A Multi-Objective Decision Making Approach for Solving Quadratic Multiple Response Surface Problems

S. J. Sadjadi, M. Habibian and V. Khaledi

Department of Industrial Engineering
Iran University of Science and Technology
Narmak, Tehran, Iran
sjadjadi@iust.ac.ir

Abstract

Many problems in engineering involve the application of optimization techniques including response surface methodology (RSM). The purpose of RSM is to model and optimize several responses of interest influenced by several variables. Such problem often is often called Multi-Objective Decision Making (MODM) approach and we normally look for an efficient solution. This paper studies different types of MODM methods and examines some of them on some real world RSM problems. The results are compared to show that even some trivial method can lead us to some promising results.

Keywords: Quadratic programming, Response Surface Modeling, Non-convex problems, Global Criterion

1 Introduction

The performance of a manufactured product is often evaluated by several quality characteristics. Optimization of a single quality characteristic or response in many cases leads to non-optimum values for the remaining ones. It is not unusual to see a typical problem having 12 or 15 response variables in some industrial setting such as semiconductor manufacturing[12]. RSM problems usually involve optimization of non-convex models. Recent developments in the theory of mathematical programming for constrained optimization have created the motivation among many statisticians, engineers and quality professionals to use RSM techniques more frequently. During the past four decades, many methods and algorithms have been developed to solve Multiple Objective
Decision Making (MODM) problems, in which some objectives are conflicting and the utility function of the Decision Maker (DM) is imprecise or fuzzy in nature. MODM is believed to be one of the fastest growing areas in management science and operations research; and the main reason for such development is that many decision making problems can be formulated in this category. Although different solution procedures have been introduced [17, 35, 28, 36], the interactive approaches are generally believed to be the most promising ones, in which the preferred information of the decision maker is progressively articulated during the solution process and incorporated into it. The purpose of MODM problems in the mathematical programming framework is to optimize $k$ different objective functions, subject to a set of system constraints. A mathematical formulation of an MODM problem is also known as the vector maximization (or minimization) problem (VMP). Generally, MODM problems can be divided into four different categories. The first group of MODM problems does not need to get any information from DM during the process of finding an efficient solution. These types of algorithms rely solely on the pre-assumptions about DM’s preferences. L-P Metric methods are among the most popular algorithms whose objectives are the minimization of deviations of the objective functions from an ideal solution named “utopian point”. Since different objectives are different in nature, they must be normalized before the process of the minimization of deviations starts. Therefore, we actually minimize a new problem which has no scale [37, 38, 39]. The second group of MODM problems includes gathering cardinal or ordinal preferred information before the solving process initiates. Some of these techniques such as Utility Function [19] and Bounded Objectives [18] are just based on gathering cardinal preferences. In the method of Utility Function, which is the most popular one, we have to determine the DM’s utility as a function of objective functions and then we maximize the overall function under the initial constraints. The other methods including, Goal Programming (GP) [26, 15] and Goal Attainment [18] are accepting a mixture of both cardinal and ordinal information. In the method of GP, which is extensively used by many researchers, the DM determines the least (the most) acceptable level of Max (Min) functions. Since attaining these values might lead to an infeasible point, the constraints are allowed to exceed, but we try to minimize these weighted deviations. The third group of MODM problems provides a set of efficient solutions in which DM has an opportunity to choose his/her preferred solution among the efficient solutions. Multiple-Objective Linear Programming (MOLP) and Multiple-Criteria Simplex [18], in this group, have been widely used. The last group provides solutions based on a continuous interaction with DM and tries to reach the preferred solution at the end of the algorithm. Based on this sound idea, there are many developed methods, categorized in this group such as: Geoffrion [14], Simplified Interactive Multiple Objective Linear Programming (SIMOLP) [25], Zionts
Multi-objective decision making approach

[40, 41], Step Method (STEM) [3], Surrogate Worth Trade Off (SWT), Sequential Multiple Objective Problem Solving (SEMOPS), Satisfactory Goals [18], and Game Theoretic Technique [36]. Although the above mentioned methods are the most famous ones, the interested readers can refer to other existing papers as well [29, 30, 31, 32, 34, 4, 2, 13].

There are many advantages on using interactive methods that we can name some of them as follows:

- There is no need to get any information from DM before the solving process initiates.
- The solution process helps DM learn more about the nature of the problem.
- Only minor preferred information are needed during the process.
- Since DM continuously contributes via analyst to the problem, he/she is more likely to accept the final solution.
- There are fewer restricting assumptions involved in these types of problems in comparison with other groups of MODM methods.

