Stieltjes Transformation on Ordered Vector Space of Generalized Functions and Abelian and Tauberian Theorems

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Abstract. Stieltjes transformation of the form
\[ \hat{f}(x) = \int_0^\infty \frac{f(t)}{(x^m + t^m)\rho} dt, \quad m, \rho > 0 \]
has been studied extensively. We apply this form of the transform to an ordered vector space of generalized functions to which the topology of bounded convergence is assigned. Some of the order properties of the transform and its inverse are studied. The asymptotic behaviour of the transform and its inverse are also studied.

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1. Introduction

In an earlier paper [1] we had associated the notion of order to multinormed spaces and their duals. The topology of bounded convergence was assigned to the dual spaces. Some properties of these spaces were also studied. As illustrative examples the spaces \( D, E, L_{a,b}, L(w,z) \) and their duals were also studied.

In the present paper, an ordered testing function space \( M_{a,b}^m \) is defined and the topology of bounded convergence is assigned to its dual. Some properties of the cone in \( M_{a,b}^m \) and in \((M_{a,b}^m)’\) are studied in section 2.

Stieltjes transformation and its properties have been studied by John J. K. [2]. In section 3 we apply some of these results to the new context when the topology of bounded convergence is assigned to the ordered vector space \((M_{a,b}^m)’\). It follows that the Stieltjes transform and its inverse are order bounded.
Order relation is defined on the space $\zeta$ of test functions of rapid descent and its dual $\zeta'$, the space of tempered distributions. The topology of bounded convergence is assigned to $\zeta'$. The notion of strongasymptotic behaviour is introduced and the strongasymptotic behaviour of $S^m_\rho(f)$ with reference to the strongasymptotic behaviour of $f$ and vice versa are studied in section 4.

2. The testing function space $M_{a,b}^m$ and its dual

Let $M_{a,b}^m$ denote the linear space of all complex-valued smooth functions defined on $(0, \infty)$. Let $(K_m)$ be a sequence of compact subsets of $R_+$ such that $K_1 \subseteq K_2 \subseteq \ldots$ and such that each compact subset of $(0, \infty)$ is contained in one $K_j$, $j = 1, 2, \ldots$. On each $K_j$ define

$$
\mu_{a,b,K_j,k}^m(\phi) = \sup_{t \in K_j} t^{m(1-a+k)}(1 + t^m)^{a-b} |(t^{1-m}\frac{d}{dt})^k \phi(t)|.
$$

$a, b \in R$, $k = 0, 1, 2, \ldots, m \in (0, \infty)$.

$\{\mu_{a,b,K_j,k}^m\}_{k=0}^\infty$ is a multinorm on $M_{a,b,K_j}^k$, where $M_{a,b,K_j}^k$ is a subspace of $M_{a,b}^m$ consisting of functions with support contained in $K_j$. The above multinorm generates the topology $\tau_{a,b,K_j}^m$ on $M_{a,b,K_j}^k$. The inductive limit topology $\tau_{a,b}^m$ as $K_j$ varies over all compact sets $K_1, K_2, \ldots$ is assigned to $M_{a,b}^m$. With respect to this topology $M_{a,b}^m$ is complete.

On each $M_{a,b,K_j}^m$ an equivalent multinorm is given by

$$
\overline{\mu}_{a,b,K_j,k}^m(\phi) = \sup_{0 \leq k' \leq k} \mu_{a,b,K_j,k'}^m(\phi).
$$

Definition 2.1. The positive cone $C$ of $M_{a,b}^m$ when $M_{a,b}^m$ is restricted to real-valued functions is the set of all non-negative functions in $M_{a,b}^m$.

Then $C + iC$ is the positive cone in $M_{a,b}^m$ which is also denoted as $C$.

Theorem 2.1. The cone $C$ of $M_{a,b}^m$ is not normal.

Proof. Suppose that $M_{a,b}^m$ is restricted to real-valued functions. Let $j$ be a fixed +ve integer and let $(\phi_i)$ be a sequence of functions in $C \cap (M_{a,b,K_j}^m)$ such that $\lambda_i = \sup \{\phi_i(t), t \in K_j\}$ converges to 0 but $(\phi_i)$ does not converge to 0 for $M_{a,b,K_j}^m$.

