Stability Behavior of Some Well-Known Stochastic Financial Models

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Abstract

In this paper, we study the stability behavior of a solution process (be it the bond value, price or interest rate) of several well-known stochastic financial models (like Black-Scholes, Vasicek and Cox-Ingersoll-Ross and others). We also investigate the stability of some of these models with jumps.

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1 Introduction

Since the publication of the celebrated paper by Black-Scholes [2] in 1973, the field of mathematical techniques applied to finance, today known as mathematical finance, has been a very active and fruitful area of research; and forms one of the major branches of study in and outside mathematics. A simple example is a bond maintained in a bank with a constant interest rate \(r > 0\). The value of this bond is governed by the differential equation:

\[
\frac{dB(t)}{dt} = rB(t), \quad t \geq 0, \quad B(0) = B_0; \quad (1.1)
\]

and the value of the bond at time \(t\), \(B(t)\) is explicitly solved as

\[B(t) = B_0 e^{rt}, \quad t \geq 0,\]

in terms of the initial value of the bond \(B_0\). However, the assumption that the interest rate \(r\) is a constant is reasonable in a small interval of time. On

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the contrary, when the time is large, it will be more realistic to consider it as a function of time, namely, \(r(t)\). In this case, the bond value at any time \(t\) in the future is given by

\[
B(t) = B_0 \exp\left\{ \int_0^t r(s)ds \right\}.
\]

Further, it is natural to consider the generalization of this situation where the bond value depends not only on time but also on random phenomenon. In that case, the bond value, now a stochastic process, is governed by a stochastic differential equation (SDE):

\[
dB(t) = B(t)[\mu(t, \omega)dt + \sigma(t, \omega)dW(t)], \quad t > 0,
\]

where \(W(t)\) is a Wiener process \([1, 7]\); and the interest rate on the other hand is again a stochastic process described by

\[
dr(t) = a(t, r(t))dt + b(t, r(t))dW(t), \quad t > 0,
\]

subject to appropriate initial conditions. We refer to Sviščuk and Kalemanova \([10]\) and the references therein for more details.

The question that curiously or naturally arises next is the behavior of these processes (bond value or interest rate) with respect to time. To put it the other way, “How stable would be the bond value over a period of time in a bank?” Surprisingly, there is not much known on the answer(s) to this question! Recently, Stoica \([9]\) Sviščuk, et al \([10]\) and Govindan, et al \([5]\) answered this question to some extent. In other words, they studied some stability behaviors, for instance, the asymptotic \(p\)-stability of solutions of several well-known stochastic financial models. Though the stability of SDEs is a well-established area of research, its applications to financial models is negligible. More so, none of the well-known texts on mathematical finance addresses this issue.

In this paper, our goal is to study principally exponential stability behavior of some well-known financial models and also their generalizations with jumps as considered in \([10]\). Note that the latter stability concept has not been dealt with in \([10]\). In the process, we shall also attempt to simplify some of the proofs from \([10]\) following more direct methods.

In the sections to follow, we shall give sufficient conditions under which the Black-Scholes, Vasicek and mean reverting models are all exponentially stable. Furthermore, we will also consider exponential stability of some of these models with jumps. Concerning the CIR model, as in \([10]\), we are able only to study the asymptotic behavior of moments of its solution. As a consequence of exponential stability, we shall study the almost sure exponential stability of the sample paths of the solution process as well.
In the rest of this section, we formulate the stability concepts. Consider the stochastic differential equation

\[ dx(t) = f(t, x(t))dt + g(t, x(t))dW(t), \quad t > t_0, \]  

(1.2)

where the functions \( f(t, x) \) and \( g(t, x) \) are deterministic and \( \{W(t)\}_{t \geq t_0} \) is a Wiener process. Let \( f(t, 0) = g(t, 0) = 0 \) a.e. \( t_0 \), so that equation (1.2) admits the trivial solution \( x(t) \equiv 0 \). Throughout this paper we assume the existence and uniqueness of a solution of all the models we study. However, we refer to [1, 7] for details.

