Approximation to Functions from the Classes of \( \bar{\psi} \) - Integrals by the Zygmund Sums

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Abstract

In this paper we investigate the approximation by the Zygmund sums in a given class of certain functions. Especially, we get asymptotic formulas which the Kolmogorov-Nikol’skii problem is solved in a given metric of the certain space for the given class of functions which satisfy the various conditions, i.e. asymptotic formulas for the value

\[ E_n(C^\bar{\psi}_\infty, Z^s_n(x)) = \sup_{f \in C^\bar{\psi}_\infty} \|f(x) - Z^s_n(f; x)\|_C, \]

under the various conditions on functions \( \psi_1(\cdot) \) and \( \psi_2(\cdot) \).

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1 Statement of the Problem.

Let \( L \) denote the space of integrable \( 2\pi \)-periodic functions, and let

\[ S[f] = \frac{a_0}{2} + \sum_{k=1}^{\infty} (a_k \cos kx + b_k \sin kx) \equiv \sum_{k=0}^{\infty} A_k(f; x) \]

be the Fourier series of a function \( f \in L \). The polynomials that have the form

\[ Z^s_n(f; x) = \frac{a_0}{2} + \sum_{k=1}^{n-1} (1 - \left(\frac{k}{n}\right)^s) A_k(f; x) \quad , s > 0 \]
are called the Zygmund sums. Within chapter IV in [6], $C^\psi_\infty$ is class of $2\pi$-periodic continuous functions which represented in the form of convolution

\[
f(x) = \frac{a_0}{2} + \frac{1}{\pi} \int_{-\pi}^{\pi} \varphi(x-t)\Psi(t)dt = \frac{a_0}{2} + (f\overline{\psi} * \Psi)(x),
\]

where $\Psi(x)$ is a certain function that has the Fourier series

\[
\sum_{k=1}^{\infty} (\psi_1(k) \cos kx + \psi_2(k) \sin kx),
\]

$\overline{\psi} = (\psi_1, \psi_2)$ is a pair of arbitrary fixed systems of numbers $\psi_1(k)$ and $\psi_2(k)$, $k = 1, 2, \cdots$. Here, the function $\varphi$ is called $\overline{\psi}$-derivative of function $f$, and is denoted by $f\overline{\psi}(\cdot)$, ess sup $|f\overline{\psi}(t)| \leq 1$, $\int_{-\pi}^{\pi} f\overline{\psi}(t)dt = 0$.

In [6], if $\psi_1(v) = \psi(v) \cos \frac{\beta \pi}{2}$ and $\psi_2(v) = \psi(v) \sin \frac{\beta \pi}{2}$, then the classes $C^\psi_\infty$ coincide with the classes $C^\psi_{\beta, \infty}$. Moreover, if $\psi(v) = v^{-r}$, then the classes $C^\psi_\infty$ coincide with the well known the classes $W^r_{\beta, \infty}$-Weil-Nagy.

We will give asymptotic results related to estimation of the value

\[
E_n(C^\psi_\infty, Z^s_n) = \sup_{f \in C^\psi_\infty} \|f(x) - Z^s_n(f; x)\|_C
\]

under various conditions on functions $\psi_1(\cdot)$ and $\psi_2(\cdot)$, where $\|\varphi\|_C = \max_x |\varphi(x)|$.

The value $E_n(\mathfrak{M}, Z^s_n)_{\mathfrak{M}}$ was investigated by many mathematicians. Some of whom are A. Zygmund [8] that investigated in case of $\mathfrak{M} = W^r_{\infty}$, $r > 0$; B. Nagy, S. A. Teljakovski [5], [7] that investigated in case of $\mathfrak{M} = W^r_{\infty}$ under various conditions on $\beta, s, r$; A. I. Stepanets, D. N. Busev [6], [1] that investigated in case of $\mathfrak{M} = C^\psi_{\beta, \infty}$ under the condition on function $\psi(\cdot)$; A. S. Federenko, [3], [4] and U. Değer, [2] that investigated in case of $\mathfrak{M} = C^\psi_\infty$ under the various conditions on functions $\psi_1(\cdot)$ and $\psi_2(\cdot)$, respectively.

