Int. J. Contemp. Math. Sci., Vol. 3, 2008, no. 21, 1041 - 1058

Groups with Many Quotients which are PC-Groups

Francesco Russo

Department of Mathematics
University of Naples "Federico II"
via Cinthia, 80126 Naples, Italy
francesco.russo@dma.unina.it

Abstract

A group G is said to be an FC-group if each element x of G has finite conjugacy classes. It is easy to see that this condition is equivalent to require that $G/C_G(\langle x\rangle^G)$ is a finite group for each element x of G. A group G is said to be a PC-group if $G/C_G(\langle x\rangle^G)$ is a polycyclic-by-finite group for each element x of G. The class of PC-groups extends the class of FC-groups.

A group G, which is not a PC-group, but all of whose proper quotients are PC-groups, is said to be a Just-Non-PC group. It has been recently opened the question about the knowledge of their structure. Here we study Just-Non-PC groups.

Mathematics Subject Classification: 20F24; 20C07; 20D10

Keywords: Just-Non-PC groups; groups with trivial center; monolithic groups

1. Introduction

A group G is called *PC-group*, or group with polycyclic-by-finite conjugacy classes, if $G/C_G(\langle x\rangle^G)$ is a polycyclic-by-finite group for each element x of G. An element x of a group G is called a *PC-element* of G if $G/C_G(\langle x\rangle^G)$ is a polycyclic-by-finite group. Of course, a group G is a PC-group if and only if each element of G is a PC-element of G. As noted in [10], the set of all PC-elements of G forms a characteristic subgroup $PC(G)$ of G which is called the *PC-center* of G.

The class of PC-groups was introduced in [2] as a generalization of FC-groups, which are those groups in which every element has finitely many conjugates.

Let $\mathfrak{X}$ be a class of groups. A group G which belongs to $\mathfrak{X}$ is said to be an $\mathfrak{X}$-group. A group G is said to be a *Just-Non-*$\mathfrak{X}$ group, or briefly a $JN\mathfrak{X}$-group, if G does not belong to $\mathfrak{X}$ but all its proper quotients are $\mathfrak{X}$-groups. Of course, every simple group which is not an $\mathfrak{X}$-group is a Just-Non-$\mathfrak{X}$ group, so that in the investigation concerning Just-Non-$\mathfrak{X}$ groups it is natural to require that they have nontrivial Fitting subgroup, i.e. that they contain a nontrivial abelian normal subgroup. The structure of Just-Non-$\mathfrak{X}$ groups has already been studied for several choices of the class $\mathfrak{X}$, so there is a well developed theory about this topic (see [1], [6], [7], [13]). The problem of studying those groups which have not a prescribed property, but all of whose proper quotients have it, was investigated also in theory of finite groups by [6], where the Lagrange property is involved. Therefore, many techniques and methods have general application.

The present paper is devoted to the investigation of Just-Non-$\mathfrak{X}$ groups, where $\mathfrak{X}$ is the class of PC-groups. Such groups are said to be Just-Non-PC groups, or briefly $JNPC$-groups.

Recently in [7] fundamental results have been summarized about the theory of infinite groups which have a prescribed property $\mathfrak{X}$ but all whose proper quotients do not have it and here has been posed for the first time the problem of studying $JNPC$-groups (question n.6, p.180).

In Section 2 some auxiliary results are listed, preparing structural theorems of the next Sections 3,4,5. Our theorems treat circumstances which are mentioned in [1], [7], [13], where Just-Non-FC groups, Just-Non-CC groups, Just-Non-(polycyclic-by-finite) groups and Just-Non-Chernikov groups have been classified. These groups are Just-Non-$\mathfrak{X}$ groups, where $\mathfrak{X}$ is respectively the class of FC-groups, CC-groups, polycyclic-by-finite groups, Chernikov groups. Briefly Just-Non-FC groups will be called $JNFC$-groups. We recall that a group G is said to be a *CC-group* or group with Chernikov conjugacy classes, if $G/C_G(\langle x\rangle^G)$ is a Chernikov group for each element x of G. This class of groups was introduced in [11] as a generalization of FC-groups. It could be useful to refer to [10] as a survey on generalized FC-groups.

A complete description of a $JNPC$-group with Fitting subgroup $Fit\, G \neq 1$ seems very hard to give, because many finitary conditions for $JNPC$-groups are local and on the entire group they are too weak restrictions. If G is a $JNPC$-group with a unique minimal normal subgroup and $G/FitG$ is locally nilpotent, then we are able to classify G; this is the object of Section 3. The notion of FC-hypercentrality is a standard hypothesis when generalized FC-groups have to be treated: this is testified for instance in [9], [[12], Chapters 4 and 5, vol.I], [14]. Requiring a qualitative condition on conjugacy classes (Definition 4.1) and that $G/FitG$ is FC-hypercentral, Theorem 4.13 allows us to reduce the study of $JNPC$-groups to the well known theory of $JNFC$-groups. This is the main result of Section 4. Finally, Section 5 regards $JNPC$-

groups which have restrictions on the abelian rank; we will discover that they are an extension of an abelian group by a polycyclic-by-finite group.

Most of our notation is standard and can be found in [9] and [12]. For general properties of PC-groups and generalized FC-groups, we refer to [2], [3], [5], [8], [10], [11], [12], [14].

2. Some auxiliary results

The following two lemmas recall properties of PC-groups which are described in [2], so the proofs have been omitted.

LEMMA 2.1. – *Let G be a group. G is a PC-group if and only if $\langle X\rangle^G$ is a polycyclic-by-finite subgroup of G, where X is a finite subset of G.*

Lemma 2.1 can be also expressed by saying that a PC-group is locally (normal and polycyclic-by-finite). It follows easily from Lemma 2.1 that $\langle x\rangle^G$ is a polycyclic-by-finite group for each nontrivial PC-element x of a group G.

LEMMA 2.2. – *Quotients, subgroups and direct products of PC-groups are PC-groups.*

A first fact is related to the properties of closure of PC-groups. [[2], Corollary 2.3, Lemma 2.4] give a weak closure by sections of PC-groups. However we know that finite extensions of FC-groups are FC-groups, but finite extensions of PC-groups can not be PC-groups. The following example is emblematic.