Let \( x(t) = x(t; t_0, x_0) \) a solution of (1.2) with the initial condition \( x(t_0) = x_0 \), where \( x_0 \in \mathbb{R} \) is a constant (or sometimes, we shall consider \( x_0 \) as a random variable independent of \( W(t) \) for \( t > t_0 \), with \( E|x_0|^2 < \infty \), where \( E(\cdot) \) denotes the mathematical expectation). Let \( x^*(t; t_0, x^*_0) \) be any other solution of (1.2) with \( x^*(t_0; t_0, x^*_0) = x^*_0 \).

The following definitions are from Hasminskii [6].

**Definition 1.1** The solution \( x(t) \) of (1.2) is called exponentially stable in the quadratic mean, or exponentially \( 2 \)–stable if there exist positive constants \( K \) and \( \gamma \) such that

\[ E|x(t) - x^*(t)|^2 \leq KE|x_0 - x^*_0|^2 \exp(-\gamma(t - t_0)), \]

for \( t \geq t_0 \).

**Definition 1.2** The trivial solution is called \( 2 \)–stable if for all \( \varepsilon > 0 \) there exists a \( \delta > 0 \) such that

\[ E|x(t)|^2 < \varepsilon, \]

for \( t \geq t_0 \) and \( |x_0| < \delta \); and is called asymptotically \( 2 \)–stable if it is \( 2 \)–stable and

\[ E|x(t)|^2 \to 0, \quad t \to \infty, \]

for sufficiently small \( |x_0| \).

**Definition 1.3** The solution \( x(t) \) of (1.2) is bounded in probability uniformly in \( t \) if

\[ \sup_{t \geq t_0} P\{|x(t)| > R\} \to 0, \quad R \to \infty, \]

for sufficiently small \( |x_0| \).
2 Black-Scholes model

In this section, we study the exponential stability of the Black-Scholes model [2] for the stock price process \( \{ S(t) \}_{t \geq 0} \) described by the SDE:

\[
dS(t) = S(t)(\mu dt + \sigma dW(t)), \quad t > 0, \\
S(0) = S_0 > 0;
\]

where \( \mu \) and \( \sigma > 0 \) constants. Consider also the bond value process \( \{ B(t) \}_{t \geq 0} \) which satisfies

\[
dB(t) = rB(t)dt, \quad t > 0,
\]

with a constant rate of interest \( r > 0 \).

Proposition 2.1 If \( \mu < -\frac{1}{2} \sigma^2 \), then the trivial solution of equation (2.1) is exponentially \( 2 \)-stable, and hence also \( 2 \)-stable.

Proof. The general solution of equation (2.1) has the form

\[
S(t) = S_0 \exp\{(\mu - \frac{\sigma^2}{2})t + \sigma W(t)\}, \quad t \geq 0.
\]

We now find the expression for the second moment of \( S(t) \):

\[
E|S(t)|^2 = E\left(|S_0|^2 \exp\{2(\mu - \frac{\sigma^2}{2})t + 2\sigma W(t)\}\right)
\]

\[
= E|S_0|^2 \exp\{2(\mu - \frac{\sigma^2}{2})t\} E(\exp\{2\sigma W(t)\})
\]

\[
= E|S_0|^2 \exp\{2(\mu - \frac{\sigma^2}{2})t + 2\sigma^2 t\}
\]

\[
= E|S_0|^2 \exp\{2(\mu + \frac{\sigma^2}{2})t\}, \quad t \geq 0.
\]

This completes the proof. \( \Box \)

Consider then the discounted value process \( X(t) = S(t)/B(t) \) which is written in the form

\[
X(t) = \frac{S_0}{B_0} \exp\{(\mu - r - \frac{\sigma^2}{2})t + \sigma W(t)\}, \quad t \geq 0,
\]

and is a solution of the equation

\[
dX(t) = X(t)((\mu - r)dt + \sigma dW(t)), \quad t > 0.
\]

As a consequence of Proposition 2.1 we obtain the following result for the stability of the solution \( X(t) \equiv 0 \) of (2.4).
Corollary 2.2 If $\mu < r - \frac{1}{2}\sigma^2$, then the solution $X(t) \equiv 0$ of equation (2.4) is exponentially 2–stable.