The value of (1) is depend on the functions $g_i(v) = v^s\psi_i(v), i = 1, 2$, which are convex upwards or convex downwards. There are five possible cases for functions $g_i(v), i = 1, 2$:

a) $g_i(v)$ are convex downwards with $\lim_{v \to \infty} g_i(v) = \infty$,

b) $g_i(v)$ are convex downwards with $\lim_{v \to \infty} g_i(v) = C > 0$,

c) $g_i(v)$ are convex downwards with $\lim_{v \to \infty} g_i(v) = 0$,

d) $g_i(v)$ are convex upwards with $\lim_{v \to \infty} g_i(v) = c > 0$,

e) $g_i(v)$ are convex upwards with $\lim_{v \to \infty} g_i(v) = \infty$. 
In [2], we gave the some asymptotic equalities in the cases of d) and e) and in this paper, we obtain some asymptotic results in case of a) for $\psi_1 \in M$ (or $-\psi_1 \in M$), and $\psi_2 \in M'$ (or $-\psi_2 \in M'$) about value (1). Here, in [6, Chpt.IV], $M$ denotes the set of continuous positive functions $\psi(\cdot)$ which are convex downwards for all $v \geq 1$ and with $\lim_{v \to \infty} \psi(v) = 0$ and $M'$ denotes the subset of functions $\psi(\cdot)$ from $M$ that satisfy in addition the following condition:

$$\int_1^\infty \frac{\psi(t)}{t} dt < \infty.$$ 

2 Main Results.

Theorem 2.1. Let $\psi_1 \in M$, $\psi_2 \in M'$ and $g_i(v) = v^s \psi_i(v)$, $s > 1$, $i = 1, 2$, be convex downwards on $v \geq b \geq 1$ with $\lim_{v \to \infty} g_i(v) = \infty$. Then as $n \to \infty$, we have

$$E_n(C^\psi, Z_n^s) = \frac{2}{\pi n^s} \int_1^n v^{s-1} \psi_2(v) dv + \frac{2}{\pi} \int_{n}^{\infty} \frac{\psi_2(v)}{v} dv + O(1) \bar{\psi}(n),$$

(2)

where $\bar{\psi}(n) = (\psi_1^2(n) + \psi_2^2(n))^{1/2}$ and $O(1)$ is a quantity uniformly bounded in $n$.

Let $\psi \in M$ and $\alpha(t) = \frac{\psi(t)}{t|\psi'(t)|}$ for $t \geq 1$. If there exist $\lim_{t \to \infty} \alpha(t)$, then let us denote value of this limit by $\alpha_0(\psi) \overset{df}{=} \lim_{t \to \infty} \alpha(t)$. Therefore we get the following corollary:

Corollary 2.2. Let $\psi_1 \in M$, $\psi_2 \in M'$ and $g_i(v) = v^s \psi_i(v)$, $s > 1$, $i = 1, 2$, be convex downwards on $v \geq b \geq 1$ with $\lim_{v \to \infty} g_i(v) = \infty$. If $\alpha_0(\psi_2) = \infty$, then as $n \to \infty$, we have the following asymptotic equality:

$$E_n(C^\psi, Z_n^s) = \frac{2}{\pi} \int_{n}^{\infty} \frac{\psi_2(v)}{v} dv + O(1) \bar{\psi}(n).$$

