Surfaces of Revolution in the 3-Dimensional Lorentz-Minkowski Space Satisfying $\Delta x^i = \lambda^i x^i$

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Abstract

In this paper the surfaces of revolution with no zero Gaussian curvature $K_G$ in the 3-dimensional Lorentz-Minkowski space are classified under the condition $\Delta x^i = \lambda^i x^i$ where $\Delta$ is the Laplace operator with respect to the induced metric and $\lambda$ is real. More precisely we prove that such surfaces are either minimal or Lorentz hyperbolic cylinders or circular cylinders or pseudospheres of real radius or pseudohyperbolic space of imaginary radius.

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1. Introduction

Let $M^2$ be a 2-dimensional surface of the Lorentz-Minkowski 3-space $\mathbb{R}^3_1$ equipped with the induced metric. By saying Lorentz-Minkowski space $\mathbb{R}^3_1$, we mean the space $\mathbb{R}^3$ with the standard metric given by

$$g = -dx_0^2 + dx_1^2 + dx_2^2$$

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where $x_0, x_1, x_2$ are the pseudo- Euclidean coordinate of type $(1, 2)$. Denote by $\vec{x}$, $\vec{H}$ and $\Delta$ respectively the position vector field, the mean curvature vector field and the Laplacian operator of $M$, with respect to the induced metric $g$ on $M^2$. Then, as it is well known [2]:

\begin{equation}
\Delta \vec{x} = -2 \vec{H}
\end{equation}

which shows, in particular, that $M^2$ is minimal surface in $\mathbb{R}^3$ if and only if its coordinate functions are harmonic (i.e. they are eigenfunctions of $\Delta$ with eigenvalue $0$). Moreover, in this context, T. Takahashi [10] states that minimal surfaces and spheres are the only surfaces in $\mathbb{R}^3$ satisfying the condition

\begin{equation}
\Delta \vec{x} = \lambda \vec{x}, \quad \lambda \in \mathbb{R}
\end{equation}

In terms of B. Y. Chen’s theory of surfaces in the Euclidean space of finite type, condition (1.2) asserts that $M^2$ is of 1-type.

Following Chen’s idea, Garay [6] determined the complete surfaces of revolution in $\mathbb{R}^3$, whose component functions are eigenfunctions of its Laplacian with possibly distinct eigenvalue, i.e.

\begin{equation}
\Delta x^i = \lambda^i x^i
\end{equation}

Later the same autor in [7] studied the hypersurfaces in $\mathbb{R}^{n+1}$ for which

\begin{equation}
\Delta \vec{x} = A \vec{x}, \quad A \in \mathbb{R}^{n+1 \times n+1}
\end{equation}

In [3], Dillen-Pas-Verstraelen pointed out that Garay’s condition is not coordinate invariant as a circular cylinder in $\mathbb{R}^3$ shows. Then they study and classify the surfaces in $\mathbb{R}^3$ which satisfy the condition

\begin{equation}
\Delta \vec{x} = A \vec{x} + B
\end{equation}

where $A \in \mathbb{R}^{3 \times 3}$ and $B \in \mathbb{R}^3$. From the formula (1.1), we know that minimal surfaces and spheres also verify the condition

\begin{equation}
\Delta \vec{H} = \lambda \vec{H}, \quad \lambda \in \mathbb{R}
\end{equation}

On the other hand, in [4] Ferrandez, Garay and Lucas proved the surfaces of $\mathbb{R}^3$ satisfying the condition (1.4) are minimal or an open part of an ordinary sphere or of a circular cylinder.
Ferrandez and Lucas in [5] classified the surface $M^2_s$ in $\mathbb{R}^3_1$ with index $s = 0, 1$ which satisfies the condition (1.3). They proved that $M^2_s$ is a zero mean curvature surface everywhere, either on a open piece of a B-scroll surface on a open piece of the surface $S^1(r) \times \mathbb{R}, S^1_1(r) \times \mathbb{R}, H^1(r) \times \mathbb{R}, H^2(r), S^1_1(r)$.

Recently, in [8] Kaimakamis and Papantoniou studied surfaces of revolution in Lorentz-Minkowski 3-space satisfying the condition

$$\Delta^{II} \mathbf{x}^v = A \mathbf{x}^u, \quad A \in \text{Mat}(3, \mathbb{R}),$$

where $\Delta^{II}$ is the Laplace operator with respect to the second fundamental form and $\text{Mat}(3, \mathbb{R})$ is the set of $3 \times 3$-real matrices.

