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A Note on the Oscillation of Second-Order Nonlinear Neutral Functional Differential Equations[1]

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Abstract

In this paper, some sufficient conditions for oscillation of second-order nonlinear neutral functional differential equation

$$[x(t) + p(t)x(t - \tau)]'' + q(t)f(x(t - \delta)) = 0, \quad t \geq 0$$

are established, for the case: $-1 \leq p(t) \leq 0$ or $0 \leq p(t) < \infty$, $q(t)$ is sign-constant. We note that most the references in the literature devoted themselves to considering the case: $0 \leq p(t) \leq 1$, while there is few paper considering the case: $0 \leq p(t) < \infty$. This paper is devoted to filling this gap. The method is based on the Riccati transformation and integral averaging technique. Finally, some illustrating examples are presented to show the feasibility and effectiveness of our results.

Keywords: Oscillation; Neutral; Second order differential equations

1. Introduction

Oscillation (including the periodic oscillation, almost periodic oscillation) of functional differential equations or difference equations has been received great attention and has been studied extensively (see e.g. [1-29]). This direction is and will be one of the main topics of the study for differential equations and difference equations. The main purpose of this paper is to find oscillation criteria for second-order nonlinear neutral functional differential equation

$$[x(t) + p(t)x(t - \tau)]'' + q(t)f(x(t - \delta)) = 0, \quad t \geq 0, \tag{1.1}$$

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where $\tau, \delta > 0$, $p, q \in C([0,\infty), (-\infty,\infty))$ and $f \in C((-\infty,\infty), (-\infty,\infty))$. As usual, a solution $x(t)$ of equation (1.1) is said to be oscillatory if it has arbitrarily large zeros, otherwise it is said to be non-oscillatory. Equation (1.1) is said to be oscillatory if its every solution is oscillatory.

Many references to some applications of equation (1.1) can be found in [15-16]. The oscillation of this type differential equations has attracted much attention of many authors in the last decade (see [1-12]). Sorted by the value of function $p(t)$, there are following four cases of research on this problem. How about the progress in this aspect? We give a list as follows:

1. $-1 \le p(t) \le 0$. Only references [10-12] are found devoted to this case.

2. $0 \le p(t) \le 1$. This case is investigated most fully of the four cases, such as [1-5,7-10].

3. $1 < p(t) < \infty$. For this case we only find reference [7].

4. $p(t)$ is oscillatory. Only reference [6] deals with this case, under condition $\lim_{t\to\infty} p(t) = 0$.

If $p(t) \equiv$ constant p, the oscillation of equation (1.1) can be studied by its characteristic equation (see, for example [8]). In the papers where $p(t) \not\equiv p$, Riccati transformation $y(t) = x(t) + p(t)x(t-\tau)$ is usually used to transform neutral equation (1.1) to some certain non-neutral equation (see e.g. [1-7, 9-12]).

Except for the Case 2, it is difficult to find much references for the other three cases respectively. The reason for this contrast may be due to the fact that it is more difficult to perform a study of the function $y(t) = x(t)+p(t)x(t-\tau)$ for $x(t) > 0$ in the Case 1, Case 3 and Case 4. Especially for the Case 3, we only find Tanaka [7] gives a sufficient and necessary condition. But as far as we know the oscillation criteria for the case $0 \le p(t) \le \infty$ has not been studied yet. In this paper, we want to fill this gap (see Theorem 3.1 in Section 3).

From the literature [1-11], one can find that all of these literatures under the conditions $xf(x) > 0$ for $x > 0$ and $q(t) \ge 0$ for $t \ge 0$ which are employed to ensure that $y''(t) = [x(t) + p(t)x(t-\tau)]'' = -q(t)f(x(t-\delta)) \le 0$ for all $x(t) > 0$ and $t \ge t_0 > 0$. In additional, either $f(x)$ is non-decreasing or $f(x)/x \ge k$ (k is a positive constant) or both of them are needed in these literature, which make it is possible to study nonlinear equation (1.1) by a linear equation or inequality. In fact, such conditions have become standard in the related literature, besides [1-11], we refer the readers to the papers [13-14].

This paper deals with equation (1.1) with the following cases (H_1) and (H_2) respectively:

(H_1) $-\mu \le p(t) \le 0$, where $\mu \in (0,1)$;
(H_2) $0 \le p(t) < \infty$.

