The Semi Orlicz Spaces

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Abstract

The aim of this paper is to introduce a new class of sequence spaces, namely the semi-Orlicz spaces. It is shown that the intersection of all semi-Orlicz spaces is semi-Orlicz. The intersection of all semi-Orlicz spaces is \( \ell_M \) and \( \ell_M \) is smallest semi-Orlicz space.

Mathematics Subject Classification: 40A05, 40C05, 40D05

Keywords: Orlicz function, semi Orlicz, entire sequence

1 Introduction

A complex sequence, whose \( k^{th} \) terms is \( x_k \) is denoted by \( \{x_k\} \) or simply \( x \). Let \( w \) be the set of all sequences \( x = (x_k) \) and \( \phi \) be the set of all finite sequences. Let \( \ell_{\infty}, c, c_0 \) be the sequence spaces of bounded, convergent and null sequences \( x = (x_k) \) respectively. In respect of \( \ell_\infty, c, c_0 \) we have \( \|x\| = \sup_k |x_k| \), where \( x = (x_k) \in c_0 \subset c \subset \ell_\infty \). Orlicz [11] used the idea of Orlicz function to construct the space \( (L^M) \). Lindenstrauss and Tzafriri
[7] investigated Orlicz sequence spaces in more detail, and they proved that every Orlicz sequence space \( \ell_M \) contains a subspace isomorphic to \( \ell_p (1 \leq p < \infty) \). Subsequently different classes of sequence spaces defined by Parashar and Choudhary[12], Mursaleen et al.[8], Bektas and Altin[1], Tripathy et al.[16], Rao and subramanian[3] and many others. The Orlicz sequence spaces are the special cases of Orlicz spaces studied in Ref[5].

Recall([5],[11]) an Orlicz function is a function \( M: [0, \infty) \rightarrow [0, \infty) \) which is continuous, non-decreasing and convex with \( M(0) = 0, M(x) > 0 \), for \( x > 0 \) and \( M(x) \rightarrow \infty \) as \( x \rightarrow \infty \). If convexity of Orlicz function \( M \) is replaced by \( M(x+y) \leq M(x)+M(y) \) then this function is called modulus function, introduced by Nakano[10] and further discussed by Ruckle[13] and Maddox[9] and many others.

An Orlicz function \( M \) is said to satisfy \( \Delta_2 \)-condition for all values of \( u \), if there exists a constant \( K > 0 \), such that \( M(2u) \leq KM(u) (u \geq 0) \). The \( \Delta_2 \)-condition is equivalent to \( M(\ell u) \leq K\ell M(u) \), for all values of \( u \) and for \( \ell > 1 \).

Lindenstrauss and Tzafriri[7] used the idea of Orlicz function to construct Orlicz sequence space

\[
\ell_M = \left\{ x \in w : \sum_{k=1}^{\infty} M\left( \frac{|x_k|}{\rho} \right) < \infty, \text{ for some } \rho > 0 \right\}.
\]  

(1)

The space \( \ell_M \) with the norm

\[
\| x \| = \inf \left\{ \rho > 0 : \sum_{k=1}^{\infty} M\left( \frac{|x_k|}{\rho} \right) \leq 1 \right\}
\]

(2)

becomes a Banach space which is called an Orlicz sequence space. For \( M(t) = t^p, 1 \leq p < \infty \), the space \( \ell_M \) coincide with the classical sequence space \( \ell_p \). Given a sequence \( x = \{x_k\} \) its \( n^{th} \) section is the sequence \( x^{(n)} = \{x_1, x_2, ..., x_n, 0, 0, ...\} \) \( \delta^{(n)} = (0, 0, ..., 1, 0, 0, ...) \), 1 in the \( n^{th} \) place and zero’s else where. An FK-space (Frechet coordinate space) is a Frechet space which is made up of numerical sequences and has the property that the coordinate functionals \( p_k(x) = x_k (k = 1, 2, 3, ...) \) are continuous. We recall the following definitions [see [17]].

An FK-space is a locally convex Frechet space which is made up of sequences and has the property that coordinate projections are continuous. An FK-space \( X \) is said to have AK (or sectional convergence) if \( x^{(n)} \rightarrow x \) as \( n \rightarrow \infty \) holds for every \( x \in X \). The space is said to have AD (or) be an AD space if \( \phi \) is dense in \( X \). We note that AK implies AD by [2].

If \( X \) is a sequence space, we define

(i) \( X' = \) the continuous dual of \( X \).
(ii) \( X^\alpha = \{ a = (a_k) : \sum_{k=1}^{\infty} |a_kx_k| < \infty, \text{ for each } x \in X \} \);
(iii) $X^\beta = \{ a = (a_k) : \sum_{k=1}^{\infty} a_k x_k \text{ is convergent, foreach } x \in X \};$

(iv) $X^\gamma = \{ a = (a_k) : \sup_n |\sum_{k=1}^{n} a_k x_k| < \infty, \text{ foreach } x \in X \};$

(v) Let $X$ be an FK-space $\supset \phi$. Then $X^f = \{ f(\delta^{(n)}) : f \in X' \}.$

$X^\alpha, X^\beta, X^\gamma$ are called the $\alpha$−(or Kőthe-Toeplitz) dual of $X$, $\beta$− (or generalized Kőthe-Toeplitz) dual of $X$, $\gamma$−dual of $X$. Note that $X^\alpha \subset X^\beta \subset X^\gamma$. If $X \subset Y$ then $Y^\mu \subset X^\mu$, for $\mu = \alpha, \beta, \text{ or } \gamma$.