However, there are some drawbacks associated with these types of algorithms that the most important ones are as follows:

- The accuracy of the final solution depends entirely upon the DM’s precise answers. In other words, if DM does not carefully interact with the analyst, the outcome(s) of the final solution may not be desirable.
- There is no guarantee to reach a desirable solution after a finite number of iterations.
- DM needs to make more effort during the process of these algorithms in comparison with other groups.

During the past decades, many have tried to discuss the strengths, the weaknesses and the comparative studies on the existing methods. The main goals of these papers are to introduce some criteria to measure the efficiency of various algorithms and to introduce the characteristics of a good method [33, 1, 8, 16, 20, 23]. Reeves and Franz [25], introduce the characteristics of a proper interactive algorithm as follows:

1. Minimum amount of information is required from DM.
2. The nature of decision making is simple.

3. If DM provides his/her answers improperly in some interactions, he/she has an opportunity to come up with it in the following interactions.

4. The number of iterations to reach the final solution is reasonable.

5. DM is familiar with the nature of judgements he/she is asked for.

6. The algorithm is suitable for solving large scale problems.

In this paper, we present a class of multiple response optimization in which different quadratic responses are optimized, simultaneously. The proposed algorithm of this paper first find the optimal solution of each response separately and all different objectives are combined together with a Lp norm and using some efficient method the optimal solution of the resulted model is determined. This paper is organized as follows. The problem statement and its modified model are discussed in section 2. In section 3, different approaches for determining optimal solution(s) are reviewed. A special case of multi-response problem involved with different undefined quadratic responses is considered. Numerical solutions are presented in the following section and results are compared to other relevant methods to evaluate the performance of the proposed modeling procedure.

2 Problem Statement and Modeling

Suppose there are $m$ responses where each response is modeled by

$$\max \tilde{y} = f^*_i(x),$$
subject to

$$g_j(x) \leq 0,$$

$$i = 1, \ldots, k,$$

$$j = 1, \ldots, m. \quad (1)$$

Let $f_i(x), i = 1, \ldots, k$ be quadratic. We further consider (1) in different forms of convex and non-convex in structure. In quality profession, the context of optimization problems usually involves such model formulation. Del Castillo and Semple [9] report that for more than three design variables the responses $y_i$s are usually non-convex. As explained in the introduction section, our primary objective is to use MODM techniques in order to find the optimal solution of our problem formulation. As explained in the previous section, there are many different MODM techniques to solve RSM problems. However, since RSM problems are normally involved in optimization of quadratic problems, we only focus to use the MODM techniques which could be easily adopted to our cases.
2.1 The Global Criterion Method

The Global Criterion (GC) method first solves each response separately in order to determine the optimal solution. Therefore, in each case, a Quadratic Programming (QP) problem must be solved. Next we use Lp norm in order to combine all different $k$ responses and at the end we find the efficient solution of the resulted combined problem. Let $f^*_i$ be the optimal solution of each response in 1. Therefore L1 norm is defined as follows,

$$
\text{min} \sum_{i=1}^{m} \frac{f^*_i - f_i}{f_i^*}, \\
\text{subject to } g_j(x) \leq 0, \\
i = 1, \ldots, k, \\
j = 1, \ldots, m.
$$

Problem (2) is a combined QP problem subject to some linear or non-linear constraints. We now show the implementation of the proposed method using a real world case study.

2.2 Case Study

Consider a chemical plan where we are interested in minimizing the production cost while we are simultaneously interested in optimizing other relevant factors such as product’s durability and adherence. Our investigation using Delphi’s method [17] indicates that the percentage of tin-coverage, $NH_4CL$ and the temperature are the most important factors which influence the objectives. Let $x_1$ to $x_4$ represent $NH_4CL$, the thickness of the tin, the temperature and the percentage of tin applied for each experiment, respectively. Therefore the response model for the cost of production is as follows,

$$
\text{The cost of production } = f_1(x) = 4438 + 4.67x_1 + 42.19x_2 + 4.17x_3 + 15.51x_4 \\
-4.5x_1^2 - 4.5x_2^2 - 4.5x_3^2 - 4.5x_4^2 \\
+6.25x_1x_4 + 3.25x_2x_4 + 6.25x_3x_4.
$$