Some illustrating examples will be given. In some sense, the results in this paper improve some previous works such as [10,11].

Throughout this paper, we assume that equation (1.1) satisfies the following hypothesis:
(H_3) $\frac{f(x)}{x} \ge k > 0$ for $x \ne 0$;
(H_4) $q(t) \ge M > 0$ for $t \ge 0$.

2. The case $-1 \le -\mu \le p(t) \le 0$

Consider first-order differential linear differential inequalities:

$$y'(t) + a(t)y(t-\tau) \le 0, \quad t \ge 0, \tag{2.1}$$

and

$$y'(t) + a(t)y(t-\tau) \ge 0, \quad t \ge 0 \tag{2.2}$$

where $a(t) \in C(R,(0,\infty))$, $\tau > 0$.

In the proof of our main results, we need the following well-known oscillation criteria for inequalities (2.1) and (2.2) (see [15]).

Lemma 2.1. **[15]** Assume that

$$\liminf_{t\to\infty} \int_{t-\tau}^{t} a(s)ds > \frac{1}{e}. \tag{2.3}$$

Then inequality (2.1) has no eventually positive solutions and inequality (2.2) has no eventually negative solutions.

Theorem 2.1. Assume that $\delta > 2\tau$ and (H_1), (H_3) hold. Assume further that $f(x)$ is nondecreasing and
(H_5) $q(t) \ge \frac{\mu}{k\,e\,\tau}$, where k is as in (H_3).
Then equation (1.1) is oscillatory.

Proof. Assume that equation (1.1) has a non-oscillatory solution $x(t)$. Without loss of generality, assume that $x(t)$ is eventually positive (the proof is similar when $x(t)$ is eventually negative). That is, let $x(t) > 0$ for $t \ge t_0 - \delta > 0$. Set

$$y(t) = x(t) + p(t)x(t-\tau).$$

Then it follows from (1.1) that

$$y''(t) = -q(t)f(x(t-\delta)) \le 0 (\not\equiv 0), \tag{2.4}$$

which implies that there exists $t_1 \ge t_0$ such that $y'(t) > 0$ or $y'(t) < 0$ for $t \ge t_1$.

1^0 If $y'(t) > 0$ for $t \geq t_1$. Then there exists $t_2 > t_1$ such that $y(t) > 0$ or $y(t) < 0$ for $t \geq t_2 - \delta$.

First let us consider the case of $y'(t) > 0$ and $y(t) > 0$ for $t \geq t_2 - \delta$. Since $x(t-\delta) \geq y(t-\delta)$ and $f(x)$ is non-decreasing, it follows from (2.4) that

$$y''(t) + q(t)f(y(t-\delta)) \leq 0, \quad t \geq t_2.$$

Integrating the above inequality from t_2 to ∞, and note that $y(t)$ is increasing on $[t_2 - \delta, \infty)$, we get

$$\begin{aligned} \infty > y'(t_2) &\geq y'(\infty) + \int_{t_2}^{\infty} q(s)f(y(s-\delta))ds \\ &\geq \int_{t_2}^{\infty} q(s)f(y(s-\delta))ds \\ &\geq f(y(t_2-\delta)) \int_{t_2}^{\infty} q(s)ds \to \infty \end{aligned}$$

which is a contradiction.

Second we claim that the case of $y'(t) > 0$ and $y(t) < 0$ for $t \geq t_2 - \delta$ doesn't hold. Otherwise, since $x(t-\delta) > -\frac{1}{\mu}y(t+\tau-\delta)$ and $f(x)$ is nondecreasing, it follows from (2.4) that

$$y''(t) + q(t)f(-\frac{1}{\mu}y(t+\tau-\delta)) \leq 0, \quad t \geq t_2.$$

Integrating the above inequality from t (here $t \geq t_2 - \delta$) to $t+\tau$, we have

$$-y'(t) + \int_t^{t+\tau} q(s)f(-\frac{1}{\mu}y(s+\tau-\delta)ds \leq -y'(t+\tau) < 0.$$