We next consider a result as in [4].

Proposition 2.3 Under the hypothesis of Corollary 2.2, the solution of equation (2.4) is almost surely exponentially stable and satisfies for $\gamma > 0$:
\[
\limsup_{t \to \infty} \frac{1}{t} \log |X(t)| \leq -\frac{\gamma}{4}, \text{ a.s.}
\]
where $\gamma = r - \frac{1}{2}\sigma^2 - \mu$.

Proof. The proof is standard and it can be obtained using arguments from [4] exploiting the Borel–Cantelli lemma. \hfill \Box

3 Vasicek interest rate model

In this section, we investigate the exponential 2–stability of the interest rate model. The Vasicek model [11] for the interest rate process $\{r(t)\}_{t \geq 0}$ is given by
\[
dr(t) = (\alpha - \beta r(t))dt + \sigma dW(t), \quad t > 0, \quad r(0) = r_0,
\]
where $\alpha, \beta, \sigma$, and $r_0$ are positive constants. The solution of equation (3.1) in a closed form is given by [8]:
\[
r(t) = r_0 e^{-\beta t} + \frac{\alpha}{\beta} (1 - e^{-\beta t}) + \sigma e^{-\beta t} \int_0^t e^{\beta u} dW(u).
\]
From well-known properties of the Itô integral, we have that $r(t)$ is normally distributed with mean
\[
Er(t) = r_0 e^{-\beta t} + \frac{\alpha}{\beta} (1 - e^{-\beta t}),
\]
and variance
\[
\text{Var } r(t) = \frac{\sigma^2}{2\beta} (1 - e^{-\beta t}).
\]
The mean $Er(t)$ tends to $\alpha/\beta$ as $t \to \infty$ and the variance $\text{Var } r(t)$ tends to $\sigma^2/2\beta$ as $t \to \infty$. The Vasicek model has the property that the interest rate is mean reverting, see [8, p. 151].

Proposition 3.1 The solution of equation (3.1) given in (3.2) is exponentially 2–stable.
**Proof.** Let $r^*(t)$ the solution of (3.1) satisfying $r^*(0) = r_0^*$. Then using the corresponding equation (3.2) for $r^*(t)$, we have

$$r(t) - r^*(t) = (r_0 - r_0^*)e^{-\beta t}, \quad t \geq 0.$$ 

Taking expectations on both sides of the last equation, we obtain

$$E|r(t) - r^*(t)|^2 = E|r_0 - r_0^*|^2e^{-\beta t}, \quad t \geq 0.$$ 

Hence, $r(t)$ is exponentially 2–stable. 

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### 4 Mean reverting model for the stock prices

In this section, we consider the following model for the stock prices process

$$dS(t) = \alpha(L - S(t))dt + \sigma S(t)dW(t), \quad t > 0,$$ \hspace{1cm} (4.1)

where $\alpha, L,$ and $\sigma$ are positive constants. We study below its exponential stability.

Applying Itô’s formula \[1\] to the process $V(t) = e^{\alpha t}S(t)$, we have

$$dV(t) = \alpha e^{\alpha t}L dt + \sigma e^{\alpha t}S(t)dW(t).$$ \hspace{1cm} (4.2)

Integration of both sides of (4.2) then yields:

$$e^{\alpha t}S(t) = S_0 + \alpha L \int_0^t e^{\alpha u}du + \sigma \int_0^t e^{\alpha u}S(u)dW(u)$$

or, equivalently,

$$S(t) = S_0 e^{-\alpha t} + L(1 - e^{-\alpha t}) + \sigma e^{-\alpha t} \int_0^t e^{\alpha u}S(u)dW(u).$$ \hspace{1cm} (4.3)

**Proposition 4.1** If $\sigma^2 < 2\alpha$, then the solution of equation (4.1) is exponentially stable in the quadratic mean.