Also, the mean and the variance of the durability associated with our experiments are as follows,

$$
\mu_{\text{durability}} = f_2(x) = 250.34 - 34.26x_1 + 43.91x_2 - 1.55x_3 + 6.75x_4 \\
-24.09x_1^2 - 22.97x_2^2 - 21.84x_3^2 - 23.22x_4^2 \\
+4.56x_1x_4 + 16.94x_2x_4 + 25.31x_3x_4
$$

$$
\sigma_{\text{durability}} = f_3(x) = 1.945 - 1.002x_1 + 1.321x_2 - 1.827x_3 + 0.093x_4 \\
+0.884x_1^2 - 0.707x_2^2 + 0.53x_3^2 + 1.061x_4^2 \\
-0.265x_1x_2 - 2.21x_2x_3 - 0.265x_2x_4.
$$
Finally, the mean and the variance of for the results of our experiments indicates the following relationship for the mean and the variance of the adherence,

\[ \mu_{\text{adherence}} = f_4(x) = 176.75 - 2.01x_1 + 30.6x_2 + 2.59x_3 + 16.87x_4 - 13.31x_1^2 - 13.19x_2^2 - 12.94x_3^2 - 12.44x_4^2 + 8.56x_1x_4 - 3.94x_2x_4 - 8.69x_3x_4 \]

\[ \sigma_{\text{adherence}} = f_5(x) = 2.172 + 0.24x_1 - 0.039x_2 + 0.226x_3 - 0.295x_4 + 1.099x_1^2 + 0.215x_2^2 + 0.215x_3^2 + 0.215x_4^2 - 1.856x_1x_2 - 0.795x_2x_4 + 0.619x_2x_3 \]

Suppose we are simultaneously interested in optimizing (3), (4) and (5) as follows,

\begin{align*}
\min f_1(x), \quad & \max f_2(x), \quad \min f_3(x), \quad \max f_4(x), \quad \min f_5(x) \\
\text{subject to} & \quad -1.8 \leq x_1 \leq 3, \quad -2.2 \leq x_2 \leq 1.8, \quad -1.67 \leq x_3 \leq 1.67, \quad -3.0 \leq x_4 \leq 3.0
\end{align*}

(6)

### 2.2.1 A Global Criterion (GC) Method

Problem (6) is an MODM problem where we are interested simultaneously in finding an efficient solution subject to some bound constraints associated with each independent variables, \( x_1 \) to \( x_4 \). As we have previously explained, in order to use Lp norm we need to detect the optimal solution of \( f_1 \) to \( f_5 \) subject to bound constraints individually. The first objective function, \( f_1 \) is a quadratic function. The Hessian matrix of \( f_1 \) yields the following eigenvalues,

\[ \lambda_1 = 9.2087, \quad \lambda_2 = 4.5, \quad \lambda_3 = 4.5, \quad \lambda_4 = -0.2087 \]

In this case we are interested in minimizing QP problem with one negative eigenvalue. Such problem is NP-Hard[24]. Best and Ding [6] show present a decomposition technique to find all local(s) and global of such problem. Applying their technique yields the following optimal solution,

\[ x_1 = -1.8, \quad x_2 = -2.2, \quad x_3 = -1.67, \quad x_4 = -3, \quad f_1^* = -3190.40 \]

Since \( f_3 \) and \( f_5 \) have similar behaviour and we can adopt Best and Ding [6] method to determine the optimal solution. However, the eigenvalues of \( f_2 \) and \( f_4 \) are all negative which means that we need to maximize concave functions subject to some linear bound constraints. Such method can be easily solved for optimal solution using active set method [7]. Therefore, the results of the
optimal solution can be summarized as follows,

\[
\begin{array}{cccc}
    x_1 &=& -1.8, & x_2 = -2.2, & x_3 = -1.67, & x_4 = -3 & f_1^* &=& -3190.40 \\
    x_1 &=& -1.3143, & x_2 = 1.8, & x_3 = 0.5892, & x_4 = 1.1394 & f_2^* &=& 326.7551 \\
    x_1 &=& 0.8365, & x_2 = 1.8, & x_3 = 1.67, & x_4 = 0.181 & f_3^* &=& 0.3111 \\
    x_1 &=& 0.2443, & x_2 = 1.8, & x_3 = -0.2127, & x_4 = 1.2294 & f_4^* &=& 218.1315 \\
    x_1 &=& 1.4107, & x_2 = 1.8, & x_3 = -1.67, & x_4 = 3.0 & f_5^* &=& 4.2703.
\end{array}
\]