Define $\psi_i = \begin{cases} 
\lambda_i, & t \in K_{j+1} \\
0, & t \notin K_{j+1}
\end{cases}.$

Let $\xi_i$ be the regularization of $\psi_i$ defined by

$$
\xi_i(t) = \int_0^\infty \theta_\alpha(t-t')\psi_i(t')dt'.
$$

Then $\xi \in M_{a,b,K_{j+2}}^m$ for all $i$.

Also, $0 \leq \phi_i \leq \xi_i$ and $(\xi_i)$ converges to 0 for $\tau_{a,b}^m$. In Proposition 1.3, chapter 1, [3], it has been proved that the positive cone in an ordered topological vector space $E(\tau)$ is normal if and only if for any two nets $\{x_\beta : \beta \in I\}$ and
\{y_\beta : \beta \in I\} in E(\tau) if 0 \leq x_\beta \leq y_\beta for all \beta \in I and if \{y_\beta : \beta \in I\} converges to 0 for \tau then \{x_\beta : \beta \in I\} converges to 0 for \tau. So we conclude that \(C\) is not normal for the Schwartz topology. It follows that \(C + iC\) is not normal for \(M_{a,b}^m\).

**Theorem 2.2.** The cone \(C\) is a strict b-cone in \(M_{a,b}^m\).

**Proof.** Let \(M_{a,b}^m\) be restricted to real-valued functions. Let \(B\) be the saturated class of all bounded circled subsets of \(M_{a,b}^m\) for \(\tau_{a,b}^m\).

Then \(M_{a,b}^m = \bigcup_{B \in B} B_c\) where \(B_c = \{(B \cap C) - (B \cap C) : B \in B\}\) is a fundamental system for \(B\) and \(C\) is a strict b-cone, since \(B\) is a collection of all \(\tau\)-bounded sets in \(M_{a,b}^m\).

Let \(B\) be a bounded circled subset of \(M_{a,b}^m\) for \(\tau_{a,b}^m\). Then all functions in \(B\) have their support in some compact set \(K_{j_0}\) and there exists a constant \(M > 0\) such that \(|\phi(t)| \leq M\), for all \(\phi \in B, t \in K_{j_0}\).

Let

\[\psi(t) = \begin{cases} M, & t \in K_{j_0+1} \\ 0, & t \notin K_{j_0+1}. \end{cases}\]

Then the regularization \(\xi\) of \(\psi\) is defined by

\[\xi(t) = \int_0^\infty \theta_\alpha(t - t')\psi(t')dt'\]

and \(\xi\) has its support in \(K_{j_0+2}\).

Also,

\[B \subset (B + \xi) \cap \{\xi\} \subset (B + \xi) \cap C - (B + \xi) \cap C\]

It follows that \(C\) is a strict b-cone. We conclude that \(C + iC\) is a strict b-cone. 

**Order and topology on the dual of \(M_{a,b}^m\).** Let \((M_{a,b}^m)^\prime\) denote the linear space of all linear functionals defined on \(M_{a,b}^m\). An order relation is defined on \((M_{a,b}^m)^\prime\) by identifying the positive cone of \((M_{a,b}^m)^\prime\) to be the dual cone \(C\) of \(C\) in \(M_{a,b}^m\). The class of all \(B^0\), the polars of \(B\) as \(B\) varies over all \(\sigma(M_{a,b}^m, (M_{a,b}^m)^\prime)\)-bounded subsets of \(M_{a,b}^m\) is a neighbourhood basis of 0 in \((M_{a,b}^m)^\prime\) for the locally convex topology \(\beta((M_{a,b}^m)^\prime, M_{a,b}^m)\). When \((M_{a,b}^m)^\prime\) is ordered by the dual cone \(C\) and is equipped with the topology \(\beta((M_{a,b}^m)^\prime, M_{a,b}^m)\) it follows that \(C\) is a normal cone since \(C\) is a strict b-cone in \(M_{a,b}^m\) for \(\tau_{a,b}^m\). Note that \(\beta((M_{a,b}^m)^\prime, M_{a,b}^m)\) is the \(G\)-topology corresponding to the class \(G\) of all complete, bounded, convex, circled subsets of \((M_{a,b}^m)^\prime\) for the topology \(\sigma(M_{a,b}^m, (M_{a,b}^m)^\prime)\). Anthony L. Perissini [3] has observed that a subset \(L(E, F)\), the vector space of all continuous linear mappings of \(E\) into \(F\) is bounded for the topology of pointwise convergence if and only if it is bounded for the \(G\)-topology corresponding to the class of all complete, bounded, convex, circled subsets of \(E(\tau)\).
Theorem 2.3. The order dual $(M^{m}_{a,b})^+$ is the same as the topological dual $(M^{m}_{a,b})'$ when $(M^{m}_{a,b})'$ is equipped with the topology of bounded convergence.