Remark 2.3. The same problem was investigated by A. S. Fedorenko in [3] and also in case of $\psi_1(v) = \psi(v) \cos \frac{\beta \pi}{2}$ and $\psi_2(v) = \psi(v) \sin \frac{\beta \pi}{2}$ by D. N. Busev, [1], but they didn’t find exact asymptotic equalities for value (1).
Corollary 2.4. Let \( \psi_1 \in \mathcal{M} \), \( \psi_2 \in \mathcal{M}' \) and \( g_i(v) = v^s \psi_i(v) \), \( s > 1 \), \( i = 1, 2 \), be convex downwards on \( v \geq b \geq 1 \) with \( \lim_{v \to \infty} g_i(v) = \infty \) or \( \lim_{v \to \infty} g_i(v) = c_i \geq 0 \). If \( \alpha_0(\psi_2) = 1/s \) then as \( n \to \infty \), we have the following asymptotic equality:

\[
\mathcal{E}_n(C_\infty^\psi, Z_n^s)_C = \frac{2}{\pi n^s} \int_1^n v^{s-1} \psi_2(v) dv + O(1) \psi(n).
\]

Corollary 2.5. Let \( \psi_1 \in \mathcal{M} \), \( \psi_2 \in \mathcal{M}' \) and \( g_i(v) = v^s \psi_i(v) \), \( s > 1 \), \( i = 1, 2 \), be convex downwards on \( v \geq b \geq 1 \) with \( \lim_{v \to \infty} g_i(v) = \infty \). If \( \alpha_0(\psi_2) \in (1/s, \infty) \), then as \( n \to \infty \), we have

\[
\mathcal{E}_n(C_\infty^\psi, Z_n^s)_C = O(1) \psi(n).
\]

If we take functions \( \psi_i(t) = \frac{1}{\ln \alpha_i(t + 2)} \), \( \alpha_i > 1 \) and \( \psi_i(t) = \frac{\ln(\sqrt{x} + c_i)}{x^r_i} \), \( 1 < r_i < s \), \( c_i > 0 \), \( i = 1, 2 \), \( \frac{\psi_2(t)}{\psi_1(t)} \neq \text{const} \), then these functions satisfy the conditions of Theorem 2.1.

3 Some Auxiliary Results.

In this section, we shall give some auxiliary results which used for the proof of the Theorem 2.1.

Proposition 3.1. Let \( \psi_1(\cdot) \in \mathcal{M} \) and let \( g_1(v) = v^s \psi_1(v) \), \( s > 1 \), be convex downwards on \( v \geq b \geq 1 \) with \( \lim_{v \to \infty} g_1(v) = \infty \). Then as \( n \to \infty \), we have

\[
\int_\infty^{-\infty} \frac{1}{\pi} \int_0^\infty \tau_1(v) \cos vt \ dv \ dt = O(1) \psi_1(n),
\]

where

\[
\tau_1(v) = \begin{cases} 
\frac{v \psi_1(1)}{\psi_1(\psi_1)}, & 0 \leq v \leq 1 \\
\frac{v^s \psi_1(v)}{\psi_1(\psi_1)}, & 1 \leq v \leq n \\
\frac{n^s}{\psi_1(v)}, & v \geq n
\end{cases}
\]

where \( O(1) \) is a quantity uniformly bounded in \( n \).
Proof. Let’s consider function \( H_n(v) \) that define the following \([0, \infty)\):

\[
H_n(v) = \begin{cases} 
  v\psi_1'(n) + \psi_1(n) - n\psi_1'(n), & 0 \leq v \leq n \\
  \psi_1(v), & v \geq n
\end{cases}
\]

\( H_n(v) \) is a continuous function that is convex downwards and monotony decreasing on \([0, \infty)\). On the other hand, it coincides with function \( \tau_1(v) \) on interval \([n, \infty)\). \( \tau_1(v) \) is a positive continuous function on interval \([0, \infty)\) that is increasing on interval \([0, n]\). Meanwhile \( \tau_1'(v) \) is continuous on interval \([0, 1]\) and \([1, n]\), and let \( \lim_{v \to \infty} \tau_1(v) = \lim_{v \to \infty} \tau_1'(v) = 0 \) on interval \([n, \infty)\).