In this article, we investigate the lorentz version of surfaces of revolution satisfying the condition (1.3).

2. Preliminaries

Let $\alpha : I = (a, b) \subset \mathbb{R} \rightarrow \Pi$ be a curve in plane $\Pi$ of $\mathbb{R}^3_1$ and let $\varepsilon$ be a straight line of $\Pi$ which does not intersect the curve $\alpha$. A surface of revolution $M^2$ in $\mathbb{R}^3_1$ is defined as a non-degenerate surface which is generated by the rigid motions $g_t : \mathbb{R}^3_1 \rightarrow \mathbb{R}^3_1$, $t \in \mathbb{R}$ around the axis $\varepsilon$. In other words, a surface of revolution $M^2$ with axis $\varepsilon$ in $\mathbb{R}^3_1$ is invariant under the one parameter subgroup of the rigid motions in $\mathbb{R}^3_1$. From this definition we obtain four types of surfaces of revolution in $\mathbb{R}^3_1$. If the axis $\varepsilon$ is space-like (resp. time-like) then there is a Lorentz transformation, by which, the axis $\varepsilon$ is transformed to the $x_2$-axis, where $Ox_0x_1x_2$ is the considered coordinate system. So, we may assume that the axis of revolution is the $x_2$-axis (resp. the $x_0$-axis). Since the surface $M^2$ is non-degenerate it suffices to consider the case that the plane $\Pi$ is space-like or time-like. Hence, without loss of generality, we may assume that $\Pi$ is $x_1x_2$-plane or the $x_0x_2$-plane. If the axis of revolution is light-like then we may assume that this is the line spanned by the vector $(1, 1, 0)$. Therefore, we distinguish the following three special cases.

I. First case. Suppose that the axis of revolution is a space-like line and without loss of generality, we may assume that the curve $\alpha$ is lying in the $x_1x_2$-plane or in the $x_0x_2$-plane. So, the curve $\alpha$ is parametrized either by

$$\alpha (u) = (0, f (u), g (u)) \quad \text{or} \quad \alpha (u) = (f (u), 0, g (u))$$

where $f, g$ are smooth functions and $f$ is a positive function. One can easily prove that the subgroup of the Lorentz group which fixes the vector $(0, 0, 1)$ consists of the matrices
\[ A(v) = \begin{pmatrix} \cosh v & \sinh v & 0 \\ \sinh v & \cosh v & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad v \in \mathbb{R} \]

Hence, the surface of revolution \( M^2 \) can be parametrized either as

\[ x(u, v) = \begin{pmatrix} \cosh v & \sinh v & 0 \\ \sinh v & \cosh v & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ f(u) \\ g(u) \end{pmatrix} \]

so

\[ (2.1) \quad x(u, v) = (f(u) \sinh v, f(u) \cosh v, g(u)) \]

or as

\[ x(u, v) = \begin{pmatrix} \cosh v & \sinh v & 0 \\ \sinh v & \cosh v & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} f(u) \\ 0 \\ g(u) \end{pmatrix} \]

so

\[ (2.2) \quad x(u, v) = (f(u) \cosh v, f(u) \sinh v, g(u)) \]

**II. Second case.** Suppose that the axis of revolution is a time-like line and without loss of generality we may assume that the curve \( \alpha \) lies in the \( x_0x_1 \)-plane. Hence, one of its parametrizations is

\[ \alpha(u) = (f(u), g(u), 0) \]

where \( f, g \) are smooth functions and \( f \) is a positive one. In this case, the subgroup of the Lorentz group, which fixes the vector \( (1, 0, 0) \), consists of the matrices

\[ A(v) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos v & -\sin v \\ 0 & \sin v & \cos v \end{pmatrix}, \quad 0 \leq v \leq 2\pi \]

