respectively $\alpha - \gamma + 1 < \beta$).
$C_\varphi D$ maps $\mathcal{B}^\alpha$ boundedly (respectively compactly)into $\mathcal{B}^\beta$, when $\alpha - \gamma + 2 \leq \beta$ (respectively $\alpha - \gamma + 2 < \beta$) and DC_φ maps $\mathcal{B}^\alpha$ boundedly (respectively compactly) into $\mathcal{B}^\beta$, when $\alpha - \gamma + 3 \leq \beta$ (respectively $\alpha - \gamma + 3 < \beta$).

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