By applying two times partial integration on the integral \( \int_0^\infty \tau_1(v) \cos vt dv \), then we have

\[
\int_0^\infty \tau_1(v) \cos vt dv = -\frac{1}{t^2} \left[ -\psi_1(1) + \psi_1(n) \cos nt - \psi_1'(1) \cos t \right] - \left( \int_1^n \tau_1''(v) \cos vt dv + \int_n^\infty \tau_1''(v) \cos vt dv \right). \tag{4}
\]

From (4), since \( g_1(v) \) is a convex downwards with \( \lim_{v \to \infty} g_1(v) = \infty \),

\[
\left| \frac{1}{\pi} \int_0^\infty \tau_1(v) \cos vt dv \right| \leq \frac{2}{\pi t^2} \left( \psi_1(1) + \psi_1'(1) \right). \tag{5}
\]

Therefore, accordingly (5), we have

\[
\int_{|t| \geq \frac{n}{2n-1}} \left| \frac{1}{\pi} \int_0^\infty \tau_1(v) \cos vt dv \right| dt = 2 \int_{\frac{n}{2n-1}}^{1/n} \left| \frac{1}{\pi} \int_0^\infty \tau_1(v) \cos vt dv \right| dt \leq c_1 \psi_1(n). \tag{6}
\]

where \( c_1 = \frac{8}{\pi} \left( \frac{\psi_1(1) + \psi_1'(1)}{\psi_1'(1)} \right) \).

Now we will show the following asymptotic statement:

\[
2 \int_{1/n}^{1/2} \left| \frac{1}{\pi} \int_0^\infty \tau_1(v) \cos vt dv \right| dt = O(1) \psi_1(n). \tag{7}
\]

\[
2 \int_{1/2}^{n/2n-1} \left| \frac{1}{\pi} \int_0^\infty \tau_1(v) \cos vt dv \right| dt = O(1) \psi_1(n). \tag{8}
\]
Let’s show the statement (7):

\[
2 \int_0^{1/n} \int_0^\infty |\tau_1(v)\cos vt| dt \leq 2 \int_0^{1/n} \int_0^1 |\tau_1(v)\cos vt| dt +
\]

\[
+ 2 \int_0^{1/n} \int_n^\infty |\tau_1(v)\cos vt| dt := I_1 + I_2
\]

Since \(g_1(v)\) is a convex downwards on \([0,n]\), we have \(|\tau_1(v)| \leq \psi_1(n)\) on interval \([0,n]\). Then for integral \(I_1\), we get

\[
I_1 \leq \frac{2}{\pi} n \psi_1(n) \frac{1}{n} = \frac{2}{\pi} \psi_1(n).
\]

(9)

For integral \(I_2\), according to function \(H_n(v)\);

\[
I_2 = 2 \int_0^{1/n} \int_0^\infty |\frac{1}{\pi} H_n(v)\cos vt| dt \leq
\]

\[
\leq 2 \int_0^{1/n} \int_0^\infty |H_n(v)\cos vt| dt + 2 \int_0^{1/n} \int_0^n |H_n(v)\cos vt| dt := I_{21} + I_{22}
\]

Firstly let’s show that \(I_{21} = O(1)\psi_1(n)\). For the simplicity we will denote

\[
I_{21} := 2 \int_0^{1/n} |I_{211}| dt
\]

where

\[
I_{211} = \frac{1}{\pi} \int_0^\infty H_n(v)\cos vt dv.
\]

By partial integration for \(I_{211}\), we have

\[
I_{211} = \frac{1}{\pi t} \int_0^\infty (-H_n'(v)\sin vt) dv
\]

Since \(H_n(v)\) is a nonincreasing and convex, \((-H_n'(v))\) is a nonnegative and nonincreasing. Thus for any \(t > 0\),

\[
\frac{1}{t} \int_0^\infty (-H_n'(v)\sin vt) dv > 0.
\]