Hence, the surface of revolution \( M^2 \) around the axis \( 0x_0 \) can be parametrized as

\[ x(u, v) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos v & -\sin v \\ 0 & \sin v & \cos v \end{pmatrix} \begin{pmatrix} f(u) \\ g(u) \\ 0 \end{pmatrix} \]
so

$$x(u, v) = (g(u), f(u) \cos v, f(u) \sin v)$$

III. Third case. Suppose that the axis of revolution is a light-like line, or equivalently the line of the plane $x_0x_1$ spanned by the vector $(1, 1, 0)$. Since the surface $M^2$ is non-degenerate, we can assume without loss of generality that the curve $\alpha$ lies in the $x_0x_1$-plane and its parametrization is given by

$$\alpha(u) = (f(u), g(u), 0), \quad u \in I$$

where $f, g$ are functions on $I$, such that $f(u) \neq g(u), \forall u \in I$. After some easy calculations it can be proved that the subgroup of the Lorentz group, which fixes the vector $(1, 1, 0)$, is given by the set of $3 \times 3$ matrices

$$A(v) = \begin{pmatrix} 1 + \frac{v^2}{2} & -\frac{v^2}{2} & v \\ \frac{v^2}{2} & 1 - \frac{v^2}{2} & v \\ v & -v & 1 \end{pmatrix}, \quad v \in \mathbb{R}$$

Therefore, the surface of revolution $M^2$ may be parametrized in the following way

$$x(u, v) = \begin{pmatrix} 1 + \frac{v^2}{2} & -\frac{v^2}{2} & v \\ \frac{v^2}{2} & 1 - \frac{v^2}{2} & v \\ v & -v & 1 \end{pmatrix} \begin{pmatrix} f(u) \\ g(u) \\ 0 \end{pmatrix}$$

so that

$$x(u, v) = \left(1 + \frac{v^2}{2}f(u) - \frac{v^2}{2}g(u), \frac{v^2}{2}f(u) + (1 - \frac{v^2}{2})g(u), f(u) - g(u)\right)$$

We denote by $E, F, G; L, M, N$ the coefficients of first and second fundamental form, respectively, of these surfaces. If $\psi(u, v)$ is a smooth function, the second differential parameter of Beltrami (Laplacian) of a function $\psi(u, v)$ with respect to the first fundamental form of $M^2$ is the operator $\Delta$ which is defined by [9] as

$$\Delta \psi = -\frac{1}{\sqrt{|EG - F^2|}} \left[ \left( \frac{G\psi_u - F\psi_v}{\sqrt{|EG - F^2|}} \right)_u - \left( \frac{F\psi_u - E\psi_v}{\sqrt{|EG - F^2|}} \right)_v \right]$$

The mean curvature $H$ and the Gaussian curvature $K_G$ are given by
\[ H = \frac{GL + EN - 2FM}{2(EG - F^2)}, \quad K_G = \frac{LN - M^2}{EG - F^2} \]

3. The main results

In this paragraph we explore the classification of the surfaces of revolution \( M^2 \) satisfying the relation (1.3). We distinguish two cases according to whether these surfaces are given by (2.1) or (2.3).

Case 1. Suppose that \( M^2 \) is given by (2.1). For the sake of simplicity, we suppose without loss of generality that the curve \( \alpha \) is parametrized by the arc length, so

(3.1) \[ f^2(u) + g^2(u) = 1, \quad \forall u \in I \]

(3.2) \[ E = 1, F = 0, G = -f^2; L = g'f'' - f'g'', M = 0, N = fg'; 2H = g'f'' - f'g'' - \frac{g'}{f} \]

where the prime denotes the derivative with respect to \( u \).

If this surface of revolution is constructed with component functions which are eigenfunctions of its Laplacian, we shall have that

(3.3) \[ \Delta (f(u) \sinh v) = \lambda_1 f(u) \sinh v; \Delta (f(u) \cosh v) = \lambda_2 f(u) \cosh v; \Delta (g(u)) = \lambda_3 g(u) \]

where \( \lambda_1, \lambda_2 \) and \( \lambda_3 \in Spec(M^2) \). This means that \( M^2 \) is at most of 3-type.