Proof. Every positive linear functional is continuous for the Schwartz topology $\tau^{m}_{a,b}$ on $M^{m}_{a,b}$.

$$(M^{m}_{a,b})^+ = C(M^{m}_{a,b}, R) - C(M^{m}_{a,b}, R)$$

where $C(M^{m}_{a,b}, R)$ is the linear subspace consisting of all non-negative order bounded linear functionals on $M^{m}_{a,b}$ of $L(M^{m}_{a,b}, R)$, the linear space of all order bounded linear functionals on $M^{m}_{a,b}$. It follows that $(M^{m}_{a,b})^+ \subseteq (M^{m}_{a,b})'$.

On the other hand, the space $(M^{m}_{a,b})'$ equipped with the topology of bounded convergence $\beta((M^{m}_{a,b})', M^{m}_{a,b})$ and ordered by the dual cone $C'$ of the cone $C$ in $M^{m}_{a,b}$ is a reflexive space ordered by a closed a normal cone. Hence if $D$ is a directed $(\leq)$ set of elements that is either majorized in $(M^{m}_{a,b})'$ or contains a $\beta((M^{m}_{a,b})', M^{m}_{a,b})$-bounded section, then $\sup D$ exists in $(M^{m}_{a,b})'$ and the filter $\mathcal{F}(D)$ of sections of $D$ converges to $\sup D$ for $\beta((M^{m}_{a,b})', M^{m}_{a,b})$.

Hence $(M^{m}_{a,b})^+ = (M^{m}_{a,b})'$.

We conclude that $(M^{m}_{a,b})'$ with respect to the topology $\beta((M^{m}_{a,b})', M^{m}_{a,b})$ is both topologically complete and order complete.

3. The Stieltjes Transformation

In this section we study some properties of the Stieltjes transformation. The inverse of the transformation is also discussed here.

For $\phi \in M^{m}_{\alpha,\beta}$ the Stieltjes transform is defined as

$$S^m_\rho (\phi) = \hat{\phi}(x) = \int_0^\infty \frac{\phi(t)}{(x^m + t^m)^\rho} dt$$

for a fixed $m > 0$, $\rho \geq 1$.

With suitable integrability properties the double integral

$$\int_0^\infty \int_0^\infty \frac{f(x)\phi(t)}{(x^m + t^m)^\rho} dt dx$$

can be evaluated in two different ways so that

$$\langle \hat{f}, \phi \rangle = \langle f, \hat{\phi} \rangle.$$

This relation is the basis for the application of the method of adjoints to the Stieltjes transformation.

We have the following theorem due to John J. K. [2].

Theorem 3.1. Let $\alpha > 1 - \frac{1}{m}$, $\beta < \rho + 1 - \frac{1}{m}$. Then the Stieltjes transform maps $C \cap M^{m}_{\alpha,\beta}$ continuously into $C \cap M^{m}_{a,b}$ if

- $a \leq 1$, $a \leq \frac{1}{m} + \alpha + \rho$ and $a < 1$ if $\alpha = \rho + 1 - \frac{1}{m}$
- $b \geq 1 - \rho$, $b \geq \frac{1}{m} + \beta - \rho$ and $b > 1 - \rho$ if $\beta = 1 - \frac{1}{m}$. 