EXAMPLE 2.3. – Let G be the locally dihedral 2-group

$$G = \mathbb{D}_{2^\infty} = \langle x\rangle \ltimes \mathbb{Z}_{2^\infty} = \langle x\rangle \ltimes P,$$

where x is an involution which acts on the quasicyclic 2-group P via $a^x = a^{-1}$, for each element $a \in P$. G is a finite extension of P by $\langle x\rangle$ and $G = \langle x\rangle^G$. Clearly $\langle x\rangle^G$ is not polycyclic-by-finite so that G is not a PC-group thanks to [[2], Theorem 2.2].□

This fact is not expected because more closure properties of PC-groups become from closure properties of the class of all polycyclic-by-finite groups. Therefore Example 2.1 proves that a group which contains a normal PC-subgroup of finite index can not be a PC-group. On the other hand a group G which contains a normal finite subgroup F whose quotient group G/F is a PC-group is certainly a PC-group. This is explained by the following statement.

LEMMA 2.4. – *If G is a $JNPC$-group, then G has no nontrivial polycyclic-by-finite normal subgroups. Moreover $PC(G) = 1$.*

Proof. – Obviously every extension of a polycyclic-by-finite group by a PC-group is likewise a PC-group, so that a $JNPC$-group cannot contain nontrivial polycyclic-by-finite normal subgroups.

Since G is a $JNPC$-group, $PC(G) \neq G$. Suppose that x is a nontrivial PC-element of G. As noted in Lemma 2.1, $\langle x \rangle^G$ is a nontrivial polycyclic-by-finite normal subgroup of G. This cannot be so that $PC(G) = 1$.□

Another interesting fact is that a $JNPC$-group is subdirectly indecomposable.

LEMMA 2.5. – *Let G be a $JNPC$-group, then every intersection of two nontrivial normal subgroups of G is nontrivial.*

Proof. – Let H and K be two nontrivial normal subgroups of G. Suppose that $H \cap K$ is trivial. G is isomorphic to a subgroup of the direct product of G/H and G/K. But G/H and G/K are PC-groups, so Lemma 2.2 implies that G is a PC-group and this contradicts the fact that G is a $JNPC$-group.□

THEOREM 2.6. – *If G is a $JNPC$-group, then $Z(G) = 1$.*

Proof. – By Lemma 2.4, G has trivial PC-center $PC(G)$. In general $PC(G)$ includes $Z(G)$ and it follows that $Z(G) = 1$.□

Unfortunately, the structure of PC-groups does not allow us to express a condition similar to [[13], Proposition 2.2]. A $JNFC$-group G can not satisfy *max-n* as testified in [7], so it is clear that a $JNPC$-group can not satisfy *max-n*. In order to adapting [[13], Proposition 2.2] and [[7], Lemma 15.1], we recall the following two notions.
If G is a group, the *Hirsch-Plotkin radical* $HP(G)$ of G is defined to be the unique largest maximal normal locally nilpotent subgroup of G (see [[12], §2, p.57-64] for details).

LEMMA 2.7. – *Let G be a locally soluble $JNPC$-group. If the Hirsch-Plotkin radical of each proper quotient group of G satisfies max-ab, then G is a Just-Non-(polycyclic-by-finite) group.*

Proof. – [[2], Theorem 3.2] implies that a locally soluble PC-group is hyperabelian, so that G has each proper quotient group which is hyperabelian. Now [[12], Theorem 3.31] implies that each proper quotient group of G is polycyclic-by-finite. The result follows. □

PROPOSITION 2.8. – *Assume that G is a $JNPC$-group, H is a nontrivial normal subgroup of G, H satisfies max-n, $HP(G/H) = R/H$. If G/H is locally soluble and R/H satisfies max-ab, then G is a Just-Non-(polycyclic-by-finite) group.*

Proof. – [[2], Theorem 3.2] implies that a locally soluble PC-group is hyperabelian, so that G/H has each proper quotient group which is hyperabelian. Now [[12], Theorem 3.31] implies that G/H is polycyclic-by-finite. Since H satisfies *max-n* and G/H is a polycyclic-by-finite group, we may conclude that G satisfies *max-n*. It follows easily from Lemma 2.1 that a PC-group which satisfies *max-n* is a polycyclic-by-finite group. Thus each proper quotient group of G is a polycyclic-by-finite group and the result follows.□

Obviously each finitely generated $JNPC$-group is a Just-Non-(polycyclic-by-finite) group. However an improvement of Lemma 2.7 can be furnished by means of [[2], Lemmas 5.10 and 5.11].

PROPOSITION 2.9. – *Let G be a locally soluble $JNPC$-group and H be a normal subgroup of G. If each infinite subset of G/H contains a pair of elements which generate a polycyclic-by-finite subgroup, then G is a Just-Non-(polycyclic-by-finite) group.*

Proof. – Since G/H is locally soluble PC-group, it is hyperabelian from [[2], Theorem 3.2]. Therefore we apply [[2], Lemma 5.10] so that G/H is a polycyclic-by-finite group. Now the result follows.□

REMARK 2.10. – *According with [7], a $JNPC$-group which satisfies the conditions of Lemma 2.7 or Proposition 2.8 or Proposition 2.9 is completely classified.*

Using wreath products we are able to construct many $JNPC$-groups: this point of view was suggested at the first time by D.J.Robinson in [13] for Just-Non-(polycyclic-by-finite) groups. This approach allows us to classify Just-Non-FC groups, Just-Non-(polycyclic-by-finite) groups and many other types of Just-Non-$\mathfrak{X}$ groups, where $\mathfrak{X}$ is a prescribed class of groups (see [7] for details).

A classical situation of a monolithic $JNPC$-group is offered by a monolithic Just-Non-(polycyclic-by-finite) group as $G = C_\infty \ltimes \mathbb{Q}_p$, where C_∞ is infinite cyclic and $\mathbb{Q}_p$ is the additive group of rational numbers with denominator a power of p for a fixed prime p. More details about this construction are mentioned in [13]. Certainly each periodic $JNPC$-group is a $JNFC$-group, since the property to be an FC-group and the property to be a PC-group coincide under periodicity. But the same example $G = C_\infty \ltimes \mathbb{Q}_p$ shows that there exist a non-periodic $JNPC$-group which is a $JNFC$-group.

In literature (see [7]) it is not known an example of a $JNPC$-group which is not a Just-Non-$\mathfrak{X}$ group, where $\mathfrak{X}$ is one of following classes of groups: polycyclic-by-finite, nilpotent, abelian, finite-by-abelian, central-by-finite, hyperfinite, hypercentral, T-groups, FC-groups, CC-groups (see [7] or [9] for details and terminology). The following example has been constructed in order to answer positively to this question.