(10)
Hence, owing to (10), we have

$$I_{21} = 2 \int_0^{1/n} |I_{211}| dt = 2 \int_0^{1/n} \frac{1}{\pi t} \int_0^\infty (-H_n'(v) \sin vt) dv dt,$$

By Fubini’s theorem, we obtain

$$I_{21} = 2 \pi \int_0^\infty (-H_n'(v)) \frac{1}{\pi t} \int_0^{1/n} \sin vt dt dv \leq H_n(0) + n|\psi_1'(n)| \leq (1 + s) \psi_1(n).$$

(11)

Secondly, let’s estimate that $I_{22} = O(1) \psi_1(n)$. Since $H_n(v)$ is a function that is monotony decreasing, then we have

$$I_{22} = 2 \int_0^{1/n} |\frac{1}{\pi} \int_0^n H_n(v) \cos vt dv| dt \leq \frac{2}{\pi} \int_0^{1/n} n \int_0^n |H_n(v)| dv dt \leq \frac{2n}{\pi} \int_0^{1/n} H_n(0) dt =$$

$$= \frac{2}{\pi} (\psi_1(n) + n|\psi_1'(n)|) \leq \frac{2(1 + s)}{\pi} \psi_1(n).$$

(12)

According to (9), (11) and (12), we get (7). Now we will show the asymptotic statement (8). By partial integration, we get

$$\frac{1}{\pi} \int_0^\infty \tau_1(v) \cos vt dv = \frac{1}{\pi t} \int_0^n \tau_1'(v) \sin vt dv + \frac{1}{\pi t} \int_n^\infty (-\tau_1'(v)) \sin vt dv := J_1 + J_2$$

Let’s show that

$$2 \int_1^{n/2n-1} |J_1| dt = O(1) \psi_1(n).$$

(13)

For this purpose, we consider the function

$$f_t(x) = \int_0^x \varphi(v) \sin vt dv, \ x > 0, \ t > 0$$

(14)

where $\varphi(v)$ is nonnegative and nondecreasing function for all $v \geq 1$. 

The function \( f_t(x) \) is a continuous for every fixed \( t \). Also, on every interval between the consecutive zeros \( v_k \) and \( v_{k+1} \) of the function \( \sin vt \) the function \( f_t(x) \) has one simple zero \( x_k \), [6, chpt. IV]. Thus by assuming that \( x_k' \) is zero nearest from the left of the point \( n \), we have \( n - v_k \leq \frac{2\pi}{t} \). In view of this, by setting \( \varphi(v) = \tau'_1(v) \) on interval \([0, n]\) in (14), we find

\[
|J_1| = \frac{1}{\pi t} \int_{x_k'}^{n} \tau'_1(v) \sin vt dv.
\]

Hence since \( \tau'_1(v) \) is nondecreasing on \([0, n]\), we get

\[
2 \int_{1/n}^{n/2n-1} \frac{1}{1/n} \int \tau'_1(n - x_k') dt \leq 4\tau'_1(n) \int_{1/n}^{n/2n-1} \frac{dt}{t^2} = 4 \left( \psi_1(n) - n|\psi'_1(n)| \right) \left( n - 1 \right) \leq 4\psi_1(n) \tag{15}
\]

Therefore we get (13). Now let’s estimate that

\[
2 \int_{1/n}^{n/2n-1} \frac{1}{1/n} \int |J_2| dt = O(1)\psi_1(n). \tag{16}
\]

Similarly to (13), we consider the function,

\[
g_t(y) = \int_{y}^{\infty} \varphi(v) \sin vt dv, \quad x > 0, \quad t > 0 \tag{17}
\]

where \( \varphi(v) \) is nonnegative and nonincreasing function for all \( v \geq 1 \).