**Proof.** Let $S^*(t)$ be another solution of (4.1) with $S^*(0) = S_0^*$. Using Itô’s formula, from (4.3), we obtain

$$[S(t) - S^*(t)]^2 = [S_0 - S_0^*]^2 e^{-2\alpha t} + 2e^{-\alpha t}[S_0 - S_0^*] \int_0^t \sigma e^{\alpha u}[S(u) - S^*(u)]dW(u)$$

$$+ \sigma^2 e^{-2\alpha t} \left[ \int_0^t e^{\alpha u}(S(u) - S^*(u))dW(u) \right]^2.$$
Thus
\[ E|S(t) - S^*(t)|^2 = E|S_0 - S_0^*|^2 e^{-2\alpha t} + \sigma^2 e^{-2\alpha t} \int_0^t e^{2\alpha u} E|S(u) - S(u)|^2 du. \]
The last equation can be written in the form:
\[ e^{\alpha t} E|S(t) - S^*(t)|^2 = E|S_0 - S_0^*|^2 + \sigma^2 \int_0^t e^{\alpha u} E|S(u) - S^*(u)|^2 du. \]
Applying Gronwall’s lemma, it follows that
\[ E|S(t) - S^*(t)|^2 \leq E|S_0 - S_0^*|^2 \exp((\sigma^2 - 2\alpha)t), \]
showing the exponential stability.

5 Cox-Ingersoll-Ross interest rate model

In this section, we consider the Cox-Ingersoll-Ross interest rate model [3]:
\[ dr(t) = (\alpha - \beta r(t))dt + \sigma \sqrt{r(t)}dW(t), \quad t > 0, \quad (5.1) \]
where \( \alpha, \beta, \) and \( \sigma \) are positive constants. Unlike the Vasicek model, the CIR model (5.1) does not have a closed-form solution. It is worth mentioning here that our approach is more direct than the one in [10] as it uses the basic tools from stochastic calculus. We are interested in showing the boundedness in probability of solutions of (5.1).

Applying Itô’s formula to the process \( X(t) = e^{\beta t}r(t) \) gives
\[ X(t) = r_0 + \frac{\alpha}{\beta}(e^{\beta t} - 1) + \sigma \int_0^t e^{\beta u} \sqrt{r(u)}dW(u). \quad (5.2) \]
Taking expectations on both sides of (5.2), we get
\[ EX(t) = r_0 + \frac{\alpha}{\beta}(e^{\beta t} - 1). \quad (5.3) \]
Next, consider the processes \( X(t) \) and \( X^*(t) = e^{\beta t}r^*(t) \), where \( r(t) \) and \( r^*(t) \) are solutions of (5.1) with the initial conditions \( r(0) = r_0, \ r^*(0) = r_0^* \), respectively. Exploiting again Itô’s formula, we obtain the following stochastic differential for \( [X(t) - X^*(t)]^2 \):
\[ d[X(t) - X^*(t)]^2 = \sigma^2 e^{\beta t} \left( \sqrt{X(t)} - \sqrt{X^*(t)} \right)^2 dt \\
+ 2\sigma e^{\beta t/2} (X(t) - X^*(t)) \left( \sqrt{X(t)} - \sqrt{X^*(t)} \right) dW(t). \quad (5.4) \]
Integrating both sides of (5.4) and then taking expectation, we get

\[ E|X(t) - X^*(t)|^2 = |r_0 - r_0^*|^2 + \int_0^t \sigma^2 e^{\beta u} E\left| \sqrt{X(u)} - \sqrt{X^*(u)} \right|^2 du. \]  

(5.5)

From equations (5.5) and (5.3), it follows that

\[ E|r(t) - r^*(t)|^2 \leq |r_0 - r_0^*|^2 e^{-\beta t} + \frac{3}{2}(r_0 + r_0^*)(1 - e^{-\beta t})e^{-\beta t} \]

\[ + \frac{3\sigma^2 \alpha}{\beta^2} \left( \frac{1}{2} - e^{-\beta t} + \frac{1}{2} e^{-2\beta t} \right), \quad t \geq 0. \]  

(5.6)

The following result is an easy consequence of (5.6).