Now let \( f \in (M_{a,b}^m)' \). For each \( \phi \in M_{a,b}^m \) we have \( S^m_{\rho}(\phi) \in M_{a,b}^m \). Then the adjoint mapping

\[
\langle S^m_{\rho}(f), \phi \rangle = \langle f, S^m_{\rho}(\phi) \rangle
\]
defines the Stieltjes transform \( S^m_{\rho}(f) \in (M_{a,b}^m)' \) of \( f \).

**Theorem 3.2.** The Stieltjes transform is strictly positive and order bounded.

**Proof.** \((M_{a,b}^m)'\) is an ordered vector space. We define an order relation on the field of complex numbers by identifying the positive cone to be the set of complex numbers \( \alpha + i \beta, \alpha > 0, \beta > 0 \). If \( f > 0 \), \( S^m_{\rho}(f) > 0 \). So \( S^m_{\rho} \) is strictly positive and hence is order bounded.

For a non-negative integer \( n \), a differential operator \( L_n \) for \( \rho + n > \frac{1}{m} \) is defined by

\[
L_n = L_{n,x}(\phi(x)) = \frac{(-1)^n m^{1-2n} \Gamma(\rho)}{\Gamma(n + \frac{1}{m}) \Gamma(\rho + n - \frac{1}{m})} \frac{d^n}{dx^n} \left( (x^{1-m} \frac{d}{dx})^{n-1} x^{2mn + m\rho - m(1-m)\frac{d}{dx}} \right) \phi(x)
\]

The Stieltjes transform can be inverted by the application of this differential operator. The formal adjoint of this operator is itself.

We make the following conclusions from the results proved by John J. K. [2].

**Result 3.1.** If \( \rho + n > \frac{1}{m} \), \( \int_0^\infty L_{n,x}(x^m + t^m)^{-\rho} dt = 1 \).

**Result 3.2.** \( L_{n,t} \) maps \( C \cap M_{a,b}^m \) continuously into \( C \cap M_{a,b}^m \), provided \( \alpha > 1 - \frac{1}{m}, \beta < \rho + 1 - \frac{1}{m}, a = \frac{1}{m} + \alpha - \rho, b = \frac{1}{m} + \beta - \rho \).

**Result 3.3.** If \( L_n \) is the differential operator and \( S \) is the Stieltjes transform operator then either \( x^{1-m\rho} L_n \) and \( S x^{m\rho-1} \) or \( L_n x^{1-m\rho} \) and \( x^{m\rho-1} S \) commute on \( M_{a,b}^m \) where

\[
a = \frac{1}{m} + \alpha - \rho, \quad b = \frac{1}{m} + \beta - \rho.
\]

That is

\[
x^{m\rho-1} S^m_{\rho} [L_{n,t}(\phi)] = x^{m\rho-1} [L_{n,t}(\phi)]^\wedge
\]

\[
= L_{n,x} \int_0^\infty (x^m + t^m)^{-\rho} t^{m\rho-1} \phi(t) dt
\]

\[
= L_{n,x} S^m_{\rho} (t^{m\rho-1} \phi), \quad \phi \in M_{a,b}^m.
\]

**Result 3.4.** If \( \alpha > 1 - \frac{1}{m}, \beta < \rho + 1 - \frac{1}{m} \), the sequence \( \{L_{n,x} \hat{\phi}(x)\} \) converges in \( C \cap M_{a,b}^m \) to \( \phi(x) \).

**Result 3.5.** Let \( a = \frac{1}{m} + \alpha - \rho, b = \frac{1}{m} + \beta - \rho \). Then \( (L_n \phi)^\wedge \) converges to \( \phi \) in \( M_{a,b}^m \) as \( n \to \infty \).
Result 3.6. Let \( f \in (M_{a,b}^m)' \). Then \( f \in C' \) if and only if for every non-negative integer \( n \), \( L_nS^m_\rho(f) \in C' \) where \( C' \) is the positive cone in \((M_{a,b}^m)'\). It follows that \( L_n \) is strictly positive and hence is order-bounded. We have the following results on inversion of the Stieltjes transform.