EXAMPLE 2.11. – For convenience of the reader we recall some elementary notions in the construction of a wreath product of two groups. Such notions can be found in each textbook of Group Theory. We follow for instance [12]. We are interested in constructing the group $G = C_p \wr Dr_{i\in\mathbb{N}} D_i$, where D_i is infinite dihedral for each positive integer $i \geq 1$ and C_p is cyclic of prime order $p \neq 2$. Let C_p and $D = Dr_{i\in\mathbb{N}} D_i$ be permutation groups acting on sets C_p and D via Cayley right action, we have that $C_p \hookrightarrow \mathbb{S}_{C_p}$ and $D \hookrightarrow \mathbb{S}_D$. Given $d \in D$, $\gamma \in C_p$ and $\delta \in D$, define permutations $\gamma(d)$ and δ^* of the set product $X = C_p \times D$ by the rules:

$$\delta^* : (c, d) \mapsto (c, d\delta)$$

$$\gamma(d) : (c, d) \mapsto (c\gamma, d) \text{ and } (\mathrm{c}, \mathrm{d}_1) \mapsto (\mathrm{c}, \mathrm{d}_1), \text{if } \mathrm{d} \neq \mathrm{d}_1.$$

It is routine to verify that $(\gamma^{-1})(d) = (\gamma(d))^{-1}$ and $(\delta^{-1})^* = (\delta^*)^{-1}$ so that $\gamma(d)$ and δ^* are permutations. The functions $* : \delta \in D \mapsto \delta^* \in \mathbb{S}_X$ and $(d) : \gamma \in C_p \mapsto \gamma(d) \in \mathbb{S}_X$ are monomorphisms of groups with images respectively $D^* \simeq D$ and $C_p(d) \simeq C_p$. The wreath product G of C_p and D is the permutation group on X generated by D^* and $C_p(d)$, for $d \in D$; in symbols $G = C_p \wr D = \langle C_p(d), D^* | d \in D\rangle$. By construction we have

$$(\circ) \qquad (\delta^*)^{-1}\gamma(d)\delta^* = \gamma(d\delta) \;\text{ and }\; (\delta^*)^{-1}\mathrm{C_p}(\mathrm{d})\delta^* = \mathrm{C_p}(\mathrm{d}\delta),$$

so if $B = Dr_{d\in D} C_p(d)$, then without loss of generality $G = D \ltimes B$, that is, G is the semidirect product of B by D in which the automorphism of B produced by an element of D is given by $(\circ)$.

$B = C_G(B) \triangleleft G$ follows by construction and $z \in Z(G)$ if and only if $z \in B$ such that for all $\delta^* \in D$, $(\delta^*)^{-1} z \delta^* = z$, that is, $z = 1$ so $Z(G) = 1$. We claim that $B = \langle b\rangle^G$ for some $b \in B$. Of course, $B = B^G \geq \langle b\rangle^G$, where $b \in B$. Note that $G = BD$ and $B = C_G(B)$, so $\langle b\rangle^G = \langle b\rangle[b, G] = \langle b\rangle[b, D]$. Now $\langle [b, D] : b \in B\rangle = B$ so that an element c of B can be written as $c = [b, d]$ for a suitable $d \in D$. In particular, $c \in \langle b\rangle^G$. Then $B = \langle b\rangle^G$.

B is characteristically simple, then a G-invariant subgroup contained in B cannot be characteristic in B. Suppose that N is a nontrivial G-invariant subgroup of B. A nontrivial element n of N has p-power order. Consider n^d, where $d \in D$. If d is of infinite order, then n^d should be of p-power order. If d is periodic, then d is of 2-power order, and still n^d should be of p-power order. In both cases, we have contradictions. From this N is trivial and B is the unique minimal normal subgroup of G.

Since B is the monolith of G, $\langle g\rangle^G \geq B$ for each nontrivial element g of G. Suppose that $\langle g\rangle^G$ is a polycyclic-by-finite group, then B is polycyclic-by-finite, in particular it has finite abelian rank and this cannot be. Therefore $\langle g\rangle^G$ is

not a polycyclic-by-finite group for each element g of G so that G is not a PC-group thanks to Lemma 2.1.
Note that $B \cap D$ is trivial. The monolithic structure of G implies that $G' \geq B$. The rules of commutators and the fact that B is normal abelian in G imply

$$G' = [G,G] = [BD,BD] = [B,BD]^D[D,BD] = [B,BD]^D[D,D][D,B]^D =$$
$$= [B,BD]^D[D,D][D^D,B^D] = [B^D,(BD)^D]D'[D,B] = [B,G]D'[D,B] =$$
$$= [B,BD]D'[B,D] = [B,D]D'[B,D] = [B,D]D'.$$

Repeating the previous steps and noting that D' is abelian, we have that

$$G'' = [[B,D]D',[B,D]D'] = [[B,D],D']D'' = [[B,D],D'].$$

Since $B \cap D = B \cap D' = 1$ and $G' = [B,D]D' \geq B$, we have $[B,D] = B$ so that

$$G''' = [[[B,D],D'],[[B,D],D']] = [[B,D'],[B,D']] \leq [B,B] = 1.$$

By monolithic structure of G, if $1 \neq N \triangleleft G$, then $N \geq B$, $D \simeq G/B \geq G/N$. By Lemma 2.2 G/N is a PC-group and G is a $JNPC$-group. The quotient G/B is a PC-group which does not satisfy $\mathfrak{X}$, where $\mathfrak{X}$ is one of the classes of groups which have been above mentioned. Thus G satisfies our requirements. It is also interesting to note that G is soluble of $derG = 3$ thanks to the series $1 = G''' \triangleleft G'' \triangleleft G' \triangleleft G$.