Noting that $-\frac{1}{\mu}y(t)$ is decreasing on $[t_2 - \tau, \infty)$, we have

$$-y'(t) + f(-\frac{1}{\mu}y(t+2\tau-\delta)) \int_t^{t+\tau} q(s)ds \ < 0.$$

By condition (H_3), we have

$$-y'(t) - y(t+2\tau-\delta)\frac{k}{\mu}\int_t^{t+\tau} q(s)ds \ < 0.$$

That is

$$y'(t) + y(t+2\tau-\delta)\frac{k}{\mu}\int_t^{t+\tau} q(s)ds \ > 0. \tag{2.5}$$

Applying Lemma 2.1 with $a(t) = \frac{k}{\mu}\int_t^{t+\tau} q(s)ds$, by (H_5) we

$$\int_{t-\tau}^{t} a(s)ds \geq \frac{1}{e},$$

therefore inequality (2.5) has no eventually negative solutions which contradicts the assumption that $y(t) < 0$ for $t \geq t_2 - \delta$.

2^0 If $y'(t) < 0$ for $t \geq t_1$. The inequality $y'(t) \leq y'(t_1) < 0$ implies that

$$\lim_{t\to\infty} y(t) = -\infty.$$

Then there exist $c > 0$ and $t_3 > t_0$ such that

$$x(t) + p(t)x(t-\tau) \leq -c, \quad t \geq t_3,$$

which yields that

$$x(t+\tau) \leq -c + \mu x(t), \quad t \geq t_3.$$

Continuing inductively, we get

$$x(t_3 + n\tau) \leq -\sum_{i=1}^{n} c\mu^{i-1} + \mu^n x(t_3).$$

Therefore $x(t_3+n\tau) < 0$ for sufficiently large n which contradicts the assumption that $x(t) > 0$ for $t \geq t_1$. This completes the proof. □

Remark 2.1. From the proof, Theorem 2.1 holds also for $\mu = -1$. But in the following theorem 2.2, $\mu \neq -1$.

Next we will give another sufficient condition for the oscillation of equation (1.1), conditions of the Theorem 2.1 are weakened. As a result, we obtain a more conservative result.

Theorem 2.2. Suppose that (H_1), (H_3) and (H_4) hold. Then every solution of equation (1.1) either oscillates or tends to zero as $t \to \infty$.

Proof. Assume that $x(t) > 0$ for $t \geq t_0 - \delta > 0$ is a solution of equation (1.1). Set

$$y(t) = x(t) + p(t)x(t-\tau).$$

Then it follows from (1.1) that

$$y''(t) = -q(t)f(x(t-\delta)) \leq 0 (\not\equiv 0), \tag{2.6}$$

which implies that there exists $t_1 \geq t_0$ such that $y'(t) > 0$ or $y'(t) < 0$ for $t \geq t_1$.

1^0 We claim that the case $y'(t) > 0$ for $t > t_1$ does not hold. Otherwise, integrating (2.6), we have

$$\infty > y'(t_1) \geq y'(t_1) - y'(t) = \int_{t_1}^{t} q(s)f(x(s-\delta))ds \geq kM \int_{t_1}^{t} x(s-\delta)ds,$$

which implies that $x(t) \in L(t_1, \infty)$. Then

$$y(t) = x(t) + p(t)x(t-\tau) \in L(t_1, \infty). \tag{2.7}$$

Since $y(t)$ is increasing and integrable, then $\lim_{t\to\infty} y(t) = 0$. We claim that $x(t)$ is bounded in $[t_1, \infty)$. If not, then $\limsup_{t\to\infty} x(t) = \infty$ so that there exists a sequence $t_n \geq t_1$ satisfying $x(t_n) \geq x(t_n - \tau)$, $x(t_n) \to \infty$ as $t \to \infty$. Then, we have

$$y(t_n) = x(t_n) + p(t_n)x(t_n - \tau) \geq x(t_n) - \mu x(t_n - \tau) = (1-\mu)x(t_n) \to \infty, \ \ t \to \infty,$$

which contract the fact that $\lim_{t\to\infty} y(t) = 0$.