Since the relation (3.1) holds, there exists a smooth function \( t = t(u) \) such that

(3.4) \[ f'(u) = \cos t(u), \quad g'(u) = \sin t(u) \quad \forall u \in I \]

(3.5) \[ E = 1, G = -f^2(u); L = -t'(u), N = f(u) \sin t(u); \quad 2H = -t'(u) - \frac{\sin t(u)}{f(u)} \]

Suppose that the surface has no zero Gaussian curvature, so

\[ t'(u) \sin t(u) \neq 0, \quad \forall u \in I \]

By using (2.4), (3.3) and (3.5) we get

\[ \frac{f'^2}{f} - t' \sin t - \frac{1}{f} = \lambda_1 f \]
\[ \frac{f'^2}{f} - t' \sin t - \frac{1}{f} = \lambda_2 f \]
\[ \frac{f'g'}{f} + t' \cos t = \lambda_3 g \]
then \( \lambda_1 = \lambda_2 \) which we shall call it simply \( \lambda \) from now on. We put \( \lambda_3 = \mu \).

Therefore, this system of equations is equivalently reduced to the system

\[
\begin{align*}
\frac{f'^2}{f} - t' \sin t - \frac{1}{f} &= \lambda f \\
\frac{f'g'}{f} + t' \cos t &= \mu g
\end{align*}
\]

This means that \( M^2 \) is at most of 2-type.

Therefore, the problem of classifying the surfaces of revolution \( M^2 \), satisfying (1.3) and (2.1) is reduced to the integration of this system of ordinary differential equations. It is remarquable that this classification is strongly dependent on the function \( t = t(u) \). Next we study this system according to the values of the constants \( \lambda, \mu \).

**A.** Let \( \lambda = \mu = 0 \).

If we multiply, the equation (3.6) by \(-\sin t\), the equation (3.7) by \( \cos t \) and add up the resulting equations, we easily get \( H = 0 \), which means that the surfaces are minimal.

**B.** Let \( \lambda = \mu \neq 0 \).

Following the same procedure as in the first case one easily proves that

\[ \lambda(f \cos t + g \sin t) = 0 \]

or

\[ f \cos t + g \sin t = ff' + gg' = 0 \]

Then

\[ f^2(u) + g^2(u) = r^2, \quad r \in \mathbb{R} \]

Therefore, in this case, the surface of revolution satisfies an equation of the form \(-x_0^2 + x_1^2 + x_2^2 = r^2, \quad r \in \mathbb{R}\), which means that these are the pseudospheres \( S^2_1(r) \) of \( \mathbb{R}^3_1 \).

**C.** Let \( \lambda \neq 0 \) and \( \mu = 0 \).

The system of equations (3.6) and (3.7), in this case, takes the form

\[
\begin{align*}
\frac{f'^2}{f} - t' \sin t - \frac{1}{f} &= \lambda f \\
\frac{f'g'}{f} + t' \cos t &= 0
\end{align*}
\]
Applying the same algebraic techniques as in the former cases we have

\[ \lambda f \cos t = 0 \]

from which \( \cos t = 0 \) then the resulting surfaces of revolution are given by the following equation

\[ x(u, v) = (c_1 \sinh v, c_1 \cosh v, u + c_2) \]

where \( c_i \in \mathbb{R}, i = 1, 2 \). These surfaces are the lorentz hyperbolic cylinders \( S^1_1(r) \times \mathbb{R} \).

**D.** Let \( \lambda = 0 \) and \( \mu \neq 0 \).

The system of equations (3.6) and (3.7) is reduced to

\[
\begin{align*}
(3.10) & \quad \frac{f'^2}{f} - t' \sin t - \frac{1}{f} = 0 \\
(3.11) & \quad \frac{f'g'}{f} + t' \cos t = \mu g
\end{align*}
\]

From this system we get \( \mu g \sin t = 0 \) and the fact that \( \sin t \neq 0 \) implies \( \mu = 0 \), which is contradiction. Therefore, there are no surfaces of revolution in this case.

**E.** Let \( \lambda \neq 0, \mu \neq 0 \) and \( \lambda \neq \mu \).

From (3.6) we get

\[
(3.12) \quad t' = \frac{1}{\sin t} \left( \frac{f'^2}{f} - \frac{1}{f} - \lambda f \right)
\]

Combining the equations (3.6) and (3.7) we have

\[
(3.13) \quad \lambda f \cos t + \mu g \sin t = 0
\]

Differentiating this equation and using (3.12) and (3.13) we get

\[
(3.14) \quad \sin^2 t(\lambda \cos^2 t + \mu \sin^2 t + \lambda) + \lambda^2 f^2 = 0
\]