Example 2.12. – G is the semidirect product of $D = Dr_{i\in\mathbb{N}}D_i$ by $Q = Dr_{j\in\mathbb{N}}Q_j$, where D_i is infinite dihedral for all integers $i \geq 1$ and Q_j is the additive group of the rational numbers for all integers $j \geq 1$.
D_i has a presentation with two generators x_i, y_i of order two and no other relations: $D_i \simeq \langle x_i\rangle * \langle y_i\rangle$. D_i has an infinite cyclic subgroup A_i of index two, generated by $a_i = x_iy_i$, so that an arbitrary element $d_i \in D_i$ can be uniquely written as $d_i = a_ix_i$ (see also [12]). For each $\alpha_i \in \mathbb{Q} \setminus \{0,1,-1\}$ the map θ_{α_i}, defined by

$$x_iy_i \in D_i \longmapsto \theta_{\alpha_i}(x_iy_i) = \begin{pmatrix} \alpha_i & 0 \\ 0 & \alpha_i^{-1} \end{pmatrix} \in GL(2,\mathbb{Q})_i$$

$$x_i \in D_i \longmapsto \theta_{\alpha_i}(x_i) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \in GL(2,\mathbb{Q})_i$$

is a monomorphism from D_i into $GL(2,\mathbb{Q})_i$. Obviously this position defines also $\theta_{\alpha_i}(d_i)$. It is easy to check that θ_{α_i} is an irreducible representation of D_i in $GL(2,\mathbb{Q})_i$. By chap. XVI of [4], $GL(2,\mathbb{Q})_1 \times GL(2,\mathbb{Q})_2 \times \ldots \leq GL(|\mathbb{N}|,\mathbb{Q})$, so we may extend θ_{α_i} to a monomorphism θ_α from D into $GL(|\mathbb{N}|,\mathbb{Q})$ via

$$d = d_1 \ldots d_n \in D \longmapsto \theta_\alpha(d) = \theta_{\alpha_1}(d_1) \ldots \theta_{\alpha_n}(d_n) \in GL(|\mathbb{N}|,\mathbb{Q}),$$

where n is a positive integer. By [[4], (c), p.250] $AutQ \simeq GL(|\mathbb{N}|,\mathbb{Q})$ and we may consider $G = D \ltimes_{\theta_\alpha} Q$. Here $C_G(Q) = Q \triangleleft G$ and Q is characteristically simple. An argument as in Example 2.11 shows that G is monolithic with monolith Q. Furthermore, an argument as in Example 2.11 shows that G is a $JNPC$-group. The quotient G/Q is a PC-group which is not an $\mathfrak{X}$-group where $\mathfrak{X}$ is one of the classes of groups which have been above mentioned.

Now we pass to list some properties which we will use to classify a $JNPC$-group.

Lemma 2.13. – *Let G be a $JNPC$-group. If N is a normal nilpotent subgroup of G, then N is abelian.*

Proof. – Let N be a nontrivial normal nilpotent subgroup of G such that $N' \neq 1$. Then G/N' is an PC-group. Now we apply [[8], Lemma 3.1] which is a generalization of the famous Hall's criterion of nilpotence (see for instance [12]). Then G is an PC-nilpotent group, that is, G has a characteristic series

$$1 = P_0 \triangleleft P_1 \triangleleft P_2 \ldots \triangleleft P_c = G,$$

where c is a positive integer,

$$P_1 = PC(G),\ P_2/P_1 = PC(G/P_1),\ \ldots,\ P_c/P_{c-1} = PC(G/P_{c-1}).$$

Further details can be found for instance in [8] and [10]. Since G is an PC-nilpotent group, $PC(G) \neq 1$ which is against Lemma 2.4. Then the result follows. □

Lemma 2.14. – *Let G be a $JNPC$-group. If $A = Fit\, G \neq 1$ then either A is a torsion-free abelian group or A is an elementary abelian p-group for some prime p.*

Proof. – Let $x, y \in A$. Then there are normal nilpotent subgroups L_x and L_y such that $x \in L_x$ and $y \in L_y$. According to Fitting's Theorem the subgroup L_xL_y is nilpotent, hence Lemma 2.13 implies that it is abelian, thus $xy = yx$ and A is abelian.
Let $T = T(A)$ be the torsion subgroup of A and let $T \neq 1$. Without loss of generality we can suppose that T is a p-group for some prime p. Assume that

$T^p \neq 1$. Then $T \neq T_1 = \Omega_1(T) = \{x \in T : x^p = 1\}$. G/T_1 is a PC-group, then Lemma 2.1 implies that $\langle g\rangle^G T_1/T_1$ is a polycyclic-by-finite group, where $g \in G \setminus T_1$.
Assume that $\langle g\rangle^G T_1/T_1$ is finite. A Sylow subgroup P/T_1 of $\langle g\rangle^G T_1/T_1$ is a finite nontrivial G-invariant subgroup of $\langle g\rangle^G T_1/T_1$. Let $H/T_1 = C_G(P/T_1)$. If $h \in H$ and $c \in P \setminus T_1$, then $[c,h]T_1 = [cT_1, hT_1] \in [P/T_1, H/T_1] = 1$ and $[c,h] \in T_1$. This allows us to say that $1 = [c,h]^p = [c^p, h]$, $c \notin T_1$, $c^p \neq 1$, then $c^p \in Z(H)$, $Z(H) \neq 1$. Since $\langle c^p\rangle$ is characteristic in H and H is normal in G, $\langle c^p\rangle$ is a nontrivial normal subgroup of G which contradicts Lemma 2.4. The contradiction implies that $\langle g\rangle^G T_1/T_1$ must be infinite.
Assume that $\langle g\rangle^G T_1/T_1$ is an infinite polycyclic-by-finite group. $\langle g\rangle^G T_1/T_1$ contains a nontrivial G-invariant torsion-free abelian subgroup which is finitely generated (see for instance [[12], Chapter 3, p.65]). We call P/T_1 such subgroup. P/T_1 contains a normal subgroup P_1/T_1 such that $(P/T_1)/(P_1/T_1) \simeq P/P_1$ is finite abelian and we may suppose that it is a p-group so that $|P/P_1| = p^n$ for some positive integer $n \geq 1$. Put $H/P_1 = C_G(P/P_1)$, $h \in H$, $1 \neq c \in P \setminus P_1$. We have again $[c,h]P_1 = [cP_1, hP_1] \in [P/P_1, H/P_1] = 1$, so $[c,h]^p = [c^p, h] \in P_1$, $c^p \neq 1$, $c \notin P_1$ and $c^p \in Z(H)$. Again $\langle c^p\rangle$ is a nontrivial normal subgroup of G which contradicts Lemma 2.4.
Such contradictions show that $T^p = 1$, that is, $T = T_1$ is an elementary abelian p-group.
On the other hand if $A \neq T$, then A can be decomposed in the sense of Prüfer as $A = T \times B$, being B a torsion-free abelian subgroup of A isomorphic with A/T. Hence $B \geq A^p$ and $A^p \cap T = 1$. Since $A \neq T$, $A^p \neq 1$. Therefore $A^p \cap T = 1$ is against Lemma 2.5. We may conclude that $A = B$ so that the lemma follows. □

