On bi-Γ-Ideal in Γ-Semirings

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Abstract

The notion of Γ-semiring was introduced by M. Murali Krishna Rao [5] as a generalization of Γ-ring as well as of semiring. We have known that Γ-semirings are a generalization of semirings. In this paper the notion of bi-Γ-ideals in Γ-semirings is introduced. We show that bi-Γ-ideals in Γ-semirings are a generalization of bi-ideals in semirings and we give some properties for bi-Γ-ideals in Γ-semirings. We give two definition as follows: A Γ-semiring $M$ is called a bi-simple Γ-semiring if $M$ is the unique bi-Γ-ideal of $M$ and a bi-Γ-ideal $B$ of $M$ is called minimal bi-Γ-ideal of $M$ if $B$ does not properly contain any bi-Γ-ideal of $M$.

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1 Preliminaries

Let $S$ and $Γ$ be two additive commutative semigroups. Then $S$ is called Γ-semiring if there exists a mapping $S \times Γ \times S \rightarrow S$ (image to be denoted by $aab$ for $a, b \in S$ and $α \in Γ$) satisfying the following conditions.

(i) $aα(b + c) = aab + aac$

(ii) $(a + b)αc = aac + bαc$

(iii) $a(α + β)b = aαb + aβb$

(iv) $aaα(bβc) = (aab)αβc$

∀ $a, b, c \in S$ and for all $α, β \in Γ$. 
**Definition 1.1.** A subsemigroup $B$ of $S$ such that $BSB \subseteq B$ is called bi-ideal of $S$. Both semigroups and semirings bi-ideals generalize quasi-ideals.

**Example 1.2.** Let $N$ be the set of all Natural numbers. $N$ is a commutative semigroup under usual multiplication. Let $B = 2N$ Thus $BNB = 4N \subseteq 2N = B$ hence $B$ is a bi-ideal of $N$.

**Definition 1.3.** Let $K$ be a nonempty subset of $S$, $K$ is called a sub $\Gamma$-semiring of $S$ if $a\gamma b \in k \quad \forall a, b \in k$ and $\gamma \in \Gamma$.

**Definition 1.4.** Let $M$ be $\Gamma$-semiring $A$ sub $\Gamma$-semiring $B$ of $M$ is called a bi-$\Gamma$-ideal of $M$ if $B \Gamma M \Gamma B \subseteq B$.

**Lemma 1.5.** Let $M$ be a $\Gamma$-semiring and $B$ is a bi-$\Gamma$-ideal of $M$ if $\cap B \neq \phi$, Then $\cap B$ is a bi-$\Gamma$-ideal of $M$.

**Proof.** Let $\cap B \neq \phi$

Let $a, b \in \cap B, m \in M$ and $\gamma, \mu \in \Gamma$ Then $a, b \in B$ since $B$ is bi-$\Gamma$-ideal of $M$

$\therefore a\gamma b \in B$ and $a\gamma \mu b \in B \Gamma B \subseteq B$

Therefore $a\gamma b \in \cap B$ and $a\gamma \mu b \in \cap B$. hence $\cap B$ is bi-$\Gamma$-ideal of $M$. $\square$

**Definition 1.6.** Let $A$ be a non empty subset of a $\Gamma$-semiring $M$ then $$(A) = A \cup A\Gamma A \cup A\Gamma M \Gamma A$$

**Proof.** Let $A$ be a nonempty subset of a $\Gamma$-semiring $M$ Also let

$$B = A \cup A\Gamma A \cup A\Gamma M \Gamma A$$

$\Rightarrow A \subseteq B$

$\square$

We have that

$$B\Gamma B = (A \cup A\Gamma A) \cup (A\Gamma M \Gamma A) \Gamma (A \cup A\Gamma A) \cup (A\Gamma M \Gamma A) \subseteq A \cup A\Gamma A \cup A\Gamma M \Gamma A \subseteq B$$

$\Rightarrow$ Hence $B$ is a sub-$\Gamma$-remiring of $M$.

Since $M$ is a $\Gamma$-semiring, all elements in $B \Gamma M \Gamma B = (A \cup A\Gamma A \cup A\Gamma M \Gamma A) \Gamma M \Gamma (A \cup A\Gamma A \cup A\Gamma M \Gamma A)$ are in the form of $a\gamma_1 m \mu_2$ for some $a, a_2 \in A, \mu_1 \gamma \in \Gamma$. Thus $B \Gamma M \Gamma B \subseteq A\Gamma M \Gamma A \subseteq B$. Therefore $B$ is a bi-$\Gamma$-ideal of $M$.

