A Simple Algorithm for Computing Analytic Functions of Real Square Matrices

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Abstract

Computation of analytic functions of a real square matrix is a useful and interesting task and many papers have been written on this subject that treat the problem in some special cases. In this paper we will give a new and relatively simple algorithm for performing this task which instead of utilizing linear algebraic tools uses a linear or nonlinear initial value problem governing the desired matrix function and also utilizes the minimal, and not the characteristic, polynomial of the corresponding matrix.

Mathematics Subject Classifications: 15-xx, 34-xx

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1. Introduction

Computation of an analytic function $f(tA)$ for a real square matrix $A$ has many applications, specially the computation of $\exp(tA)$ is very important in connection with the solution of initial value problems, and so many methods are given by some authors for these computations[1,2,4,5,6,10]. Most of the given methods use a lot of linear algebraic tools and methods and also have high accumulated errors in the process of finding eigenvectors and generalized eigenvectors of $A$ or differentiating a function several times[4]. However linear algebraic methods are very interesting and well structured in first glance and for doing the job manually, and so they are the most convenient methods for the authors of the textbooks on differential equations and linear algebra[7,10].
In this paper a very simple algorithm is represented, which utilizes simple methods and tools of linear algebra and differential equations to compute \( f(tA) \) on its domain of convergence.

In section 2 we consider the linear case and give the main results. In section 3 some examples are given to illustrate our algorithm in the linear case. In the final section we have solved a nonlinear example.

2. Main Results

In this section we first briefly review some useful results from linear algebra, functional analysis and the theory of analytic functions. Then by utilizing these facts we obtain the main results of the paper in the linear case.

Suppose a positive real number \( R \) and a real analytic function \( f : (-R,R) \rightarrow \mathcal{R} \) are given. Moreover suppose \( A \) is a nonzero \( n \times n \) real matrix with \( ||A|| = \alpha > 0 \), where \( ||.|| \) is the functional norm induced on \( \mathcal{L}(\mathbb{R}^n) \) by an arbitrary norm of \( \mathbb{R}^n \). Then putting \( R' = R/\alpha \), the matrix function \( B(t) = f(tA) \) is analytic on the interval \((-R',R')\). Direct computation of \( B(t) \) by using Maclaurin series of \( f \) and the conventional tools and methods of linear algebra is sometimes possible, but obviously it may involve long computations and has very high accumulated errors[4,7,10]. Moreover in contrast with the special case of exponential functions, where after finding the commutative diagonal and nilpotent parts of \( A \), the function can be simply computed, in the general case it may happen that separating the diagonal and nilpotent parts of \( A \) doesn’t help much the computation by any means. Consequently a new method which doesn’t use any advanced linear algebraic tools and methods, is proposed.

To explain our method we need some assumptions. Suppose \( A,f \) and \( B \) are as mentioned above and moreover suppose \( A \) has the real minimal polynomial

\[
m_A(x) = x^k - \sum_{j=0}^{k-1} a_j x^j, \quad a_0 \neq 0
\]

and that the matrix function \( B(t) \) satisfies the linear homogeneous differential equation of order \( m \) with constant coefficients depending on \( A \):

\[
L_A\{B(t)\} = 0,
\]
and \( m \) given initial conditions

\[
B^{(p)}(0) = D_p, \quad 0 \leq p \leq m - 1
\]

where \( D_p, 0 \leq p \leq m - 1 \), are given real \( n \times n \) matrices, and that \( L_A \) is a linear differential operator of the form

\[
L_A = \sum_{l=0}^{m} P_l(A) \frac{d^l}{dt^l}
\]

and each \( P_l(A) \) is a polynomial in \( A \) of the form

\[
P_l(A) = \sum_{q=0}^{k-1} g_{q,l} A^q, \quad 0 \leq l \leq m
\]

We will try to find a closed formula for \( B(t) \). For this purpose we first give an elementary but important theorem:

**Theorem 2.1:** Suppose \( A \) is a nonzero real \( n \times n \) matrix with \( ||A|| = \alpha > 0 \) whose minimal polynomial is given by (2.1), \( R \) is a positive real number and \( f : (-R, R) \to \mathcal{R} \) be an analytic function, and put \( R' = R/\alpha \). Moreover suppose \( a \) be a nonzero real number and \( b(t) = f(at) \) be a parametric function which satisfies the linear parametric differential equation

\[
L_a\{b(t)\} = \sum_{l=0}^{m} P_l(a) b^{(l)}(t) = 0,
\]

on \((-R_1, R_1)\), where \( R_1 = R/|a| \) and each \( P_l \) is a polynomial in \( a \) in the form

\[
P_l(a) = \sum_{q=0}^{k-1} g_{q,l} a^q, \quad 0 \leq l \leq m
\]

Then the matrix function \( B(t) = f(tA) \) is analytic on \((-R', R')\) and satisfies the linear homogeneous differential equation

\[
L_A(B(t)) = \sum_{l=0}^{m} P_l(A) \frac{d^l}{dt^l} B(t) = 0
\]

which has coefficients depending on \( A \). Moreover there are unique analytic functions

\[
r_j : (-R', R') \to \mathcal{R},
\]

such that

\[
B(t) = \sum_{j=0}^{k-1} r_j(t) A^j
\]
Proof: We first note that since the minimal polynomial of $A$ over $\mathcal{R}$ is of degree $k$ the set $S = \{I, A, A^2, \ldots, A^{k-1}\}$ is independent on $\mathcal{R}$ and moreover for each nonnegative integer $i$ there is a unique real sequence

$$C_{i,j}, \ 0 \leq j \leq k - 1$$

such that:[6]

$$A^i = \sum_{j=0}^{k-1} C_{i,j} A^j, \quad (2.12).$$

Moreover we have

$$B(t) = f(tA) = \sum_{i=0}^{\infty} f_i(tA)^i$$

where $f_i, 0 \leq i < \infty$ are Maclaurin coefficients of $f$. So by substituting (2.12) in (2.13) we obtain

$$B(t) = \sum_{i=0}^{\infty} f_i t^i \sum_{j=0}^{k-1} C_{i,j} A^j = \sum_{i=0}^{\infty} \sum_{j=0}^{k-1} f_i C_{i,j} t^i A^j,$$

(2.14)

Now since the above series is uniformly and absolutely convergent over $-R' < t < R'$, we can change the order of summations and obtain

$$B(t) = \sum_{j=0}^{k-1} \sum_{i=0}^{\infty} f_i C_{i,j} t^i A^j,$$

(2.15)

Then putting

$$r_j(t) = \sum_{i=0}^{\infty} f_i C_{i,j} t^i, \ 0 \leq j \leq k - 1,$$

(2.16)

it is obvious that each of these real functions is analytic over the interval $(-R', R')$. Moreover from (2.6) and the chain rule it follows that $B(t)$ satisfies (2.2) and the theorem is proved.

Now according to the above discussion and other assumptions about $B(t)$ we can find a method for computing $r_j(t), 0 \leq j \leq k - 1$, and finally $B(t)$ is obtainable by relation(2.10). In fact we have

$$L_A\{B(t)\} = \sum_{l=0}^{m} P_l(A) \frac{d^l}{dt^l} \left( \sum_{j=0}^{k-1} r_j(t) A^j \right)$$

$$= \sum_{l=0}^{m} \sum_{j=0}^{k-1} A^j P_l(A) \frac{d^l}{dt^l} r_j(t).$$
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\[
= \sum_{l=0}^{m} \sum_{j=0}^{k-1} A^{j} \sum_{q=0}^{k-1} B_{q,l} A^{q}(r_{j}^{(l)}(t))
\]

\[
= \sum_{l=0}^{m} \sum_{j=0}^{k-1} \sum_{q=0}^{k-1} B_{q,l} r_{j}^{(l)}(t) A^{j+q}
\]

(2.17)

and by successive applications of the relation

\[
A^{k} = \sum_{j=0}^{k-1} a_{j} A^{j},
\]