**Proposition 5.1** For the CIR model (5.1), we have

\[ \limsup_{t \to \infty} E|r(t) - r^*(t)|^2 \leq \frac{3\sigma^2 \alpha}{2\beta^2}. \]

**Proposition 5.2** For the CIR model (5.1), we have that \( r(t) \) is bounded in probability uniformly in \( t \).

**Proof.** For the equation (5.1), consider the generating operator \( L \) [6, p. 164] applied to the function \( V(r) = r^2, \quad r > 0 \):

\[ L(V(r)) = V_r(r)f(r) + \frac{1}{2} g^2(r)V_{rr}(r) \]

\[ = (2\alpha + \sigma^2)r - 2\beta r^2. \]  

(5.7)

From (5.7), we have that

\[ L(V(r)) \leq 0 \quad \text{for} \quad r \geq \frac{2\alpha + \sigma^2}{2\beta}. \]  

(5.8)

Now, taking expectations in the Itô’s formula for \( EV(r(t)) \) and using (5.8), we obtain that if \( r(t) \geq \frac{2\alpha + \sigma^2}{2\beta} \), then

\[ E|r(t)|^2 = EV(r(t)) = E(V(r_0)) + \int_0^t EL(V(r(t)))dt \]

\[ \leq EV(r_0) = E|r_0|^2. \]

Finally from the last inequality, if \( R \geq \frac{2\alpha + \sigma^2}{2\beta} \), then

\[ P\{|r(t, \omega)| > R\} \leq \frac{1}{R^2} E|r(t)|^2 \leq \frac{1}{R^2} E|r_0|^2, \]

which means that \( r(t) \) is bounded in probability uniformly in \( t \).
6 A model of (B, S) securities market with jumps

Assume the bond process as given in (1.2) and that the stock price process is varying according to (2.1) on the intervals \([\tau_i, \tau_{i+1})\), \(i = 1, 2, \ldots\), while at the random times \(\tau_i\), the values jump, namely,

\[ S(\tau_i) - S(\tau_i-) = S(\tau_i-)U_i, \]

or

\[ S(\tau_i) = (1 + U_i)S(\tau_i-). \] (6.1)

We assume that the total number of jumps on the interval \([0, t]\), denoted by \(N(t)\), is a Poisson process with intensity \(\lambda > 0\). We also assume that the jumps \(\{U_i\}_{i \geq 0}\) form a sequence of i.i.d. random variables taking values in \((-1, +\infty)\).

In [10], the process that satisfies (6.1) is characterized as follows:

**Proposition 6.1** [10] The stock value process described in (6.1) is of the form

\[ S(t) = S_0 \left( \prod_{i=1}^{N(t)} (1 + U_i) \right) \exp \left\{ (\mu - \frac{\sigma^2}{2})t + \sigma W(t) \right\}, \quad t \geq 0. \] (6.2)

The stock value process can be represented as a stochastic integral equation, namely,

\[ S(t) = S_0 + \int_0^t S_0(\mu du + \sigma dW(u)) + \sum_{i=1}^{N(t)} S(\tau_i-)U_i. \] (6.3)

The following result gives the exponential stability.

**Proposition 6.2** Assume that \(E(1 + U_i)^2 < \infty\). If \(\mu < -\frac{1}{2}\sigma^2 - \frac{1}{3}(E(1 + U_i)^2 - 1)\), then the solution \(S(t) \equiv 0\) of (6.3) is exponentially 2–stable.