1.1 Lemma

(See (17, Theorem 7.27)). Let $X$ be an FK-space $\supset \phi$. Then

(i) $X^\gamma \subset X^f.$

(ii) If $X$ has AK, $X^\beta = X^f.$

(iii) If $X$ has AD, $X^\beta = X^\gamma.$

Because of the historical roots of summability in convergence, conservative space and matrices play a special role in its theory. However, the results seem mainly to depend on a weaker assumption, that the spaces be semi conservative. (See[17]). Snyder and Wilansky[14] introduced the concept of semi conservative spaces. Snyder[15] studied the properties of semi conservative spaces. Later on, in the year 1996 the semi replete spaces were introduced by Chandrasekhar Rao and Srinivasalu[4].

In a similar way, in this result we define semi-Orlicz spaces, and show that $\ell_M$ is smallest semi-Orlicz space.

2 Results: Properties of semi-Orlicz spaces

2.1 Definition

An FK-space $X$ is called ”semi-Orlicz” if its dual $X^f \subset \ell_N$. In other words $X$ is semi-Orlicz if $f(\delta^{(k)}) \in \ell_N \forall f \in X'$ for each fixed $k$.

2.2 Example

$\ell_M$ is semi-orlicz. Indeed if $\ell_M$ is the space of all Orlicz sequence, then by Lemma 2.3 $(\ell_M)^f = \ell_N$.

2.3 Lemma

$(\ell_M)^f = \ell_N$.

Proof: $(\ell_M)^\beta = \ell_N$ in Kamthan and Gupta[6]. But $\ell_M$ has AK in Kamthan and Gupta[6]. Hence $(\ell_M)^\beta = (\ell_M)^f$. Therefore $(\ell_M)^f = \ell_N$. This completes the proof.
2.4 Proposition

Let $Y$ be any FK-space $\supset \phi$. Then $Y \supset \ell_M$ if and only if the sequence $\{\delta^{(k)}\}$ is a member of the complementary Orlicz space.

**Proof:** $Y \supset \ell_M \iff y^f \subset (\ell_M)^f \iff Y^f \subset \ell_N$, since $(\ell_M)^f = \ell_N$ [Kamthan and Gupta[6]] $\iff (\delta^{(k)})$ is a member of the complementary Orlicz space. We recall

2.5 Lemma

(See 17, Theorem 4.3.7) Let $z$ be a sequence. Then $(z^\beta, P)$ is an AK space with $P = (P_k : k = 0, 1, 2, \ldots )$, where $P_0 (x) = \sup |\sum_{k=1}^m z_kx_k|, P_n (x) = |x_n|$. For any $k$ such that $z_k \neq 0, P_k$ may be omitted. If $z \in \phi, p_0$ may be omitted.

2.6 Proposition

Let $z$ be a sequence $z^\beta$ is semi-Orlicz if and only if $(z^\beta)^f \subset \ell_N$. Proof: Step 1. Suppose that $z^\beta$ is semi-Orlicz. $z^\beta$ has AK by Lemma 2.5. Therefore $Z^{\beta\beta} = (z^\beta)^f$ by Theorem 7.2.7 of Wilansky [17]. So $Z^\beta$ is semi-Orlicz if and only if $z^{\beta\beta} \subset \ell_N$. Step2: Conversely, suppose that $z \in \ell_N$. Then $z^\beta \supset \{\ell_N\}^\beta$ and $z^{\beta\beta} \subset \{\ell_N\}^{\beta\beta} = (\ell_M)^\beta = \ell_N$, because $(\ell_M)^\beta = \ell_N$. But $(z^\beta)^f = z^{\beta\beta}$. Hence $(z^\beta)^f \subset \ell_N$. Therefore $z^\beta$ is semi-Orlicz. This completes the proof.

2.7 Proposition

Every semi-Orlicz space contains $\ell_M$.

**Proof:** Let $X$ be any semi-Orlicz space. Hence $X^f \subset \ell_N$. Therefore $f (\delta^{(k)}) \in \ell_N \forall f \in X^f$. So, $\{\delta^{(k)}\}$ is a member of the complementary Orlicz space with respect to $X$. Hence $X \supset \ell_M$ by Proposition 2.4. This completes the proof.

2.8 Proposition

The intersection of all semi-Orlicz spaces $\{X_n : n = 1, 2, \ldots \}$ is semi-Orlicz.

**Proof:** Let $X = \bigcap_{n=1}^\infty X_n$. Then $X$ is an FK-space which contains $\phi$. Also every $f \in X^f$ can be written as $f = g_1 + g_2 + \ldots + g_m$, where $g_k \in X_n$ for some $n$ and for $1 \leq k \leq m$. But then $f (\delta^k) = g_1 (\delta^k) + g_2 (\delta^k) + \ldots + g_m (\delta^k)$. Since $X_n (n = 1, 2, \ldots)$ are semi-Orlicz spaces, it follows that $g_i (\delta^k) \in \ell_N$ for all $i = 1, 2, \ldots, m$. Therefore $f (\delta^k) \in \ell_N$. Hence $X$ is semi-Orlicz. This completes the proof.
2.9 Proposition

The intersection of all semi-Orlicz space is $\ell_{M}$.

Proof: Let $I$ be the intersection of all semi-Orlicz spaces. Then the intersection $I \subset \bigcap \{z^{\beta} : z \in \ell_{N}\}$, (by Proposition 2.6) = $\{\ell_{N}\}^{\beta}$

$$= \ell_{M} \quad (3)$$

By Proposition 2.8 it follows that $I$ is semi Orlicz. Consequently

$$\ell_{M} \subset I\text{(by Proposition 2.7)} \quad (4)$$

From (3) and (4) we get $I = \ell_{M}$. This completes the proof.

2.10 Corollary

The smallest semi-Orlicz space is $\ell_{M}$.

References


Received: April 30, 2008