(2.18)

and changing the order of summations it is obvious that there are unique linear differential operators \( L_{i} \), \( 0 \leq i \leq k-1 \) with constant coefficients, each of order at most \( k - 1 \), such that

\[
L_{A} \{ B(t) \} = \sum_{i=0}^{k-1} L_{i}(a_{0}, a_{1}, ..., a_{k-1}, r_{0}(t), r_{1}(t), ..., r_{k-1}(t)) A^{i},
\]

(2.19)

Now from (2.2), and the linear independence of \( S \) over \( \mathcal{R} \), we obtain the homogeneous linear system of differential equations with constant coefficients

\[
L_{i}(a_{0}, a_{1}, ..., a_{k-1}, r_{0}(t), r_{1}(t), ..., r_{k-1}(t)) = 0, 0 \leq i \leq k - 1,
\]

(2.20)

of order at most \( m \), moreover using the initial conditions for \( B(t) \) and linear independence of \( S \) the corresponding initial conditions for \( r_{j}, 0 \leq j \leq k - 1 \) are obtained. Solving this system of linear differential equations and using the corresponding initial conditions, we can compute \( r_{j}, 0 \leq j \leq k - 1 \) and finally a closed formula for \( B(t) \) is obtained.

**Corollary 2.2:** It is evident that in the above computations advanced tools and methods of linear algebra have been avoided. Moreover as it can be seen in the examples of the next section, this method is useful in computation of various functions.

\section*{3. Some Examples}

In this section we will show the strength of our method by solving some illustrative examples.

**Example 3.1:** Compute \( B(t) = \exp(tA) \) where \( A \) is a real \( n \times n \) matrix with the minimal polynomial

\[
= \sum_{l=0}^{m} \sum_{j=0}^{k-1} A^{j} \sum_{q=0}^{k-1} B_{q,l} A^{q}(r_{j}^{(l)}(t))
\]

\[
= \sum_{l=0}^{m} \sum_{j=0}^{k-1} \sum_{q=0}^{k-1} B_{q,l} r_{j}^{(l)}(t) A^{j+q}
\]

(2.17)
\[ m_A(x) = x^k - \sum_{j=0}^{k-1} a_j x^j, \quad a_0 \neq 0 \tag{3.1} \]

and compute \( B(t) = \exp(tA) \).

Solution: By putting

\[ B(t) = \sum_{j=0}^{k-1} r_j(t)A^j \tag{3.2} \]

and using the fundamental properties of the exponential functions we have[9]:

\[ r_1(t) = \mathcal{L}^{-1}\left\{ \frac{1}{m_A(s)} \right\} \tag{3.3} \]

where \( \mathcal{L} \) is the operator of Laplace transform. Moreover we have[9]:

\[ r_0'(t) = a_0 r_{k-1}(t) \tag{3.4} \]

\[ r_j'(t) = r_{j-1}(t) + a_j r_{k-1}(t), \quad 1 \leq j \leq k - 1 \tag{3.5} \]

\[ r_0(0) = 1 \tag{3.6} \]

\[ r_j(0) = 0, \quad 1 \leq j \leq k - 1 \tag{3.7} \]

As it is seen finding \( \exp(tA) \) by this method is very simple and elegant. For better illustration of our method we give an example:

**Example 3.2:** Compute \( \exp(tA) \) for
\[
A = \begin{bmatrix}
0 & -2 & -1 & -1 \\
1 & 2 & 1 & 1 \\
0 & 1 & 1 & 0 \\
0 & 0 & 0 & 1
\end{bmatrix}.
\]