Let $C$ be any bi-$\Gamma$-ideal of $M$ containing $A$. Since $C$ is a sub $\Gamma$-semiring of $MA \subseteq C, A\Gamma A \subseteq C$. Since $C$ is a bi-$\Gamma$-ideal of $M$ and $A \subseteq CA\Gamma MA \subseteq C$. Therefore $B = A \cup A\Gamma A \cup A\Gamma M \Gamma A \subseteq C$. Hence $B$ is the smallest bi-$\Gamma$-ideal of $M$ containing $A$. Therefore $(A) = B = A \cup A\Gamma A \cup A\Gamma M \Gamma A$ as required.
Theorem 1.7. Let $M$ be a $\Gamma$-semiring. Let $B$ be a bi-$\Gamma$-ideal of $M$ and $A$ be a non empty subset of $M$ then following are true.

(i) $B\Gamma A$ is a bi-$\Gamma$-ideal of $M$

(ii) $A\Gamma B$ is a bi-$\Gamma$-ideal of $M$.

Proof. (i) We see that $(B\Gamma A)\Gamma(B\Gamma A) = (B\Gamma A\Gamma B)\Gamma A$ and

$$(B\Gamma A)\Gamma M\Gamma(B\Gamma A) = (B\Gamma A\Gamma M\Gamma B)\Gamma A$$

Since $B$ is a bi-$\Gamma$-ideal of $M$.

$$(B\Gamma A)\Gamma(B\Gamma A) = (B\Gamma A\Gamma B)\Gamma A \subseteq B\Gamma A$$

and

$$(B\Gamma A)\Gamma M\Gamma(B\Gamma A) \subseteq (B\Gamma M\Gamma B)\Gamma A \subseteq B\Gamma A.$$ 

Therefore $B\Gamma A$ is a bi-$\Gamma$-ideal of $M$.

(ii) Similar to (i) $\square$

Definition 1.8 (Dulta, [2]). A $\Gamma$-semiring $M$ is called simple if $M\Gamma M \neq \{0\}$ and $M$ has no ideals other then 0 and $M$.

Definition 1.9. Let $M$ be a $\Gamma$-semiring , $M$ is called simple $\Gamma$-semiring if $M$ is the Unique bi-$\Gamma$-ideal of $M$.

Theorem 2.0. Let $M$ be a $\Gamma$-semiring then $M$ is a simple $\Gamma$-semiring if $M = m\Gamma M\Gamma m$ for all $m \in M$.

Proof. Let $M$ is a simple $\Gamma$-semiring let $m \in M$. By Theorem 1.7 (i) $m\Gamma M\Gamma M$ is a bi-$\Gamma$-ideal of $M$. Then $M = m\Gamma M\Gamma m$.

Let $B$ be a bi-$\Gamma$-ideal of $M$. Let $b \in B$, by above we get $M = b\Gamma M\Gamma b \subseteq B\Gamma M\Gamma B \subseteq B$

Hence $M = B$.

So $M$ is a simple $\Gamma$-semiring. $\square$

Theorem 2.1. Let $M$ be a $\Gamma$-semiring and $B$ a bi-$\Gamma$-ideal of $M$ then $B$ is minimal bi-$\Gamma$-ideal of $M$ if and only if $B$ is a simple $\Gamma$-semiring.

Proof. Let $B$ a minimal bi-$\Gamma$-ideal of $M$. Let $C$ be a bi-$\Gamma$-ideal of $B$. Then $CTB\Gamma C \subseteq C$. Since $B$ is a bi-$\Gamma$-ideal of $M$ by the Theorem 1.7(i) $CTB\Gamma C$ is a bi-$\Gamma$-ideal of $M$. Since $B$ is a minimal bi-$\Gamma$-ideal of $M$. So $CTB\Gamma C = B$.

Hence $B = CTB\Gamma C \subseteq C$. 
This implies that \( B = C \). Then \( B \) is a simple \( \Gamma \)-semiring.

Conversely let \( B \) be a simple \( \Gamma \)-semiring, let \( C \) be a bi-\( \Gamma \)-ideal of \( M \) such that \( C \subseteq B \). Then \( \text{CT}_B C \subseteq \text{CT}_M C \subseteq C \).

Therefore \( C \) is a bi-\( \Gamma \)-ideal of \( B \). Since \( B \) is a simple \( F \) semiring so \( B \) is a minimal bi-\( \Gamma \)-ideal of \( M \) as required. \( \square \)

**Definition 2.2 (Dutta, [2]).** Let \( M \) be a \( \Gamma \)-remiring and \( \Gamma \) be the free additive commutative semigroup generated by \( M \times \Gamma \). Then left and right operator semiring is defined.