LEMMA 2.15. – *Assume that G is a group, A is a normal subgroup of G and p is a prime. If A is either a maximal elementary abelian p-group or a maximal torsion-free abelian group, then $C_G(A)$ is a maximal abelian subgroup of G.*

Proof. – Of course, $A \leq C_G(A)$. Conversely, $A \geq C_G(A)$, because A is maximal abelian. Then the result follows with $A = C_G(A)$. □

COROLLARY 2.16. – *Let G be a $JNPC$-group with nontrivial Fitting subgroup $A = Fit(G)$. Then $A = C_G(A)$.*

Proof. – By Lemma 2.14 either A is a torsion-free abelian group or A is an elementary abelian p-group. A is a maximal abelian subgroup in G, then $C_G(A) = A$ by Lemma 2.15. □

3. Monolithic Just-Non-PC groups

In a Just-Non-$\mathfrak{X}$ group, where $\mathfrak{X}$ is a prescribed class of groups (see [7]), the action of the Fitting subgroup is fundamental to obtain structural conditions on the whole group. Here we will see that a monolithic $JNPC$-group splits on its Fitting subgroup: the following result of D.J.Robinson will be useful [[7], Theorem 4.5].

THEOREM 3.1. – *Let G be a group with an abelian subgroup A satisfying the minimal condition on its G-invariant subgroups and let K be a normal subgroup of G satisfying the following conditions:*

(i) *$K \geq A$ and K/A is locally nilpotent;*

(ii) *the FC-hypercenter of $G/C_K(A)$ includes $K/C_K(A)$;*

(iii) *$A \cap Z(K) = 1$.*

Then G contains a free abelian subgroup X such that the index $|G : XA|$ is finite and $X \cap A = 1$ (nearly splitting of G on A). Moreover the complements of A in G fall into finitely many conjugacy classes.

Also the following notion can be useful in order to formulate our main results of classification of monolithic $JNPC$-groups.

DEFINITION 3.2. – *According to Lemma 2.14, we will say that a $JNPC$-group G with $1 \neq FitG = A$ has char$A$=0 if A is a torsion-free abelian group. We will say that G has char$A$=p, for some prime p, if A is an elementary abelian p-group.*

The following two results classify monolithic $JNPC$-groups.

THEOREM 3.3. – *Let G be a monolithic group with $1 \neq A = FitG$. If G is a $JNPC$-group, char$A = 0$ and G/A is locally nilpotent, then*

(i) *A is torsion-free abelian;*

(ii) *$A = C_G(A)$ is the monolith of G;*

(iii) *G contains a free abelian subgroup X such that $|G : XA|$ is finite and $X \cap A = 1$ (nearly splitting of G on A). If G splits over A, the complements of A fall into finitely many conjugacy classes.*

Proof. – (i). Follows by Lemma 2.14.
(ii). By Corollary 2.16 $A = C_G(A)$ so we must prove that $A = M$ where M

is the monolith of G, that is, M is the unique minimal normal subgroup of G. By definition of M, $A \geq M$, so M is abelian. Conversely we suppose that M is nontrivial abelian and $M > A$. Then $M > C_G(A)$ and $C_G(A)$ is not a maximal abelian subgroup of G, against Lemma 2.15.
(iii). By the previous steps (i) and (ii), A is an abelian subgroup of G which satisfies the minimal condition on its G-invariant subgroups. G/A is a locally nilpotent PC-group such that $G/C_G(A) = G/A$. Theorem 4.38]. Now Theorem 2.6 implies that $Z(G) \cap A = 1$. We may apply Theorem 3.1 so that (iii) is proved. □

Theorem 3.4. – *Let G be a monolithic group with $1 \neq A = FitG$. If G is a $JNPC$-group, $char A = p$ for some prime p and G/A is locally nilpotent, then*

(i) *A is p-elementary abelian;*

(ii) *$A = C_G(A)$ is the monolith of G;*

(iii) *G contains a free abelian subgroup X such that $|G : XA|$ is finite and $X \cap A = 1$ (nearly splitting of G on A). If G splits over A, the complements of A fall into finitely many conjugacy classes.*

Proof. – A similar argument as in Theorem 3.3 can be applied.□

4. Non-monolithic Just-Non-PC groups

If R is a ring, an infinite R-module M is called just-infinite if all its proper factor modules are finite and the intersection of all its nontrivial R-submodules is trivial. Let G be a $JNPC$-group with nontrivial Fitting subgroup A. Then A is abelian by Lemma 2.15, so it can be regarded as a module over the ring $\mathbb{Z}H$, where $H = G/A$ is a PC-group. In this sense it is natural to ask when A is a just-infinite $\mathbb{Z}H$-module. We will see that the positive answer to this question plays an important role in the study of non-monolithic $JNPC$-groups. Here we treat in details such problem.

Definition 4.1. – *A PC-group G is said to be strongly PC if for each element g of G, either $\langle g\rangle^G$ is finite or it contains a normal subgroup N such that $\langle g\rangle^G/N$ is infinite cyclic.*

By previous definition an FC-group is strongly PC, but Example 5.5 in [2] shows that there are FC-groups which are not strongly PC. In other terms a PC-group G is strongly PC if for each element g of G there exists a positive integer n and a finite series

$$1 = G_0 \lhd G_1 \lhd G_2 \lhd \ldots \lhd G_n = \langle g \rangle^G$$

such that either $\langle g \rangle^G$ is finite or, G_1 is finite and $G_2/G_1, \ldots, G_n/G_{n-1}$ are infinite cyclic.
Let G be a PC-group which is strongly PC and g be a nontrivial element of G. The symbol $h(\langle g \rangle^G)$ will denote the Hirsch length of $\langle g \rangle^G$. Elementary properties of such invariant are described both in [9] and [12]. While each polycyclic-by-finite group is a PC-group, there are some polycyclic-by-finite groups which are not strongly PC, for instance the infinite dihedral group. Certainly each finitely generated nilpotent group with no proper chief factors of order 2 is always strongly PC. The same holds for supersoluble groups with no proper chief factors of order 2. We will see that the condition to be strongly PC allows us to reduce the theory of $JNPC$-groups to the theory of $JNFC$-groups. Next lemmas adapt the main statements of [[7], Chapter 16].

