A New Approach to the Numerical Solution of Linear Volterra Integral Equations of the Second Kind

A. Tahmasbi *

Department of Applied Mathematics
Damghan University of Basic Sciences, Damghan, Iran

Abstract

In this paper, we present a numerical method for solving linear Volterra integral equations of the second kind based on the power series method. The proposed method provides the Taylor expansion of the exact solution of the integral equation using simple computations with quite acceptable approximate solution. In the case of equations with polynomial solutions, the proposed method gives exactly the same solutions as the analytical method does. Comparisons with other methods prove that our method is very effective and convenient.

Mathematics Subject Classification: 35D05

Keywords: Linear Volterra Integral Equation; Numerical Method; Power Series Method

1. Introduction

There are many numerical methods for solving linear Volterra integral equations; Rashidinia and Zarebnia [1] obtained a numerical solution of the integral equation by Sinc-collection method, in [2] Babolian and Davari solved the integral equation numerically based on Adomian decomposition method and recently in [3], Saberi and Heidari used a quadratic method with variable step for solving linear Volterra integral equation of the second kind. In this paper, we present a novel and very
simple numerical method based upon power series, the main advantage of which is
the capability to obtain the exact solution of linear Volterra integral equation when
the analytical solution is a polynomial. In Section 2, the details of the proposed
method are explained.

2. The method of solution

Consider the linear Volterra integral equation of the second kind as follows:
\[ y(x) = g(x) + \int_0^x k(x,t) y(t) \, dt, \quad 0 \leq x \leq T \]  
(2.1)

where, \( y(x) \) is an unknown function; \( g(x) \) and \( k(x,t) \) are given functions, which
\( k(x,t) \) is called the kernel of integral equation. We assume that (2.1) have a unique
solution. Also, we suppose, without any loss of generality, that \( g(x) \) and \( k(x,t) \) are
polynomials, since any function can be approximated by a power series. Our
proposed method requires a starting point i.e. initial condition. In order to obtain the
initial condition for the integral equation, we set \( x = 0 \) in (2.1) and obtain
\( y(0) = g(0) \). In other words, \( y_0 = y(0) \) will be the initial condition to (2.1).

Now, we suppose that the solution of (2.1) with the initial condition \( y_0 = g(0) \) is as
follows, where, \( e \) is an unknown parameter.
\[ y(x) = y_0 + ex \]  
(2.2)

Substituting (2.2) in (2.1) gives the following algebraic equation:
\[ (e-a)x + Q(x) = 0 \]  
(2.3)

here, \( a \) is a known constant and \( Q(x) \) is a polynomial of order greater or equal
two.

By ignoring \( Q(x) \) in (2.3) and solving the equation \( e-a = 0 \), the unknown \( e \) will
be obtained. We put \( y_1 = e \). In the next step, we assume that the solution of equation
(2.1) to be
\[ y(x) = y_0 + y_1x + ex^2 \]  
(2.4)

Again, by substituting (2.4) into (2.1) we have the following algebraic equation:
\[ (e-a)x^2 + Q(x) = 0 \]  
(2.5)

where, \( Q(x) \) is a polynomial with order greater or equal three.

By neglecting \( Q(x) \) and solving \( e-a = 0 \), we obtain the unknown \( e \) and set \( y_2 = e \).

Having repeated the above procedure for \( N \) iterations, a power series of the
following form is derived:
\[ y(x) = y_0 + y_1x + y_2x^2 + ... + y_N x^N. \]  
(2.6)

Equation (2.6) will be an approximation for the exact solution \( y(x) \) of integral
equation (2.1).
3. Numerical examples

In this Section, we apply the presented method for solving integral equation (2.1) in four different examples using MATLAB® 7.3.

a. Example1. Consider the integral equation [2].

\[ y(x) = 1 - x - \frac{x^2}{2} + \int_0^x (x-t)y(t)\,dt, \quad 0 \leq x \leq 1. \]  

The theoretical solution is \( y(x) = 1 - \sinh x \). we solved the integral equation (3.1) by Adomian decomposition method discussed in [3] and the proposed method, the numerical results are given in Table 1.

<table>
<thead>
<tr>
<th>( x_i )</th>
<th>( y(x) ) \text{Adomian decomposition method}</th>
<th>\text{The proposed method by N=15}</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>\text{Approximation}</td>
<td>\text{Absolute error}</td>
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<td>1</td>
<td>0</td>
</tr>
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<td>0.2</td>
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<td>2.202 \times 10^{-3}</td>
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<td>0.8</td>
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</tr>
<tr>
<td>1</td>
<td>-0.17476298</td>
<td>4.382 \times 10^{-3}</td>
</tr>
</tbody>
</table>

b. Example2. Consider the integral equation [5]:

\[ y(x) = 1 + \int_0^x (t-x)y(t)\,dt, \quad 0 \leq x \leq 1, \]

Numerical results appear in Table 2 with N=13.

<table>
<thead>
<tr>
<th>( x_i )</th>
<th>\text{Exact} ( y(x) )</th>
<th>\text{Approximation} ( y(x) )</th>
<th>\text{Absolute error}</th>
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</thead>
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<td>1</td>
<td>1</td>
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<td>0.540302305879562</td>
<td>1.142 \times 10^{-11}</td>
</tr>
</tbody>
</table>

Exact solution of example 2 is \( y(x) = \cos(x) \).
4. Conclusion

The numerical examples solved in this work prove that the method introduced here can be simply implemented to linear Volterra integral equations of the second kind. The simple, easy-to-apply and fast algorithm of the proposed method is the main advantages over other existing methods.

References


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