For \( f \in (M_{a,b}^m)' \), \( \phi \in M_{a,b}^m \)
\[
\langle S^m_\rho(L_n f), \phi \rangle = \langle f, L_n(S^m_\rho \phi) \rangle \to \langle f, \phi \rangle \text{ as } n \to \infty.
\]

For \( f \in (M_{a,b}^m)' \), \( \phi \in M_{a,b}^m \)
\[
\langle L_n(S^m_\rho f), \phi \rangle = \langle f, S^m_\rho(L_n \phi) \rangle \to \langle f, \phi \rangle.
\]

4. Abelian and Tauberian Theorems

We have made a study of the spaces \( D, E \) and their duals with order relation defined on these spaces and with the topology of bounded convergence assigned to the dual spaces [1]. Here we consider the space \( \zeta \) of test functions of rapid descent with an order relation defined on it and its dual \( \zeta' \), the space of tempered distribution. An order relation is defined on \( \zeta' \) and the topology of bounded convergence is assigned to it. We introduce the notion of strong asymptotic behaviour of \( f \) and study the strong asymptotic behaviour of \( S^m_\rho(f) \) with reference to the strong asymptotic behaviour of \( f \) and vice versa in the present context.

We have the relations \( D \subseteq \zeta \) and \( \zeta' \subseteq D' \) [4].

Definition 4.1. Restricting \( \zeta \) to the real-valued functions an order relation is defined on \( \zeta \) by identifying the positive cone to be the set of all non-negative functions in \( \zeta \). The cone \( C + iC \) is the positive cone of \( \zeta \) which is also denoted as \( C \).

Result 4.1. The restriction of \( C \) to \( D \) is the positive cone of \( D \).

Definition 4.2. An order relation is defined on \( \zeta' \) by identifying the positive cone of \( \zeta' \) to be the dual cone \( C' \) of \( C \) in \( \zeta \).

Result 4.2. The topology of \( \zeta' \) is the topology of bounded convergence which is the same as the subspace topology on \( \zeta' \) when \( D' \) is assigned the topology of bounded convergence.

Proof. We prove that \( B_\zeta^{\zeta'} = B_\zeta^{D'} \cap \zeta' \) where \( B_\zeta^{\zeta'} \) is the polar of \( B_\zeta' = \{ \psi \in \zeta' : |f(\psi)| < \epsilon t \text{ for some } f \in \zeta' \} \), for some \( \epsilon > 0 \), for all \( t > s \), \( t, s \in R \) and \( B_\zeta^{D'} \) is the polar of \( B_D \), the corresponding set of elements on \( D \). We know that the elements of \( \zeta' \) are precisely those members of \( D' \) that have continuous extensions to \( \zeta' \). So if \( f_1 \in B_\zeta^{D'}, f_1 \in \zeta' \), \( |f_1(\psi)| < 1 \) for all \( \psi \in B_\zeta' \). If \( \psi \in B_D \), \( |f(\psi)| < \epsilon t \) for some \( f \in D' \), \( \psi \in B_\zeta' \) and \( |f(\psi)| < 1 \) for all \( \psi \in B_D \). Since \( f_1 \in \zeta' \), \( f_1 \in D' \) such that \( f_1 \) has a continuous extension to \( \zeta' \), so \( f_1 \in B_\zeta^{D'} \). Thus \( B_\zeta^{\zeta'} = B_\zeta^{D'} \cap \zeta' \).

Conversely, if \( f_1 \in B_\zeta^{D'} \cap \zeta' \), then \( f_1 \in \zeta' \) and \( |f_1(\psi)| < 1 \), for all \( \psi \in B_D \). If \( \psi \in B_D \), \( |f(\psi)| < \epsilon t \) for \( f \in D' \). Since \( \zeta' \subseteq D' \), \( \psi \in B_\zeta' \) and so \( f_1 \in B_\zeta' \) and we have \( B_\zeta^{D'} \cap \zeta' \subseteq B_\zeta^{\zeta'} \). \( \square \)
The elements $f$ of $\zeta'$ whose support is contained in $[0, \infty)$ form a subspace of $\zeta'$ which is denoted as $\zeta'_+$. $\zeta'_+$ denotes the space of $C^\infty$-functions on $[0, \infty)$ equipped with the topology induced by the topology on $\zeta'$.