LEMMA 4.2. – *Let G be a non-monolithic $JNPC$-group. If $1 \neq A = FitG$, $a \in A$, $B = \langle a \rangle^G$, $H = G/A$, $charA = 0$ and each proper quotient of G is strongly PC, then B includes a nontrivial G-invariant subgroup C such that B/C is polycyclic-by-finite and C is a just-infinite $\mathbb{Z}H$-module.*

Proof. – Let K be a nontrivial G-invariant subgroup of B. G/K is a PC-group, so BK/K is a polycyclic-by-finite group by Lemma 2.1, in particular B/K is a polycyclic-by-finite group.
Let U be a nontrivial G-invariant subgroup of B such that $h(B/U) \geq 0$ is maximal. The existence of such a maximum is due to the fact that B/U is polycyclic-by-finite. When $h(B/U) = 0$, B/U is finite so for each nontrivial G-invariant subgroup W of U we have $h(U/W) \leq h(B/W) = 0$, hence U/W is finite.
Let $h(B/U) \geq 1$. Each proper quotient of G is strongly PC, so we may suppose in a first moment that BU/U is a finite-by-cyclic group. Then $h(BU/U) = h(B/U) = 1$ and $h(U/W) \leq h(B/W) - h(B/U) = h(B/W) - 1$, but the choice of U implies that $h(B/W) = h(B/U)$. We have $h(U/W) = 0$ and U/W is finite. In this case $h(B/U) = 1$ and U is a $\mathbb{Z}$-irreducible $\mathbb{Z}H$-module. Given a nontrivial element $u \in U$, $C = \langle u \rangle^G$ is the required subgroup.
Assume now that $h(B/U) > 1$, in particular, let $h(B/U) = 2$. We have that $BU/U = (F/U)(PU/U)$ contains a normal series

$$U \lhd F \lhd P_1U \lhd BU$$

with F/U normal finite subgroup of BU/U such that both P_1U/F and BU/P_1U are infinite cyclic. The existence of such series is given by the condition of strongly PC on G/U. Clearly $1 < h(B/U) \leq 2$ so $h(B/U) = 2$. The choice of U implies $h(B/W) \leq h(B/U) = 2$ then $0 \leq h(U/W) \leq h(B/W) - h(B/U)$ gives $h(U/W) = 0$ and again U/W is finite. By induction we may generalize such argument for each positive integer $h(B/U)$, concluding that U/W is finite. Again we obtain that U is a $\mathbb{Z}$-irreducible $\mathbb{Z}H$-module. Given a nontrivial element $u \in U$, $C = \langle u \rangle^G$ is the required subgroup. □

LEMMA 4.3. – *Let G be a non-monolithic $JNPC$-group. If $1 \neq A = FitG$, $a \in A$, $B = \langle a \rangle^G$, $H = G/A$, $charA = p$ for some prime p, then B is a just-infinite $\mathbb{Z}H$-module.*

Proof. – For all nontrivial G-invariant subgroups U of B, we have that $B/U \leq BU/U$. By Lemma 2.1 and Lemma 2.2 BU/U is a periodic and polycyclic-by-finite group, so it is finite.□

LEMMA 4.4. – *Let G be a non-monolithic $JNPC$-group. If $1 \neq A = FitG$, $H = G/A$ is FC-hypercentral, each proper quotient of G is strongly PC, $charA = 0$ and A includes a nontrivial G-invariant subgroup B such that B is a $\mathbb{Z}$-irreducible $\mathbb{Z}H$-module, then A is a $\mathbb{Z}$-irreducible $\mathbb{Z}H$-module.*

Proof. – It is sufficient to show that A/B is periodic. Suppose that it is false and choose a nontrivial element $aB \in A/B$ of infinite order. Since G/B is strongly PC, $C/B = \langle a \rangle^G B/B$ includes a normal subgroup U/B whose quotient C/U is infinite cyclic.
We will repeat the argument which has been used in the proof of Lemma 4.2. Firstly we suppose that C/B is finite-by-(infinite cyclic). U/B is a $\mathbb{Z}$-irreducible $\mathbb{Z}H$-module, since $|G : C_G(U/B)| \leq 2$. Therefore U is a $\mathbb{Z}$-irreducible $\mathbb{Z}H$-module, since $B \leq U$. By Lemma 2.4 G has trivial PC-center, then $G/C_G(B)$ is infinite. [[7], Corollary 4.4] implies that C includes a G-invariant subgroup L such that $L \cap U \neq 1$, against Lemma 2.5. Then we may suppose that C/B contains a normal series $B \triangleleft U \triangleleft U_1 \triangleleft C$ where U_1/B is a finite-by-(infinite cyclic) group and C/U_1 is infinite cyclic. By previous step U_1/B is a $\mathbb{Z}$-irreducible $\mathbb{Z}H$-module, $|G : C_G(C/U_1)| \leq 2$ and C/U_1 is a $\mathbb{Z}$-irreducible $\mathbb{Z}H$-module. As before we contradict Lemma 2.5. By induction we get to our final contradiction, concluding that A is a $\mathbb{Z}$-irreducible $\mathbb{Z}H$-module.□

LEMMA 4.5. – *Let G be a non-monolithic $JNPC$-group. If $1 \neq A = FitG$, $a \in A$, $B = \langle a \rangle^G$, $H = G/A$, $charA = 0$, H is FC-hypercentral and each proper quotient of G is strongly PC, then the $\mathbb{Z}H$-module B is just-infinite. In particular the $\mathbb{Z}H$-module A is $\mathbb{Z}$-irreducible.*

Proof. – Lemma 4.2 implies that B includes a nontrivial G-invariant subgroup C such that B/C is polycyclic-by-finite and C is a just-infinite $\mathbb{Z}H$-module. Put T/C the torsion subgroup of B/C, T is a $\mathbb{Z}$-irreducible $\mathbb{Z}H$-module. By Lemma 4.4 A is a $\mathbb{Z}$-irreducible $\mathbb{Z}H$-module, then B is a just-infinite $\mathbb{Z}H$-module.□

Corollary 4.6. – *Let G be a non-monolithic $JNPC$-group. If $1 \neq A = FitG$, $a \in A$, $B = \langle a\rangle^G$, $H = G/A$, $charA = 0$, H is FC-hypercentral and each proper quotient of G is strongly PC, then $C_G(B) = A$.*