Solution: We have

\[ m_A(x) = (x-1)^3 = x^3 - 3x^2 + 3x - 1, \tag{3.8} \]

and we have \( a_0 = 1, \ a_1 = -3 \) and \( a_2 = 3 \). So putting

\[ \exp(tA) = r_2(t)A^2 + r_1(t)A + r_0(t)I \tag{3.9} \]

we have

\[ r_2(t) = \mathcal{L}^{-1}\left\{ \frac{1}{(s-1)^3} \right\} = \frac{1}{2} t^2 e^t \tag{3.10} \]

and therefore

\[ r_0'(t) = a_0 r_2(t) = \frac{1}{2} t^2 e^t \tag{3.11} \]
and
\[ r_0(0) = 1 \]
consequently
\[ r_0(t) = \frac{1}{2}(t^2 - 2t + 2)e^t \]
and finally we have
\[ r'_1(t) = r_0(t) + a_1r_2(t) = \frac{1}{2}(t^2 - 2t + 2)e^t - \frac{3}{2}t^2e^t = -(t^2 + t - 1)e^t \]
and
\[ r_1(0) = 0 \]
so we obtain
\[ r_1(t) = -(t^2 - t)e^t \]
and then according to (3.2) we obtain
\[
\exp(tA) = r_2(t)A^2 + r_1(t)A + r_0(t)I \\
= \frac{1}{2}t^2e^t \begin{bmatrix} 0 & -2 & -1 & -1 \\ 1 & 2 & 1 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}^2 - (t^2 - t)e^t \begin{bmatrix} 0 & -2 & -1 & -1 \\ 1 & 2 & 1 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \\
+ \frac{1}{2} \left( t^2 - 2t + 2 \right)e^t \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \\
= e^t \begin{bmatrix} 1 - t - t^2/2 & -2t - t^2/2 & -t - t^2/2 & -t - t^2/2 \\ t & 1 + t & t & t \\ t^2/2 & t + t^2/2 & 1 + t^2/2 & t^2/2 \\ 0 & 0 & 0 & 1 \end{bmatrix}
\]
which is just the result given in [10].
The above examples were about the exponential function of matrices and have
been solved by other methods in some text books. But the following examples show the strength of the method of this paper more precisely:

**Example 3.3:** Compute the function

\[ B(t) = \sin(tA), -\infty < t < \infty \]  

(3.18)

where \( A \) is a real \( n \times n \) matrix with minimal polynomial

\[ m_A(x) = x^2 - x - 1 \]  

(3.19)

**Solution:** It is simply seen that \( B(t) \) is the unique solution of the following initial value problem

\[ B''(t) + A^2B(t) = 0, \]  

(3.20)

\[ B(0) = 0, \]  

(3.21)

\[ B'(0) = A, \]  

(3.22)

Moreover according to theorem (2.1) there are unique entire functions \( a(t) \) and \( b(t) \) such that

\[ B(t) = a(t)I + b(t)A, \]  

(3.23)

Inserting (3.23) into (3.20) and using the relation

\[ A^2 = I + A, \]  

(3.24)

we obtain

\[ a''(t) + a(t) + b(t) = 0, \]  

(3.25)

\[ a(t) + b''(t) + 2b(t) = 0, \]  

(3.26)

and according to (3.21) and (3.22) we have

\[ a(0) = b(0) = a'(0) = 0, \]  

(3.27)

and

\[ b'(0) = 1, \]  

(3.28)

This is an initial value problem with constant coefficients which by putting \( c(t) = a'(t) \) and \( d(t) = b'(t) \) becomes a first order linear differential equation with constant coefficient matrix

\[ A = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & -1 & 0 & 0 \\ -1 & -2 & 0 & 0 \end{bmatrix}. \]

and with initial conditions

\[ a(0) = b(0) = c(0) = 0, \]  

(3.29)
and
\[ d(t) = 1, \]  
(3.30).

The minimal polynomial of \( A \) is given by
\[ m_A(x) = x^4 + 3x^2 + 1, \]  
(3.31).

This problem is solvable by the method given in example (3.1).