**Definition 4.3.** A function $\nu(k)$ which is positive and continuous on $(0, \infty)$ is said to be regularly varying of order $r$, $r$ real, if for any $a > 0$ the limit

$$
\lim_{k \to \infty} \frac{\nu(ak)}{\nu(k)} = a^r
$$

exists. In the sense of convergence in $\zeta'$ with respect to the topology of bounded convergence, provided $F \neq 0$. $F$ is a homogeneous function of order $r$ and hence $F \in \zeta'$ and support of $F \subseteq (0, \infty)$. By saying that $\lim_{k \to \infty} \frac{f(kt)}{\nu(k)} = F(t)$ exists in $\zeta'$ with respect to the topology of bounded convergence what we mean is that if $F(t) \in B^o$, where $B^o$ is a basis element for the topology of bounded convergence, $\frac{f(kt)}{\nu(k)} \in B^o$ for $k \geq k_0$.

**Lemma 4.1.** Let $a = \frac{1}{m} + \alpha - \rho$, $b = \frac{1}{m} + \beta - \rho$. Then the set $A = \{S^m_\rho(\phi) : \phi \in M_{a,b}^m\}$ is dense in $M_{a,b}^m$.

**Proof.** Let $\phi \in M_{a,b}^m$. Then $\phi_n = S^m_\rho(L_{n,t}\phi)$ are elements of $A$ and $\phi_n \to \phi$ in $M_{a,b}^m$.

**Theorem 4.1.** Let $f \in (M_{a,b}^m)'$ and $\nu(k)$ be a regularly varying function of order $r > (-ma)$. The following statements are equivalent

(i) $f$ has in $C' \cap (M_{a,b}^m)'$ strongasymptotic behaviour at $\infty$ with respect to $\nu(k)$.

(ii) $S^m_\rho(f)$ has in $C' \cap (M_{a,b}^m)'$ strongasymptotic behaviour at $\infty$ with respect to $k^{1-m}\nu(k)$ and $\frac{1}{k^{1-m}\nu(k)}L_{n,t}S^m_\rho(f)(kx)$, for $k > 0$ is uniformly continuous in the topology of bounded convergence in $C' \cap (M_{a,b}^m)'$ for all values of $n$.

**Proof.** (i) $\Rightarrow$ (ii)

Let $\phi \in M_{a,b}^m$, $\phi_1 \in M_{a,b}^m$. Also let $\lim_{k \to \infty} \frac{f(kt)}{\nu(k)} = g(t)$. Then we have