The function \( g_t(y) \) is a continuous for every fixed \( t \). Also, on every interval between the consecutive zeros \( v_k \) and \( v_{k+1} \) of the function \( \sin vt \) the function \( g_t(y) \) has one simple zero \( y_k \). Thus, by assuming that \( y_k' \) is zero nearest from the right of the point \( n \), we have \( n \leq y_k' \leq n + \frac{2\pi}{t} \). Since the function \( (-\tau'_1(v)) \) is nonnegative and nonincreasing, by setting \( \psi_1(v) = -\tau'_1(v) \) in (17), we find

\[
|J_2| = \frac{1}{t} \int_{n}^{\infty} (-\tau'_1(v)) \sin vt dv \leq \frac{1}{t} \int_{n}^{n+2\pi/t} |\tau'_1(v)| dv \leq \frac{2\pi|\psi'_1(n)|}{t^2}
\]

Hence

\[
2 \int_{1/n}^{n/2n-1} |J_2| dt \leq 4|\psi'_1(n)| \int_{1/n}^{n/2n-1} \frac{dt}{t^2} \leq 4s\psi_1(n) \tag{18}
\]
Therefore we have (16). By combining (15) and (18), we get (8). According to (7) and (8), (3) is proved.

**Proposition 3.2.** Let $\psi_2(\cdot) \in \mathcal{W}$ and let $g_2(v) = v^s \psi_2(v)$, $s > 1$, be convex downwards on $v \geq b \geq 1$ with $\lim_{v \to \infty} g_2(v) = \infty$. Then as $n \to \infty$, we have

$$\int_{-\infty}^{\infty} \frac{1}{\pi} \int_{0}^{\infty} \tau_2(v) \sin vt dv|dt = \frac{2}{\pi n^s} \int_{1}^{n^s} \psi_2(v) dv + \frac{2}{\pi n} \int_{n^s}^{\infty} \frac{\psi_2(v)}{v} dv + O(1) \psi_2(n),$$

where

$$\tau_2(v) = \begin{cases} \frac{v \psi_2(1)}{n^s}, & 0 \leq v \leq 1 \\ \frac{v^s \psi_2(v)}{v^s}, & 1 \leq v \leq n \\ \frac{n^s}{\psi_2(v)}, & v \geq n \end{cases}$$

and $O(1)$ is a quantity uniformly bounded in $n$.

**Proof.** Since the rest of the proof of the proposition 3.2 is get similarly by proof of proposition 2 in [2], except the following integral; here we will estimate only this one for $\pi/2n \leq t \leq \pi/2$:

$$\frac{1}{t} \int_{\pi/2t}^{\pi/2t} \tau_2'(v) \cos vt dv = \frac{1}{t} \int_{0}^{\pi/2t} \tau_2'(v) \cos vt dv + \frac{1}{t} \int_{\pi/2t}^{n} \tau_2'(v) \cos vt dv dt. \quad (20)$$

In [2], we know that

$$\int_{\pi/2n}^{\pi/2} \frac{1}{t} \int_{0}^{\pi/2t} \tau_2'(v) \cos vt dv|dt = \frac{1}{n^s} \int_{1}^{n} v^{s-1} \psi_2(v) dv + O(\psi_2(n)).$$

Now by considering the second part of the (20) equality, we will proof the following asymptotic statement:

$$\int_{\pi/2n}^{\pi/2} \frac{1}{t} \int_{\pi/2t}^{n} \tau_2'(v) \cos vt dv|dt = O(\psi_2(n)). \quad (21)$$