From this claim, there exists $c \geq 0$ such that $\limsup_{t\to\infty} x(t) = c$. Then

$$\begin{aligned} 0 &= \limsup_{t\to\infty}[x(t) + p(t)x(t-\tau] \\ &\geq \limsup_{t\to\infty}[x(t) - \mu x(t-\tau)] \\ &\geq \limsup_{t\to\infty} x(t) + \liminf_{t\to\infty}[-\mu x(t-\tau)] \\ &= \limsup_{t\to\infty} x(t) - \mu \limsup_{t\to\infty} x(t-\tau) \\ &= (1-\mu)c. \end{aligned}$$

Therefore $\limsup_{t\to\infty} x(t) = c = 0$, which implies that $\lim_{t\to\infty} x(t) = 0$.

2^0 The case $y'(t) < 0$ for $t \geq t_1$ does not hold, the proof is the same to that of Theorem 2.1. This completes the proof. □

3. The case $0 \leq p(t) < \infty$

In this section, we establish the oscillation criteria for equation (1.1) in the case $0 \leq p(t) < \infty$. Comparing the following theorem 3.1 with Theorem 2.2, maybe one can get some idea about why it is more difficult to deal with $-1 \leq p(t) \leq 0$ than $0 \leq p(t) \leq 1$, as we point out in our introduction.

Theorem 3.1. Suppose that (H_2)-(H_4) hold. Then equation (1.1) is oscillatory.

Proof. Assume that $x(t) > 0$ for $t \geq t_0 - \delta > 0$ is a solution of equation (1.1). Set

$$y(t) = x(t) + p(t)x(t-\tau).$$

Then $y(t) > 0$, and it follows from (1.1) that

$$y''(t) = -q(t)f(x(t-\delta)) \leq 0 (\not\equiv 0)$$

which implies that there exists $t_1 \geq t_0$ such that $y'(t) > 0$ or $y'(t) < 0$ for $t \geq t_1$.

1^0 If $y'(t) > 0$ for $t > t_1$. Then the same to the proof of (2.7), we get

$$y(t) = x(t) + p(t)x(t - \tau) \in L(t_1, \infty).$$

But inequality $y(t) \geq y(t_1) > 0$ implies that $y(t) \notin L(t_1, \infty)$. We obtain a contradiction.

2^0 If $y'(t) < 0$ for $t > t_1 \geq t_0$, then

$$\lim_{t\to\infty} y(t) = -\infty$$

which contradicts to the fact that $y(t) > 0$ for $t \geq t_0 - \delta > 0$.

□

4. Examples

In this section, we will give some applications of our oscillation criteria for equation (1.1). For the case $-1 \leq -\mu \leq p(t) \leq 0$, we only find literature [10-12]. In reference [11] the author gives a counterexample to show that the theorem in reference [12] is false. We will give two oscillatory equations based on our theorems, but it cannot be demonstrated by [10-11] as follows:

Example 4.1. Consider the equation

$$[x(t) - x(t - 0.4))]'' + (t + 1)[x(t - 1) + x^3(t - 1)] = 0, \quad t \geq 0. \tag{4.1}$$

Here $p(t) \equiv 1, q(t) = t + 1$ and $f(x) = x + x^3$. By our remark 2.1, equation (4.1) is oscillatory. The results in [10] fail to apply to equation (1.1) with $p(t) \equiv -1$. There are two oscillation criteria in [11], Theorem 2.1 [11] fails to the case $q(t) = t + 1$, and $\int_0^\varepsilon f(x)dx = +\infty$ for any $\varepsilon > 0$ makes Theorem 3.1 [11] invalid to equation (4.1).

Example 4.2. Consider the equation

$$[x(t) - x(t - \tau)]'' + (t + 1)[2x(t - \delta) + x(t - \delta)\sin x(t - \delta)] = 0, \quad t \geq 0, \tag{4.2}$$

where $\tau, \delta \geq 0$. By Theorem 2.2, every solution $x(t)$ is oscillatory or $\lim_{t\to} x(t) = 0$. Here $f(x) = 2x + x \sin x$ is nondecreasing which is one of conditions in the oscillation theorems in [11].

We will give an application example for Theorem 3.1 as follows:

Example 4.3. Consider the equation

$$[x(t) + (3 + \sin t)x(t - \tau)]'' + e^{\cos t}[2x(t - \delta) + x(t - \delta)\sin x(t - \delta)], \quad t \geq 0, \tag{4.3}$$

for any $\tau, \delta \geq 0$. By Theorem 3.1, equation (4.3) is oscillatory.

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