$$
\lim_{k \to \infty} \frac{1}{\nu(k)} \langle f(kt), \phi(t) \rangle = \langle g(t), \phi(t) \rangle
$$
If \( \phi_1 \in M^m_{a,\beta} \), \( S^m_\rho(\phi_1) \in M^m_{a,b} \).
Hence
\[
\langle g(t), S^m_\rho(\phi_1) \rangle = \lim_{k \to \infty} \frac{1}{\nu(k)} \langle f(kt), S^m_\rho(\phi_1)(t) \rangle = \lim_{k \to \infty} \frac{1}{k^{1-m}\nu(k)} \langle S^m_\rho(f)(kx), \phi_1(x) \rangle.
\]
But \( \langle g(t), S^m_\rho(\phi_1) \rangle = \langle S^m_\rho(g)(x), \phi_1(x) \rangle \).
So we conclude that
\[
\lim_{k \to \infty} \frac{1}{k^{1-m}\nu(k)} S^m_\rho(f)(kx) = S^m_\rho(g)(x).
\]
This means that \( S^m_\rho \) has in \( C' \cap (M^m_{a,\beta})' \) strong asymptotic behaviour at infinity with respect to \( k^{1-m}\nu(k) \).
\[
\left| \frac{1}{k^{1-m}\nu(k)} \langle L_n S^m_\rho(f)(kx), \phi(x) \rangle \right| = \left| \frac{f(kt)}{\nu(k)} S^m_\rho L_n(\phi)(t) \right|.
\]
Since \( \lim_{k \to \infty} \frac{f(kt)}{\nu(k)} = F(t) \) in \( \zeta' \), \( \left| \frac{f(kt)}{\nu(k)}, \psi \right| < 1 \), for all \( \psi \in B, k \geq k_0 \) whenever \( |\langle F(t), \psi \rangle| < 1 \) for all \( \psi \in B \) where \( B = \{ \psi \in \zeta' : |f(\psi)| < t\epsilon \text{ for some } f \in \zeta' \} \) for some \( \epsilon > 0 \), for all \( t > s, t, s \in R \). Since \( \frac{f(kt)}{\nu(k)} \) is a continuous linear functional, there exists a positive constant \( C_1 \) and integers \( j, q \) such that
\[
\left| \frac{1}{k^{1-m}\nu(k)} \langle L_n S^m_\rho(f)(kx), \phi(x) \rangle \right| \leq C_1 \mu^m_{a,b,K,j,q}(S^m_\rho L_n(\phi))
\]
\[
\bar{\mu}^m_{a,b,K,j,q}(S^m_\rho L_n(\phi)) \leq \bar{\mu}^m_{a,b,K,j,q}(S^m_\rho L_n(\phi) - \phi) + \bar{\mu}^m_{a,b,K,j,q}(\phi).
\]
Using results 3.3, 3.4 and 3.5,
\[
\bar{\mu}^m_{a,b,K,j,q}(S^m_\rho L_n(\phi) - \phi) = \bar{\mu}^m_{a,b,K,j,q}(k^{1-m}\rho(L_n S^m_\rho(t^{m-1}\phi(t)) - \phi(x)))
\]
\[
\leq C_2 \bar{\mu}^m_{a,\alpha,K,j,q}(L_n S^m_\rho(\phi_0) - \phi_0)
\]
where \( \phi_0(t) = t^{m-1}\phi(t) \in M^m_{a,\beta,K,j} \)
\[
\leq \epsilon_n C_2 \bar{\mu}^m_{a,\alpha,K,j,q+1}(\phi_0)
\]
\[
\leq \epsilon_n C_3 \bar{\mu}^m_{a,b,K,j,q+1}(\phi).
\]
Consequently, \( \bar{\mu}^m_{a,b,K,j,q}(S^m_\rho L_n(\phi)) \leq C_4 \bar{\mu}^m_{a,b,K,j,q+1}(\phi) \).
Hence
\[
\left| \frac{1}{k^{1-m}\nu(k)} \langle L_n S^m_\rho(f)(kx), \phi \rangle \right| \leq C_5 \bar{\mu}^m_{a,b,K,j,q+1}(\phi)
\]
for all \( n \), uniformly for \( k \geq k_0 \) where \( C_5 \) is a constant depending only on \( f, j \) and \( q \).

(ii) \( \Rightarrow \) (i)

Let
\[
\lim_{k \to \infty} \frac{S^m_\rho(f)(kt)}{k^{1-m}\nu(k)} = S^m_\rho(g)(t), \quad \text{say}.
\]
Then
\[ \lim_{k \to \infty} \frac{1}{k^{1-m_p \nu(k)}} (S^m(f)(kx), \phi_1(x)) = \langle S^m(f)(x), \phi_1(x) \rangle = \langle g(t), S^m_\rho(\phi_1)(t) \rangle \]

But
\[ \lim_{k \to \infty} \frac{1}{k^{1-m_p \nu(k)}} (S^m(f)(kx), \phi_1(x)) = \lim_{k \to \infty} \frac{1}{\nu(k)} (f(kt), S^m_\rho(\phi_1)(t)) \]