**Example 3.4:** Compute the function
\[ B(t) = \exp(At)\cos(At), \]  
(3.32)
where \( A \) is a real \( n \times n \) matrix with minimal polynomial
\[ m_A(x) = x^2 - 2x + 1, \]  
(3.33)

**Solution:** It is easily seen that \( B(t) \) is the unique solution of the initial value problem
\[ B''(t) - 2AB'(t) + 2A^2B(t) = 0, \quad B(0) = I, \quad B'(0) = A, \]  
(3.34)
(3.35)  
(3.36)

moreover we have
\[ A^2 = 2A - I, \]  
(3.37)

and so there are unique entire functions \( a(t) \) and \( b(t) \) such that
\[ B(t) = a(t)I + b(t)A, \]  
(3.38)

Inserting (3.38) in (3.34) we obtain
\[ a''(t) - 2a(t) + 2b'(t) - 4b(t) = 0, \]  
(3.39)
\[ 4a(t) + b''(t) - 4b'(t) + 6b(t) = 0, \]  
(3.40)

moreover according to (3.35) and (3.36) we have
\[ a(0) = 1, \quad b(0) = 0, \]  
(3.41)
\[ a'(0) = 0, \quad b'(0) = 1, \]  
(3.42)

Solving this initial value problem we easily obtain the desired function.

**Comment 3.5:** By the above examples it seems that although our algorithm doesn’t seem to be as user friendly as the linear algebraic methods, but it is
more rapid and less erroneous, and also it is simply programmable.

**Comment 3.6:** It can be easily shown that in the general case we must solve an \( m \)th order initial value problem of dimension \( k \), and also by considering new variables for the derivatives of the functions \( r_j(t) \), we can reduce the general problem to a problem of the form given in example (3.1) but with a new coefficient matrix[10]. Moreover the size of \( n \) is not very important.

## 4. A Nonlinear Example

We have already analytically computed some analytic matrix functions that satisfy linear initial value problems. Now we give some nonlinear examples.

**Example 4.1:** Compute the function

\[
B(t) = (I + tA)^{-1},
\]

where \( A \) is a real \( n \times n \) matrix with minimal polynomial

\[
m_A(x) = x^2 - 2x + 1,
\]

**Solution:** It is easily seen that \( ||A|| = 1 \) and so \( B(t) \) is analytic on \( |t| < 1 \). Moreover \( B(t) \) is the unique solution of the nonlinear initial value problem

\[
B'(t) = -A[B(t)]^2,
\]

\[
B(0) = I,
\]

So there are unique analytic functions \( a, b : (-1, 1) \to \mathbb{R} \), such that

\[
B(t) = a(t)I + b(t)A,
\]

Inserting (4.5) in (4.3) and using \( A^2 = 2A - I \) and \( A^3 = 3A - 2I \) we obtain

\[
a'(t)I + b'(t)A = 2\{a(t)b(t) + b^2(t)\}I - \{a^2(t) + 4a(t)b(t) + 3b^2(t)\}A,
\]

and so we have the nonlinear initial value problem

\[
a'(t) = 2\{a(t)b(t) + b^2(t)\},
\]

\[
b'(t) = -\{a^2(t) + 4a(t)b(t) + 3b^2(t)\},
\]
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\[ a(0) = 1, \quad (4.9) \]
\[ b(0) = 0, \quad (4.10) \]

Adding (4.7) and (4.8) we obtain

\[ a'(t) + b'(t) = -\{a(t) + b(t)\}^2, \quad (3.53) \]

and so using (4.9) and (4.10) we have

\[ a(t) + b(t) = \frac{1}{t + 1}, \quad (4.12) \]

and finally we find

\[ a(t) = \frac{2t + 1}{(t + 1)^2}, \quad (4.13) \]
\[ b(t) = \frac{-t}{(t + 1)^2}, \quad (4.14) \]

which means that

\[ (I + tA)^{-1} = \frac{2t + 1}{(t + 1)^2} I - \frac{t}{(t + 1)^2} A, \quad (4.15) \]

The previous example shows the strength of our method, even in connection with nonlinear problems. But in the nonlinear cases the degrees of the irreducible factors of the minimal polynomial are very important, but I think it is solvable with a little more effort. I have given the above example only to show the strength of our methods. We are able to work on some more nonlinear examples which may be very interesting.

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References


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