For this aim, we consider the function

$$\phi_t(x) = \int_{\pi/2t}^{x} \varphi(v) \cos vt dv, \quad t > 0, \ x > 0 \quad (22)$$
where \( \varphi(v) \) is nonnegative and nondecreasing function for all \( v \geq 1 \). The function \( \phi_t(x) \) is a continuous function for every fixed \( t \). Also, on every interval between the consecutive zeros \( v_k \) and \( v_{k+1} \) of the function \( \cos vt \) the function \( \phi_t(x) \) has one simple zero \( x_k \). Thus let’s assume that \( x'_k \) is zero the nearest from the left of the point \( n \). In view of this, by setting \( \varphi(v) = \tau'_2(v) \) on interval \([1, n]\) in (22), we find

\[
\frac{1}{t} \int_{\pi/2}^{\pi/2t} \tau'_2(v) \cos vt \, dv = \frac{1}{t} \int_{x_k'}^{n} \tau'_2(v) \cos vt \, dv
\]

\[
\int_{\pi/2n}^{\pi/2} \int_{\pi/2t}^{\pi/2} \tau'_2(v) \cos vt \, dv \, dt \leq \int_{\pi/2n}^{\pi/2} \tau'_2(n) \frac{n - x_k}{t} \, dt \leq 4s \psi_2(n) \frac{(n-1)}{n} \leq 4s \psi_2(n).
\]

Hence we get (21). Therefore we have (19).

\( \square \)

4 Proof of the Theorem 2.1.

Proof. In [2], we know that

\[
\mathcal{E}_n(C_\infty, Z^n_s) = \int_{-\infty}^{\infty} |\hat{\tau}_n(t)| \, dt + \gamma(n) \quad s > 0
\]

(23)

where \( \gamma(n) \leq 0 \) and

\[
|\gamma(n)| = O(\int_{|t| \geq \frac{\pi}{2}} |\hat{\tau}_n(t)| \, dt)
\]

By using (23) and Proposition 3.1-3.2, we will proof the Theorem 2.1. Firstly, let us estimate \( \gamma(n) \):

\[
|\gamma(n)| \leq O(1) \int_{|t| \geq \frac{\pi}{2}} |\hat{\tau}_n(t)| \, dt \leq O(1) \int_{|t| \geq \frac{\pi}{2}} \left[ \frac{1}{\pi} \int_{0}^{\infty} \tau_1(v) \cos vt \, dv \right] \, dt +
\]

\[
+ O(1) \int_{|t| \geq \frac{\pi}{2}} \left[ \frac{1}{\pi} \int_{0}^{\infty} \tau_2(v) \sin vt \, dv \right] \, dt := \gamma_1 + \gamma_2.
\]

We get completely analogously to estimations of (6) that \( \gamma_1 = O(1)\psi_1(n) \) and we know that \( \gamma_2 = O(1)\psi_2(n) \) from [2], as well. For this reason, we have \( |\gamma(n)| \leq O(1)\psi(n) \). Finally, according to Proposition 3.1-3.2, we get (2). Therefore, the proof of the Theorem 2.1 is completed. \( \square \)
Proof of Corollary 2.2-2.5. By L’Hopital’s and Leibniz rules we obtain the following relations:

\[
\lim_{x \to \infty} \frac{\psi_2(v)}{v} \int_x^\infty \psi_2(v) \, dv = \lim_{x \to \infty} \frac{\psi_2(x)}{x |\psi'_2(x)|},
\]

\[
\lim_{x \to \infty} \frac{\psi_2(x)}{x^s} \frac{1}{\int_1^x v^{s-1} \psi_2(v) \, dv} = \lim_{x \to \infty} s - \frac{x |\psi'_2(x)|}{\psi_2(x)},
\]

and

\[
\lim_{x \to \infty} \frac{1}{\int_x^\infty \psi_2(v) \, dv} \frac{x^{s-1} \psi_2(v) \, dv}{\int_1^\infty \psi_2(v) \, dv} = -1 + \lim_{x \to \infty} \frac{1}{1 - \frac{x |\psi'_2(x)|}{s \psi_2(x)}}.
\]

Therefore, the proofs of the Corollary 2.2-2.5 are easily get by relations (2) and (24)-(26).

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