**Proof.** Using equation (6.2) and Lemma 1 [10], we have:

\[
E|S(t)|^2 = E|S_0|^2E\left( \prod_{i=1}^{N(t)} (1 + U_i)^2 \exp \left\{ (\mu - \frac{\sigma^2}{2})t + 2\sigma W(t) \right\} \right) = E|S_0|^2 \exp \left\{ (2\mu + \sigma^2 + \lambda(E(1 + U_i)^2 - 1))t \right\},
\]

and the exponential 2–stability follows. \(\square\)

Next, for the discounted process with random jumps \(X(t) = S(t)/B(t)\) we have

\[ X(t) = \frac{S_0}{B_0} + \int_0^t X(u)((\mu - r)du + \sigma dW(u)) + \sum_{i=1}^{N(t)} X(\tau_i-)U_i. \] (6.4)
The solution of (6.4) can be written in the form
\[ X(t) = \frac{S_0}{B_0} \left( \prod_{i=1}^{N(t)} (1 + U_i) \right) \exp \left\{ (\mu - r - \frac{\sigma^2}{2})t + \sigma W(t) \right\}, \quad t \geq 0. \]

**Corollary 6.3** If \( \mu < -\frac{\sigma^2}{2} - \frac{1}{2}(E(1+U_1)^2 - 1) + r \), then the solution \( X(t) \equiv 0 \) of equation (6.4) is exponentially 2-stable.

### 7 The Vasicek model of interest rates with jumps

In this last section, we consider the exponential stability of the interest rate model with jumps. Assume in the Vasicek model that the interest process is continuous on time intervals \([\tau_i, \tau_{i+1})\), \(i = 1, 2, \ldots\). At random times \(\tau_i\), the interest jumps
\[ r(\tau_i) = (1 + U_i)r(\tau_i-). \]

The number of jumps in the interval \([0, t]\), denoted by \(N(t)\) is assumed to be a Poisson process with intensity \(\lambda > 0\). The jumps \(\{U_i\}_{i \geq 0}\) form a sequence of i.i.d. random variables taking values in \((-1, +\infty)\).

Applying the results from [10], the process \(r(t)\) can be represented in the following form:
\[ r(t) = \prod_{i=1}^{N(t)} (1 + U_i) \left( e^{-\beta t} r_0 + \frac{\alpha}{\beta} (1 - e^{-\beta t}) + \sigma e^{-\beta t} \int_0^t e^{\beta s} dW(s) \right), \quad t \geq 0. \quad (7.1) \]

**Proposition 7.1** Assume that \(E(1+U_1)^2 < \infty\). If \(\lambda(E(1+U_1)^2 - 1) - 2\beta < 0\), then the solution (7.1) of the Vasicek model with jumps is exponentially 2-stable.

**Proof.** Let \(r^*(t)\) be any other solution of the Vasicek model with jumps which satisfies the initial condition \(r^*(0) = r_0^*\). Then from (7.1), we obtain
\[ r(t) - r^*(t) = \prod_{i=1}^{N(t)} (1 + U_i)(r_0 - r_0^*)e^{-\beta t}, \quad t \geq 0. \]

Using Lemma 1 from [10], we have
\[ E|r(t) - r^*(t)|^2 = E|r_0 - r_0^*|^2 e^{-2\beta t} \exp \left\{ \lambda t(E(1+U_1)^2 - 1) \right\} \]
\[ = E|r_0 - r_0^*|^2 \exp \left\{ (\lambda(E(1+U_1)^2 - 1) - 2\beta)t \right\}. \]

The last equality yields the desired conclusion. \(\square\)
8 Conclusions

In this work, we studied the exponential stability of Black-Scholes, Vasicek and mean reverting financial models by giving sufficient conditions that guarantee their stability behavior. Indeed, we also did address the stability issue of the Vasicek and (B,S) securities model with jumps. In the case of CIR model, we could only obtain the boundedness in probability of its solution which could be viewed as stability in some sense. Finally, we also investigated the almost sure exponential stability of the sample paths of the Black-Scholes process.

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