**Definition 2.3.** An ideal \( P \) of a \( \Gamma \)-semiring \( M \) is Prime if for any ideals \( A, B, \subseteq M, A \Gamma B \subseteq P \) implies \( A \subseteq P \) or \( B \subseteq P \).

For a subset \( Q \subseteq R \) where \( R \) is the right operator semiring of a gamma semiring define by [6] as:

\[
Q^* = \{ a \in M \mid [\Gamma, a] = [\Gamma, \{ a \}] \subseteq Q \}
\]

and for a subset \( P \subseteq M \)

\[
P^* = \{ r \in R \mid Mr \subseteq P \}.
\]

Similarly for \( L, a \) left operator semiring we have if \( S \subseteq L \) then

\[
S^+ = \{ a \in M \mid [a, \Gamma] = [\{ a \}, \Gamma] \subseteq S \}
\]

and for a \( p \subseteq M \)

\[
P^+ = \{ \ell \in L \mid \ell m \subseteq P \}.
\]

\[ \Rightarrow \] If \( P \) is an ideal of gamma semiring \( M \) then \( P^+ \) is an ideal of \( L \)

**Theorem 2.4.** Let \( P, Q \) and \( S \) be Prime ideals of a \( \Gamma \)-semiring \( M \), a Prime ideal at the right operator semiring \( R \) and a prime ideal at the left operator semiring \( L \) respectively. Then \( P^* \) is a prime ideal of \( R, P^+ \) is a prime ideal of \( L, Q^* \) and \( S^+ \) are Prime ideals of \( M \).

**Proof.** Let \( I \) and \( J \) be ideals of \( R \) such that \( IJ \subseteq P^* \) where \( P^* = \{ r \in R \mid Mr \subseteq P \} \). Since \( iJ \) is an ideal therefor \( i\Gamma M J = iR J \subseteq IJ \) and Then \( i\Gamma M J \subseteq P^* \).

Thus \( M i \Gamma M J \subseteq P \). But since \( P \) is prime this implies that \( MI \subseteq P \) or \( MJ \subseteq P \). Hence \( I \subseteq P^* \) or \( J \subseteq P^* \), which proves \( P^* \) is Prime. Similarly It can be verified that \( P^+ \) is Prime ideal of Left operator semiring of a \( \Gamma \)-semiring.

Let \( A, B \) be ideals of \( M \) such that \( A \Gamma B \subseteq Q^* \), where \( Q^* = \{ x \in M \mid [\Gamma, x] \subseteq Q \} \). Then by [6] \( [\Gamma, A][\Gamma, B] = [\Gamma, A \Gamma B] \subseteq Q \), where \( [\Gamma, A], [\Gamma, B] \) are ideals of \( M \). Since \( Q \) is Prime, \( [\Gamma, A] \subseteq Q \) or \( [\Gamma, B] \subseteq Q \),

\[ \Rightarrow A \subseteq Q^* \text{ or } B \subseteq Q^* , \]

\( \Rightarrow Q^* \) is Prime

similarly It can be verified that \( S^+ \) is Prime. \( \square \)
Proposition 2.5. The intersection of an Arbitrary set of bi-$\Gamma$-ideals $B\lambda$ ($\lambda \in \Lambda$) of a $\Gamma$-semiring $S$ is again a bi-ideal of $S$.

Proof. Let $B$ be a bi-ideal of a $\Gamma$-semiring $S$. Thus $B\Sigma B \subseteq B$ and $B\Gamma S B = BSB \subseteq B$

Hence $B$ is a bi-$\Gamma$-ideal of $S$ set $B = \bigcap_{\lambda \in \Lambda} B\lambda$ clearly $B$ is a subsemiring of $S$. By inclusion $B\lambda \Gamma S B\lambda \subseteq B\lambda$ and $B \subseteq B\lambda (\forall \lambda \in \Lambda)$ It follows that

$B\Gamma S B \subseteq B\lambda S B\lambda \subseteq B\lambda \forall \lambda \in \Lambda$

Consequently we have

$B\Gamma S B \subseteq B$

\[\square\]

Proposition 2.6. The intersection of a bi-$\Gamma$-ideal $B$ of a $\Gamma$-semiring $S$ and a subsemiring $A$ of $S$ is always a bi-$\Gamma$-ideal of the $\Gamma$-semiring $S$.

Proof. Let us Assume that $C = B \cap A$

Since $A$ is subsemiring and $C \subseteq A$ we have

$CAC \subseteq AAA \subseteq A$

On the other hand $CAC \subseteq BAB \subseteq BSB \subseteq B$, hence $CAC \subseteq B \cap A = C$.

\[\square\]

References


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