Finally, we use (2) to find the efficient solution for the resulted model. Let \( \hat{f}(x) \) be the \( L_1 \) norm. Therefore we have,

\[
\hat{f}(x) = 8.7939 - 3.052x_1 + 3.9492x_2 - 5.8282x_3 - 0.4709x_4 \\
    + 3.235x_1^2 + 2.4551x_2^2 + 1.8816x_3^2 + 3.5903x_4^2 \\
    - 0.0552x_1x_4 - 1.0728x_2x_4 - 0.1193x_3x_4 - 6.9589x_2x_3. 
\] \hspace{1cm} (7)

Problem (7) is QP problem and must be minimized subject to regular bound constraints. Therefore we have,

\[
\hat{x} = \left\{ \min \hat{f}(x) \right| -1.8 \leq x_1 \leq 3, \quad -2.2 \leq x_2 \leq 1.8, \quad -1.67 \leq x_3 \leq 1.67, \quad -3.0 \leq x_4 \leq 3 \right\}. \hspace{1cm} (8)
\]

Problem (8) is a non-convex QP problem with one negative eigenvalue. The resulted problem can be solved for efficient solution using the approach by Best and Ding [6] which yields,

\[
x_1 = 0.1577, \quad x_2 = 1.5939, \quad x_3 = 1.67, \quad x_4 = 0.3327, \quad \hat{f}^* = -2.1586. \]

Applying efficient solution on (3) to (5) yields,

\[
f_1 = 4499, \quad f_2 = 215.4327, \quad f_3 = -1.8052, \quad f_4 = 157.3709, \quad f_5 = 4.3836. \]

### 2.2.2 A Desirability Approach

Another MODM approach to solve the case study formulation is to find a set of efficient solution and select the one which is the most appropriate. In this case we may choose to minimize the cost of production while we consider the following relationships between the \( \mu \) and \( \sigma \) for both durability and adherence as follows,

\[
CP_{\text{durability}} = \min \left\{ \frac{\infty - \mu_{\text{durability}}}{3\sigma_{\text{durability}}}, \frac{\mu_{\text{durability}} - \mu_{\text{durability}}}{3\sigma_{\text{durability}}} \right\} = \frac{\mu_{\text{durability}} - \mu_{\text{durability}}}{3\sigma_{\text{durability}}} = \frac{f_2 - \mu_{\text{durability}}}{3f_3}. \hspace{1cm} (9)
\]

and

\[
CP_{\text{adherence}} = \min \left\{ \frac{\infty - \mu_{\text{adherence}}}{3\sigma_{\text{adherence}}}, \frac{\mu_{\text{adherence}} - \mu_{\text{adherence}}}{3\sigma_{\text{adherence}}} \right\} = \frac{\mu_{\text{adherence}} - \mu_{\text{adherence}}}{3\sigma_{\text{adherence}}} = \frac{f_4 - \mu_{\text{adherence}}}{3f_5}. \hspace{1cm} (10)
\]
where $\bar{\mu}_{\text{durability}} = 200$ and $\bar{\mu}_{\text{adherence}} = 130$. Therefore we are interested in the following optimization model,

$$
\begin{align*}
\min & \quad f_1(x) \\
\text{subject to} & \quad -1.8 \leq x_1 \leq 3, \\
& \quad -2.2 \leq x_2 \leq 2.2, \\
& \quad -1.67 \leq x_3 \leq 1.67, \\
& \quad -3.0 \leq x_4 \leq 3.0,
\end{align*}
$$

where $\tilde{CP}$ and $\hat{CP}$ are the lower bounds for $CP_{\text{durability}}$ and $CP_{\text{adherence}}$, respectively. Model (11) is a general optimization problem and can be solved using Sequential Quadratic Programming (SQP) approach. We choose different values of the lower bounds for $\tilde{CP}$ and $\hat{CP}$ and solve the model for each case. Table (2) summarizes the results the implementation of the desirability approach when different values for $\tilde{CP}$ and $\hat{CP}$ are chosen. Note that due to nonlinearity occurs in the resulted constrains we may not guarantee that the solutions are global minimum. On the other hand, the results of We may wish to compare the results of Global Criterion with Desirability Approach. The GC method has different stages where we are interested only in optimization of QP problem. On the other hand the second approach combine a problem into a uniform generalized nonlinear form of optimization. We may use different techniques in order to globally determine the optimal solution of QP problem. However, it is not easy to find the global solution for a general optimization problem. In many cases, finding a local optimum for a general form of optimization problem is considered to be sufficient for many practitioners. As we may also observe, due to a better procedure we could use to solve in the first approach, three out of five objective functions have better values in the first model. Therefore, although the GC method is one of very simple and trivial method but it seems that the results of its implementation may be far better than other methods since we may detect the global solution of the resulted model in each part using the advances in optimization for QP problems.