It follows that \( \lim_{k \to \infty} \frac{f(kt)}{\nu(k)} = g(t) \) on a dense set of elements of the space \( M^m_{a,b} \). Now we prove that the set \( \{ f(kt)/\nu(k) : k \geq k_0 \} \) is bounded in \( (M^m_{a,b})' \). By the theorem of uniform convergence it follows that \( f \) has in \( (M^m_{a,b})' \) strongasymtotic behaviour at \( \infty \) with respect to \( \nu(k) \). Let \( \phi \in M^m_{a,b} \). Since
\[ \frac{1}{k^{1-m_p \nu(k)}} L_n S^m_\rho(f)(kx), k \geq k_0 \]
is uniformly continuous in \( (M^m_{a,b})' \) there exists a constant \( C_1 > 0 \) and positive integers \( j, q \) such that
\[ \left| \frac{1}{k^{1-m_p \nu(k)}} (L_n S^m_\rho(f)(kx), \phi_1(x)) \right| \leq C_1 \mu^m_{a,b,K,j,q}(\phi) \]
for \( k \geq k_0 \), for all \( n \in \mathbb{Z}_+ \). Since
\[ \left| \frac{1}{k^{1-m_p \nu(k)}} (L_n S^m_\rho(f)(kx), \phi(x)) \right| = \left| \frac{1}{\nu(k)} (f(kt), S^m_\rho(L_n \phi)(t)) \right| \]
it follows that
\[ \left| \frac{1}{\nu(k)} (f(kt), \phi(t)) \right| \leq C_2 \mu^m_{a,b,K,j,q+1}(\phi). \]

Hence the theorem.

**Corollary 4.2.** If \( (f_a)_{a \in J} \) is a monotone net in \( C' \cap (M^m_{a,b})' \) having strongasymtotic behaviour at \( \infty \) with respect to a regularly varying function \( \nu(k) \) then \( (S^m_\rho(f_a))_{a \in J} \) converges to a function \( f \in C' \cap (M^m_{a,b})' \) having strongasymtotic behaviour with respect to \( k^{1-m_p \nu(k)} \).

Also, \( \frac{1}{k^{1-m_p \nu(k)}} L_n S^m_\rho(f)(kx) \) for \( k \geq k_0 \) is uniformly continuous with respect to the topology of bounded convergence in \( C' \cap (M^m_{a,b})' \) for all values of \( n \).

**Proof.** Follows from Theorem 4.1 since \( (M^m_{a,b})' \) is order complete.

**Theorem 4.3.** Let \( f \in (M^m_{a,b})' \) and \( \nu(k) \) be a regularly varying function of order \( \gamma > (-ma) \). If \( S^m_\rho(f) \) has strongasymtotic behaviour at \( \infty \) with respect to \( k^{1-m_p \nu(k)} \) in \( C' \cap (M^m_{a,b})' \) then \( \frac{1}{k^{1-m_p \nu(k)}} L_n S^m_\rho(f)(kx) \) for \( k \geq k_0 \) is uniformly continuous with respect to the topology of bounded convergence in \( C' \cap (M^m_{a,b})' \) for all values of \( n \).
Corollary 4.4. Let \((f_\alpha)_{\alpha \in J}\) be a monotone net in \(C' \cap (M_{a,b}^m)'\) such that \((S_\rho^m(f_\alpha))_{\alpha \in J}\) has strong asymptotic behaviour at \(\infty\) with respect to \(k^{1-m_\rho} \nu(k)\) in \(C' \cap (M_{a,b}^m)'</span> where \(\nu(k)\) is regularly varying function of order \(r > (\sigma - ma)\). Then \((S_\rho^m(f_\alpha))_{\alpha \in J}\) converges to a function \(f \in C' \cap (M_{a,b}^m)'</span> and \(\frac{1}{k^{1-m_\rho} \nu(k)} L_{n,x} S_\rho^m(f)(kx)\) for \(k \geq k_0\) is uniformly continuous with respect to the topology of bounded convergence in \(C' \cap (M_{a,b}^m)'</span> for all values of \(n\).

Proof. Follows from Theorem 4.3 since \((M_{a,b}^m)'\) is order complete.

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References


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