If we differentiate once again this equation and use (3.12) we have

\[
(3.15) \quad -2 \cos t \sin^2 t \left( \mu + (\lambda - \mu) \cos^2 t \right) + f^2 \cos t \left( \lambda^2 - 2\mu \lambda - 2\lambda(\lambda - \mu) \cos^2 t \right) = 0
\]

From this equation and (3.14) we conclude that
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\[(\lambda - \mu) \cos t \sin^2 t \left[ 2(\lambda - \mu) \cos^4 t - (\lambda - 4\mu) \cos^2 t - \lambda - 2\mu \right] = 0\]

\(\cos t\) cannot be equal to 0 because if it were, then equation (3.13) would give \(\mu g \sin t = 0\) which is contradiction. Now since \(\lambda - \mu \neq 0\), then

\[2(\lambda - \mu) \cos^4 t - (\lambda - 4\mu) \cos^2 t - \lambda - 2\mu = 0\]

This equation is valid only for some values of the function \(t = t(u)\) and of the constants \(\lambda, \mu\). Therefore, there are no surfaces of revolution in this case.

We are ready to state the following theorem

**THEOREM 3.1.** Let \(M^2\) be a surface of revolution given by (2.1) in \(\mathbb{R}^3_1\). Then \(\Delta x^i = \lambda^i x^i\) if and only if the following statements hold true

1. \(M^2\) has zero mean curvature
2. \(M^2\) is either the Lorentz hyperbolic cylinder \(S^1_1(r) \times \mathbb{R}\) or the pseudosphere \(S^2_1(r)\) of positive real radius \(r\).

**Case 2.** Suppose now that the immersed surface \(M^2\) in \(\mathbb{R}^3_1\) is given by (2.3). The tangent vector of the revolving curve satisfies the relation

\[\langle \alpha'(u), \alpha'(u) \rangle = f'^2(u) - g'^2(u) = \pm 1, \forall u \in I\]

Consider that

\[(3.16)\hspace{1cm} f'^2(u) - g'^2(u) = -1, \forall u \in I\]

On the other hand it is easily calculated that

\[(3.17)\hspace{1cm} E = -1, F = 0, G = f'^2; L = g'f' - f'g', M = 0, N = fg'; 2H = g'f' - f'g' + \frac{g'}{f}\]

where the prime denotes the derivative with respect to \(u\).

If this surface of revolution is constructed with component functions which are eigenfunctions of its Laplacian, we will have

\[(3.18)\hspace{1cm} \Delta (g(u)) = \lambda_1 g(u); \Delta (f(u) \cos v) = \lambda_2 f(u) \cos v; \Delta (f(u) \sin v) = \lambda_3 f(u) \sin v.\]

where \(\lambda_1, \lambda_2\) and \(\lambda_3 \in Spec(M^2)\). This means that \(M^2\) is at most of 3-type.

Since the relation (3.1) holds, there exists a smooth function \(t = t(u)\) such that

\[(3.19)\hspace{1cm} f'(u) = \sinh t(u), g'(u) = \cosh t(u) \hspace{0.5cm} \forall u \in I\]
(3.20) \[ E = -1, G = f^2(u); L = -t'(u), N = f(u) \cosh t(u); \ 2H = t'(u) + \frac{\cosh t(u)}{f(u)} \]

and since the surface has no zero Gaussian curvature, then

\[ t'(u) \neq 0, \ \forall u \in I \]

Using (2.4), (3.18) and (3.20) we get

\[
\frac{f'^2}{f} + t' \cosh t + \frac{1}{f} = -\lambda_2 f
\]

\[
\frac{f'^2}{f} + t' \cosh t + \frac{1}{f} = -\lambda_3 f
\]

\[
\frac{f'g'}{f} + t' \sinh t = -\lambda_1 g
\]

then \( \lambda_2 = \lambda_3 \) (called simply \( \lambda \)). We put \( \lambda_1 = \mu \).

Therefore, this system of equations is equivalently reduced to the system

(3.21) \[ \frac{f'^2}{f} + t' \cosh t + \frac{1}{f} = -\lambda f \]

(3.22) \[ \frac{f'g'}{f} + t' \sinh t = -\mu g \]

This means that \( M^2 \) is at most of 2-type.