Proof. – Lemma 4.4 implies A/B periodic. Let $b \in A$, $g \in C_G(B)$, $b_1 = b^g$. There is a positive integer n such that $b^n \in B$. We have $b_1^n = (b^g)^n = (b^n)^g = b^n$. It follows that $b = b_1$, since $charA = 0$. Hence $C_G(B) = C_G(A) = A$.□

Corollary 4.7. – *Let G be a non-monolithic $JNPC$-group. If $1 \neq A = FitG$, $a \in A$, $B = \langle a\rangle^G$, $H = G/A$ and $charA = p$ for some prime p, then $C_G(B) = A$.*

Proof. – A/B is p-group and let $b \in A$, $g \in C_G(B)$, $b_1 = b^g$. There is a positive integer m such that $1 \neq b^{p^m} \in B$. Now $b_1^{p^m} = (b^g)^{p^m} = (b^{p^m})^g = b^{p^m}$, so $b_1^{p^m} \in C_G(B) \cap A$. Hence $C_G(B) = C_G(A) = A$. □

Proposition 4.8. – *Let G be a non-monolithic $JNPC$-group. If $1 \neq A = FitG$, $H = G/A$ is FC-hypercentral, each proper quotient of G is strongly PC and $charA = 0$, then H is an FC-group.*

Proof. – Put $1 \neq a \in A$ and $B = \langle a\rangle^G$ Lemma 4.5 implies that the $\mathbb{Z}H$-module B is just-infinite, while Corollary 4.6 yields to $C_G(B) = A$. Since G is non-monolithic, B contains a nontrivial G-invariant subgroup C. Assume that $x \in H$, $X = \langle x\rangle^H$ and R is the subgroup of X generated by all the normal subgroups Y of X with infinite index $|X : Y|$. When $R \leq A$, X is finite and we finish. Suppose that $R \not\leq A$. Certainly $R \cap C_G(B/C) \leq R$, but B/C is finite hence $|G : C_G(B/C)|$ is finite. In particular there is an element $r \in R$ and an integer $n > 1$ such that $r^n \in C_G(B/C)$, so $R \cap C_G(B/C) = R$. We have proved that $R \leq C_G(B/C)$ for each G-invariant subgroup C of B, then $[B, R] \leq C$, but $\bigcap_{i\in I} C_i = 1$ when C_i is a G-invariant subgroup of B, so $[B, R] = 1$. We get $R \leq C_G(B) = A$ and there is a contradiction. We conclude that H is an FC-group. □

Proposition 4.9. – *Let G be a non-monolithic $JNPC$-group. If $1 \neq A = FitG$, each proper quotient of G is strongly PC and $charA = p$ for some prime p, then H is an FC-group.*

Proof. – Put $1 \neq a \in A$ and $B = \langle a\rangle^G$ Lemma 4.4 implies that the $\mathbb{Z}H$-module B is just-infinite, while Corollary 4.7 yields to $C_G(B) = A$. Since G is non-monolithic, B contains a nontrivial G-invariant subgroup C. We finish thanks to the argument of Proposition 4.8. □

Corollary 4.10. – *Let G be a non-monolithic $JNPC$-group. If $1 \neq A =$*

$FitG$, $H = G/A$ is FC-hypercentral, each proper quotient of G is strongly PC and $char A = 0$, then H is a central-by-finite group and $Z(H)$ includes a torsion-free subgroup of finite index.

Proof. – Put $1 \neq a \in A$ and $B = \langle a \rangle^G$. Lemma 4.5 shows that the $\mathbb{Z}H$-module B is just-infinite. Corollary 4.6 implies $C_G(B) = A$. Proposition 4.9 implies that H is an FC-group. It is sufficient to apply [[7], Corollary 6.16]□

Corollary 4.11. – *Let G be a non-monolithic $JNPC$-group. If $1 \neq A = FitG$, $H = G/A$ is FC-hypercentral, each proper quotient of G is strongly PC, $char A = 0$ and there is a nontrivial element $zA \in Z(G/A)$ such that $U = C_G(\langle z \rangle^G B/B)$, then the $\mathbb{Z}G$-module $U \cap A$ is just-infinite.*

Proof. – By Corollary 2.16 $A = C_G(A)$. The mapping $\phi : a \in A \mapsto [a, z] \in A$ is a $\mathbb{Z}G$-homomorphism of A with G-invariant subgroups $Ker\phi = C_A(z)$ and $Im\phi = [A, z]$. Lemma 4.4 implies that the $\mathbb{Z}H$-module A is $\mathbb{Z}$-irreducible, then $A/Ker\phi$ has to be finite. On the other hand $Ker\phi$ is a pure group (see [4]), otherwise it would be an abelian finitely generated group, against Lemma 2.4. We conclude that $Ker\phi = 1$ so $A \simeq [A, z]$, in particular $U \cap A \simeq [U \cap A, z]$. The choice of z implies $[U \cap A, z] \leq B$. This means that the $\mathbb{Z}G$-module $[U \cap A, z]$ is just-infinite, hence the $\mathbb{Z}G$-module $U \cap A$ is just-infinite.□

Proposition 4.12. *Let G be a non-monolithic $JNPC$-group. If $1 \neq A = FitG$, G/A is FC-hypercentral, each proper quotient of G is strongly PC and $char A = 0$, then the $\mathbb{Z}G$-module A is just-infinite.*

Proof. – Corollary 2.16 says that $A = C_G(A)$ and Corollary 4.10 implies that G/A is central-by-finite such that $Z(G/A)$ includes a torsion-free subgroup of finite index. Let $1 \neq a \in A$, $B = \langle a \rangle^G$ and zA nontrivial element of $Z(G/A)$ with infinite order such that $U/B = C_G(\langle z \rangle^G B/B)$. Corollary 4.11 implies that the $\mathbb{Z}G$-module $U \cap A$ is just-infinite, in particular $U \cap A/B$ is finite.
Now $U/(U \cap A) \simeq UA/A \leq G/A$ and G/A is central-by-finite, thus U/B is an FC-group. Since $zA \in Z(U/B)$, U/B is non-periodic and $(U/B)/Z(U/B)$ is a periodic FC-group (see [[12], Theorem 4.32]). By Corollary 4.10 $U/U \cap A$ is central-by-finite. In particular the torsion subgroup $T/(U \cap A)$ of $U/(U \cap A)$ is finite, but also $(U \cap A)/B$ is finite, so we have that the torsion subgroup T/B of U/B is finite.
Put $Z(G/B) = Z/B$, the Prüfer decomposition of Z/B implies that

$$Z/B = X/B \times (Z/B \cap T/B)$$

for X/B torsion-free abelian. Let $t = |(Z/B) \cap (T/B)|$, then $(Z/B)^t = Y/B \leq X/B$. In particular Y/B is a nontrivial torsion-free abelian normal subgroup of G/B. Now $Y/B \leq Z(U/B) = Z(G/B) \cap U/B \leq Z(G/B)$, so $C_G(yB) = G/B$ for a suitable element $yB \in Y/B$. By Corollary 4.11 we conclude that the