<table>
<thead>
<tr>
<th>Case</th>
<th>$CP$</th>
<th>$CP$</th>
<th>$x_1$</th>
<th>$x_2$</th>
<th>$x_3$</th>
<th>$x_4$</th>
<th>$f_1(x)$</th>
<th>$f_2(x)$</th>
<th>$f_3(x)$</th>
<th>$f_4(x)$</th>
<th>$f_5(x)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>2.0</td>
<td>2.0</td>
<td>-0.56</td>
<td>-0.8</td>
<td>0.03</td>
<td>-0.04</td>
<td>4396.89</td>
<td>212.3</td>
<td>2.05</td>
<td>140.0</td>
<td>1.67</td>
</tr>
<tr>
<td>2</td>
<td>1.7</td>
<td>2.0</td>
<td>-0.55</td>
<td>-0.82</td>
<td>0.03</td>
<td>0.11</td>
<td>4396.85</td>
<td>210.5</td>
<td>2.05</td>
<td>138.5</td>
<td>1.75</td>
</tr>
<tr>
<td>3</td>
<td>1.4</td>
<td>2.0</td>
<td>-0.54</td>
<td>-0.83</td>
<td>0.06</td>
<td>-0.13</td>
<td>4396.17</td>
<td>208.6</td>
<td>2.06</td>
<td>138.4</td>
<td>1.84</td>
</tr>
<tr>
<td>4</td>
<td>2.0</td>
<td>1.7</td>
<td>-0.59</td>
<td>-0.81</td>
<td>0.0</td>
<td>-0.05</td>
<td>4396.87</td>
<td>212.5</td>
<td>2.08</td>
<td>138.0</td>
<td>1.57</td>
</tr>
<tr>
<td>5</td>
<td>1.7</td>
<td>1.7</td>
<td>-0.58</td>
<td>-0.84</td>
<td>0.05</td>
<td>0.11</td>
<td>4396.17</td>
<td>208.6</td>
<td>2.08</td>
<td>138.4</td>
<td>1.64</td>
</tr>
<tr>
<td>6</td>
<td>1.4</td>
<td>1.7</td>
<td>-0.57</td>
<td>-0.84</td>
<td>0.04</td>
<td>-0.13</td>
<td>4395.48</td>
<td>210.6</td>
<td>2.08</td>
<td>138.5</td>
<td>1.73</td>
</tr>
<tr>
<td>7</td>
<td>2.0</td>
<td>1.4</td>
<td>-0.61</td>
<td>-0.81</td>
<td>0.1</td>
<td>-0.21</td>
<td>4395.48</td>
<td>212.8</td>
<td>2.13</td>
<td>136.2</td>
<td>1.46</td>
</tr>
<tr>
<td>8</td>
<td>1.7</td>
<td>1.4</td>
<td>-0.61</td>
<td>-0.83</td>
<td>0.04</td>
<td>-0.13</td>
<td>4394.19</td>
<td>210.8</td>
<td>2.11</td>
<td>136.5</td>
<td>1.55</td>
</tr>
<tr>
<td>9</td>
<td>1.4</td>
<td>1.4</td>
<td>-0.61</td>
<td>-0.85</td>
<td>0.03</td>
<td>-0.05</td>
<td>4393.97</td>
<td>208.9</td>
<td>2.11</td>
<td>136.8</td>
<td>1.63</td>
</tr>
</tbody>
</table>

Table 1: The Results for Desirability Approach
3 Conclusions

We have analyzed quadratic multiple response problems under different conditions. A modified QP algorithm for a class of multiple response problems that could determine global optimum under two different conditions has been presented. In the authors’ view, the convex optimization case has already been well addressed by researchers and therefore we have concentrated on non-convex problems. An efficient algorithm has been studied to find isolated local(s) including the global and possibly some of non-isolated local minimum(s) when the problem had only one single negative eigenvalue. We have also discussed a class of multi-response problem where objective function is non-convex quadratic and constraints are linear. A method has been explained for standard QP problem with full rank and undefined Hessian matrix to detect all local and global of the problem under the condition that matrix $C$ has at least two eigenvalues of opposite signs.

References


Received: April 19, 2008