Therefore, the problem of classifying the surfaces of revolution \( M^2 \), satisfying (1.3) and (2.3) is reduced to the integration of this system of ordinary differential equations. Applying similar algebraic methods, used in the first case, we study this system according to the values of the constants \( \lambda, \mu \).

A. Let \( \lambda = \mu = 0 \).

If we multiply (3.21) by \( \sinh t \), (3.21) by \( \cosh t \) and then subtract the resulting equations we easily get \( H = 0 \), which means that the surfaces are minimal.

B. Let \( \lambda = \mu \neq 0 \).

Applying the same algebraic techniques foregoing as in the cases we have

\[ -\lambda (f \sinh t - g \cosh t) = 0 \]

or equivalently
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\[ ff' - gg' = 0 \]

which, integrated, gives us

\[ f^2(u) - g^2(u) = \pm r^2, \quad r \in \mathbb{R}^+ \]

Therefore, the revolving surface is either the pseudosphere \( S^2_1(r) \) of real radius \( r \), given by the equation \(-x_0^2 + x_1^2 + x_2^2 = r^2\), or the pseudo-hyperbolic space \( H^2(r) \) with imaginary radius, given by the equation \(-x_0^2 + x_1^2 + x_2^2 = -r^2\).

C. Let \( \lambda \neq 0, \mu = 0 \)

Following the earlier procedure we get

\[ -\lambda f \sinh t = 0 \]

from which \( \sinh t = 0 \) then the resulting surfaces of revolution are given by the following equation

\[ x(u, v) = (u + c_1, c_2 \cos v, c_2 \sin v) \]

where \( c_i \in \mathbb{R}, i = 1, 2 \). These surfaces are the circular cylinders.

D. Let \( \lambda = 0, \mu \neq 0 \).

In this case the system (3.21), (3.22) reduces to

\[
\begin{align}
(3.23) & \quad \frac{f'^2}{f} + t' \cosh t + \frac{1}{f} = 0 \\
(3.24) & \quad \frac{f'g'}{f} + t' \sinh t = -\mu g
\end{align}
\]

From this system, we get \( \mu g \cosh t = 0 \) which is clearly a contradiction. Hence, there are no surfaces of revolution in this case.

E. Let \( \lambda \neq 0, \mu \neq 0 \) and \( \lambda \neq \mu \).

From (3.21) we have

\[
(3.25) \quad t' = -\frac{1}{\cosh t} \left( \lambda f + \frac{f'^2 + 1}{f} \right)
\]

Combining the equations (3.21), (3.22) we get
(3.26) \[-\lambda f \sinh t + \mu g \cosh t = 0\]

Now, if we differentiate this equation and use (3.25) and (3.26) we get

(3.27) \[\cosh^2 t \left( \mu \cosh^2 t - \lambda \sinh^2 t + \lambda \right) + \lambda^2 f^2 = 0\]

Differentiating once again this equation and using (3.25) allow us to obtain

\[
\sinh t \left[ \cosh^2 t (2\lambda + 2(\mu - \lambda) \cosh^2 t) + f^2 (\lambda^2 + 2\lambda (\mu - \lambda) \cosh^2 t) \right] = 0
\]

But \(\sinh t\) cannot be equal to 0 since if it were, then equation (3.26) would give us \(\mu g \cosh t = 0\). Hence we get a contradiction. Then

\[
\cosh^2 t (2\lambda + 2(\mu - \lambda) \cosh^2 t) + f^2 (\lambda^2 + 2\lambda (\mu - \lambda) \cosh^2 t) = 0
\]

This equation by using (3.27) is reduced to

\[
(\mu - \lambda) \cosh^2 t \left( 2(\mu - \lambda) \cosh^2 t + 3\lambda \right) = 0
\]

This equation is valid only for some values of the function \(t = t(u)\) and of the constants \(\lambda, \mu\). Therefore, there are no surfaces of revolution in this case.

The above analysis, enables us to state the following theorem:

THEOREM 3.1. Let \(M^2\) be a surface of revolution given by (2.1) in \(\mathbb{R}^3\). Then \(\Delta x^i = \lambda^i x^i\) if and only if the following statements hold true
1. \(M^2\) has zero mean curvature
2. \(M^2\) is either the circular cylinder or the pseudosphere \(S^2_1(r)\) of real radius or the pseudohyperbolic space \(H^2(r)\) with imaginary radius.

References


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