$\mathbb{Z}G$-module A is just-infinite.□

THEOREM 4.13. – *Let G be a non-monolithic $JNPC$-group with $1 \neq A = FitG$ and $H = G/A$.*

(i) *If H is FC-hypercentral, each proper quotient of G is strongly PC and $charA = 0$, then G is a $JNFC$-group.*

(ii) *If $charA = p$ for some prime p and each proper quotient of G is strongly PC, then G is a $JNFC$-group.*

PROOF. – (i). Let $charA = 0$. By Corollary 4.10, H is a central-by-finite group and Proposition 4.12 implies that the $\mathbb{Z}G$-module A is just-infinite. Let N be a nontrivial normal subgroup of G. By Lemma 2.5, $K = N \cap A \neq 1$, then A/K is finite, so G/K is a finite extension of a central-by-finite group, then G/K is again a central-by-finite group, in particular it is an FC-group. We conclude that G/N is an FC-group, so that the statement follows.
(ii). Let $charA = p$. By Proposition 4.9 we know that H is an FC-group. Let N be a nontrivial normal subgroup of G and $x \in G \setminus N$. Clearly $\langle x\rangle^G N/N \leq A/N$ is a periodic policyclic-by-finite group so that $\langle x\rangle^G N/N$ is finite. As before we may conclude that G/N is an FC-group. □

5. Special cases of Just-Non-PC groups

We end with a general theorem on the structure of a $JNPC$-group whose factor group $G/FitG$ has finite abelian rank.

PROPOSITION 5.1. – *Assume that G is a $JNPC$-group, $1 \neq A = FitG$ and $G/A = H$. If $Z(H) = 1$ and H has finite abelian rank, then G is an abelian-by-(polycyclic-by-finite) group.*

Proof. – By Lemma 2.15 A is either an elementary abelian p-group for some prime p or a torsion-free abelian group, so there is a subgroup H of G such that $H \simeq G/A$ and $G = AH$. By [[2], Theorem 4.7] it exists a monomorphism of groups

$$\mu : H \mapsto Dr_{i\in I} P_i,$$

where P_i is a polycyclic-by-finite group for each element i of a nonempty index set I. $H \simeq \mu(H) = Dr_{j\in J} H_j$ for H_j which is a polycyclic-by-finite group, J nonempty subset of I, $j \in J$. Suppose that J is infinite, then the ascending series $1 \triangleleft H_1^{(n_1-1)} \triangleleft H_1^{(n_1-1)} \times H_2^{(n_2-1)} \triangleleft \dots$ allows us to consider the abelian subgroup $Dr_{j\in J} H_j^{(n_j-1)}$ which has infinite abelian rank against that H has finite abelian rank. Therefore, J is finite and H is a polycyclic-by-finite group. Now it is clear that G is a abelian-by-(polycyclic-by-finite) group.□

Proposition 5.2. – *Assume that G is a $JNPC$-group, $1 \neq A = FitG$ and $H = G/A$. If $|H/Z(H)|$ is countable and H has finite abelian rank, then G is an abelian-by-(polycyclic-by-finite) group.*

Proof. – It can be found in [2] that a PC-group L whose quotient $L/Z(L)$ is at most countable can be embedded in a direct product of polycyclic-by-finite groups. Thanks to this result, we may use again the argument of the proof of Proposition 5.1.□

References

[1] S. Franciosi, F. de Giovanni and L. Kurdachenko, Just-Non-FC groups, *J. Algebra* **186** (1996), 544-577.

[2] S. Franciosi, F. de Giovanni and M. J. Tomkinson, Groups with Polycyclic-by-finite conjugacy classes, *Boll. U.M.I.* **7** (1990), 35-55.

[3] S. Franciosi, F. de Giovanni and M. J. Tomkinson, Groups with Chernikov conjugacy classes, *J. Austral. Math. Soc.* **50** series A (1991), 1-14.

[4] L. Fuchs, *Infinite Abelian Groups*, vol.I and vol.II, Academic Press,Inc., New York, 1970 and 1973.

[5] M. Gonzalez and J. Otál, The extension of results due to Gorchakov and Tomkinson from FC-groups to CC-groups, *J. Algebra* **185** (1996), 314-328.

[6] H. Heineken, Groups with all quotient groups lagrangian. *Arch. Mat.* **64** (1995), 97-102.

[7] L. Kurdachenko, J. Otál and I. Subbotin, In: *Groups With Prescribed Quotient Subgroups and Associated Module Theory*, World Scientific, Singapore, 2002.

[8] T. Landolfi, On generalized central series of groups, *Ricerche Mat.* **XLIV** (1995), 337-347.

[9] J. C. Lennox and D. J. Robinson, *The Theory of Infinite Soluble Groups*, Oxford University Press, Oxford, 2004.

[10] R. Maier and J. R.Rogério, $\mathfrak{X}C$-elements in groups and Dietzmann classes, *Beitr. Algebra Geom.* **40** (1999), 243-260.

[11] Ya. D. Polovicky, Groups with extremal classes of conjugated elements, *Siberian Math. J.* **5**(1964), 891-895.

[12] D. J. Robinson, *Finiteness Conditions and Generalized Soluble Groups*, vol. I and vol.II, Springer Verlag, Berlin, 1972.

[13] D. J. Robinson and J. Wilson, Soluble groups with many polycyclic quotients, *Proc. London Math. Soc.* **48** (3) (1984), 193-229.

[14] M. J. Tomkinson, *FC groups*, Research Notes in Mathematics, 96, Pitman, Boston,1994.

Received: January 1, 2008